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Recurrent conformal 2-forms on pseudo-Riemannian manifolds

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Dedicated to Professor Witold Roter on his 82nd Birthday

In this paper, we introduce the notion of recurrent conformal 2-forms on a pseudo-Riemannian manifold of arbitrary signature. Some theorems already proved for the same differential structure on a Riemannian manifold are proven to hold in this more general contest. Moreover other interesting results are pointed out; it is proven that if the associated covector is closed, then the Ricci tensor is Riemann compatible or equivalently, Weyl compatible: these notions were recently introduced and investigated by one of the present authors. Further some new results about the vanishing of some Weyl scalars on a pseudo-Riemannian manifold are given: it turns out that they are consequence of the generalized Derdziński–Shen theorem. Topological properties involving the vanishing of Pontryagin forms and recurrent conformal 2-forms are then stated. Finally, we study the properties of recurrent conformal 2-forms on Lorentzian manifolds (space-times). Previous theorems stated on a pseudo-Riemannian manifold of arbitrary signature are then interpreted in the light of the classification of space-times in four or in higher dimensions.

Keywords: Recurrent conformal 2-forms; conformal curvature tensor; conformally recurrent; pseudo-Riemannian manifolds; Weyl compatible and Pontryagin forms; Petrov types; Lorentzian metrics.

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1. Introduction

Recurrent manifolds have been of great interest and were investigated by many geometers (see for example [1, 24, 25, 32, 33, 52, 74]). In particular, Walker studied manifolds on which the Riemann curvature tensor is recurrent [74] while conformally recurrent manifolds were investigated by Adati and Miyazawa [1] as well as other geometers (see for example [24, 25, 52, 58–61]). In 1972, McLenaghan and Leroy [46] and then McLenaghan and Thompson [47] embarked on a detailed investigation of space-times with complex-recurrent conformal curvature tensor. They showed that such spaces belong to types D and N of the Petrov–Penrose diagram, and found the metric forms of these spaces in the case in which the recurrence vector is real. In [36] the authors introduced the notion of K-recurrent manifold. A Riemannian manifold with generalized curvature tensor is said to be K-recurrent tensor $(KRM)_n$ if it is nonzero and satisfies the following condition (see [18, 60]):

$$\nabla_i K^m_{jkl} = \alpha_i K^m_{jkl}, \tag{1}$$

where α_i is a non-null covector.

Recently, the present authors (see [42–44]) introduced the notion of recurrent forms on a Riemannian manifold. They stated the following:

Definition 1. Let M be an n-dimensional Riemannian manifold. The curvature 2-form $\Omega^m_{(K)l} = K^m_{jkl} dx^j \wedge dx^k$ is said to be recurrent if there exist a nonzero scalar 1-form α for which:

$$D\Omega^m_{(K)l} = \alpha \wedge \Omega^m_{(K)l}, \tag{2}$$

being $\alpha = \alpha_i dx^i$ the associated 1-form and $D\Omega^m_{(K)l} = \nabla_i K^m_{jkl} dx^i \wedge dx^j \wedge dx^k$ is the covariant exterior derivative [35] associated to the connection ∇ with respect to the (positive definite) metric g_{kl}.

In particular, they investigated the properties of *recurrent conformal* 2-forms on an n-dimensional Riemannian manifold, i.e.:

$$D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}, \tag{3}$$

being $\Omega^m_{(C)l} = C^m_{jkl} dx^j \wedge dx^k$ the curvature 2-form associated to the conformal curvature tensor, defined in local components as [56]:

$$\begin{aligned} C^m_{jkl} &= R^m_{jkl} + \frac{1}{n-2}(\delta^m_j R_{kl} - \delta^m_k R_{jl} + R^m_j g_{kl} - R^m_k g_{jl}) \\ &\quad - \frac{R}{(n-1)(n-2)}(\delta^m_j g_{kl} - \delta^m_k g_{jl}). \end{aligned} \tag{4}$$

In the previous expression the Ricci tensor is defined as $R_{kl} = -R^m_{mkl}$ [75] and the scalar curvature as $R = g^{ij} R_{ij}$ (it may be scrutinized that the conformal curvature tensor vanishes identically for $n = 3$ [56]). Some useful theorems about recurrent curvature 2-forms were stated and the particular case with harmonic conformal curvature tensor i.e. for which $\nabla_m C^m_{jkl} = 0$ [5] was investigated. In the same paper was introduced the notion of recurrence of a generalized Ricci 1-form, i.e. for the vector-valued 1-form defined as:

$$\Lambda_{(K)l} = K_{kl} dx^k, \tag{5}$$

being $K_{kl} = -K^m{}_{mkl}$ the symmetric contraction of the generalized curvature tensor. The following definition was stated:

Definition 2. Let M be an n-dimensional Riemannian manifold. The generalized Ricci 1-form $\Lambda_{(K)l} = K_{kl}dx^k$ is said to be recurrent if there exist a nonzero scalar 1-form β for which:

$$D\Lambda_{(K)l} = \beta \wedge \Lambda_{(K)l}, \tag{6}$$

being $\beta = \beta_i dx^i$ the associated 1-form.

In [41] and subsequently in [37] a new generalized $(0,2)$ symmetric tensor was introduced and studied; precisely the new tensor was defined as:

$$Z_{kl} = R_{kl} + \varphi g_{kl}, \tag{7}$$

where φ is an arbitrary scalar function and was named *generalized Z tensor*. These authors pointed out several interesting properties of such tensor. From the results in [37, 41] the Z tensor may be used to write the Einstein's field equations of general relativity (see [15, 29, 65, 66, 73]). In fact the equation $Z_{kl} = kT_{kl}$ being $k = \frac{8\pi G}{c^4}$ the Einstein's gravitational constant and the condition $\nabla^l Z_{kl} = 0$ coming from the stress-energy tensor give $\nabla_k(\frac{R}{2} + \varphi) = 0$ that is $\varphi = -\frac{R}{2} + \Lambda$. The term Λ is thus the cosmological constant and Einstein's equations take the form $R_{kl} - \frac{R}{2}g_{kl} + \Lambda g_{kl} = kT_{kl}$. These equations relate the Ricci tensor of a space-time to the matter content described by the stress-energy tensor T_{kl}. Moreover, a Z form associated to the Z tensor was introduced in [42]. Then the notion of Z recurrent form was defined [42] and several properties of Riemannian manifolds on which the Z form is recurrent was studied in depth. The notion of Z recurrent form incorporates both pseudo-Z-symmetric and weakly Z-symmetric manifolds [37, 41].

In this paper, we investigate the properties of recurrent conformal 2-forms on pseudo-Riemannian manifolds. In Sec. 2, some general properties already proven in the Riemannian case are readily extended to the case of arbitrary signature and some new results are stated; moreover it is proven that if the associated covector is closed then the Ricci tensor is *Riemann- and Weyl-compatible*: these notions were recently introduced and investigated by one of the present authors (see [38–40]).

In Sec. 3, we study conformal changes of metric and recurrent 2-forms; a nice result is pointed out. In Sec. 4, a deep account of Riemann and Weyl compatibility of $(0,2)$ symmetric tensors is given: some new results about the generalized Derdziński–Shen theorem are investigated. It is shown that this implies the vanishing of some Weyl's scalars. In Sec. 5, topological properties involving Pontryagin forms and recurrent conformal 2-forms are stated. Finally, in Sec. 6, we study the properties of recurrent conformal 2-forms on Lorentzian manifolds (space-times). Previous theorems stated on a pseudo-Riemannian manifold of arbitrary signature are then interpreted in the light of the classification of space-times in four or in n dimensions. Throughout the paper, all manifolds under considerations are assumed to be connected Hausdorff manifolds endowed with a non-degenerate metric of arbitrary signature, i.e. n-dimensional pseudo-Riemannian manifolds; where necessarily we will specialize to a metric of signature $s = n-2$, i.e. to $n(\geq 4)$-dimensional Lorentz manifolds [29].

2. Recurrent Conformal Forms: General Properties

In this section, we extend the notion of recurrent conformal 2-forms to pseudo-Riemannian manifolds of arbitrary signature. First we recall some basic definitions about generalized curvature tensors and associated forms: consider a class of tensor K of type $(1,3)$, with the local components K_{jkl}^{m}, defined on an n-dimensional vector space with the usual symmetries of the Riemann tensor satisfying the first Bianchi identity. Specifically, we admit a generalized curvature tensor satisfying the following relations [34, 36, 68]:

$$\begin{aligned} &\text{(a)} \quad K_{jkl}^{m} + K_{klj}^{m} + K_{ljk}^{m} = 0, \quad K_{jkl}^{m} = -K_{kjl}^{m}, \\ &\text{(b)} \quad \nabla_i K_{jkl}^{m} + \nabla_j K_{kil}^{m} + \nabla_k K_{ijl}^{m} = B_{ijkl}^{m}, \end{aligned} \tag{8}$$

where B_{ijkl}^{m} is a tensor source in the second Bianchi identity. Moreover, we may also define an associated completely covariant $(0,4)$ tensor $K_{jklm} = g_{mh}K_{jkl}^{h}$ with the following further properties [34]:

$$\begin{aligned} &K_{jklm} = -K_{kjlm} = -K_{jkml}, \\ &K_{jklm} = K_{lmjk}. \end{aligned} \tag{9}$$

In this way the contraction $K_{kl} = -K_{mkl}^{m}$ defines a symmetric generalized Ricci tensor [43]. An n-dimensional pseudo-Riemannian manifold is said to be K-flat if $K_{jkl}^{m} = 0$ and K-symmetric if $\nabla_i K_{jkl}^{m} = 0$, K-harmonic if $\nabla_m K_{jkl}^{m} = 0$ and K-recurrent if $\nabla_i K_{jkl}^{m} = \alpha_i K_{jkl}^{m}$ [36]. Now the vector-valued form associated to a generalized curvature tensor is given by:

$$\Omega_{(K)l}^{m} = K_{jkl}^{m} dx^j \wedge dx^k. \tag{10}$$

If we consider the symmetric contraction $K_{kl} = -K_{mkl}^{m}$ a generalized Ricci 1-form may be defined [43] as follows:

$$\Lambda_{(K)l} = K_{kl} dx^k. \tag{11}$$

Hereafter, we consider an n-dimensional pseudo-Riemannian manifold admitting nonzero tensor K. The forms $\Omega_{(K)l}^{m}, \Lambda_{(K)l}$ are said to be closed if $D\Omega_{(K)l}^{m} = 0$, $D\Lambda_{(K)l} = 0$ being D the exterior covariant derivative. This implies respectively

$$\nabla_i K_{jkl}^{m} + \nabla_j K_{kil}^{m} + \nabla_k K_{ijl}^{m} = 0 = B_{ijkl}^{m}$$

and

$$\nabla_i K_{kl} - \nabla_k K_{il} = 0,$$

for the previously defined forms (see [43, 44]).

We recall that a generalized curvature tensor K with vanishing tensor B^m_{ijkl} is called a proper generalized curvature tensor ([50, p. 27]).

The notion of recurrent curvature form enlarges the closedness condition and the ordinary recurrence of curvature tensors. Here, we extend such definitions to pseudo-Riemannian manifolds.

Definition 3. Let M be an n-dimensional pseudo-Riemannian manifold. The curvature 2-form $\Omega^m_{(K)l} = K^m_{jkl} dx^j \wedge dx^k$ is said to be recurrent if there exist a nonzero scalar 1-form α for which:

$$D\Omega^m_{(K)l} = \alpha \wedge \Omega^m_{(K)l}, \tag{12}$$

being $\alpha = \alpha_i dx^i$ the associated 1-form.

It is easy to see that the previous condition is a generalization of the notion of K-recurrency. In fact if we write Eq. (12) in local components we have:

$$(\nabla_i K^m_{jkl} - \alpha_i K^m_{jkl}) dx^i \wedge dx^j \wedge dx^k = 0. \tag{13}$$

If $\alpha = 0$ we recover the closedness of $\Omega^m_{(K)l}$. The following theorem stated in [44] explains the meaning of this kind of recurrence.

Theorem 4 ([44]). *Let M be an n-dimensional pseudo-Riemannian manifold. The curvature 2-form $\Omega^m_{(K)l} = K^m_{jkl} dx^j \wedge dx^k$ satisfies condition (12) if and only if:*

$$B^m_{ijkl} = \nabla_i K^m_{jkl} + \nabla_j K^m_{kil} + \nabla_k K^m_{ijl} = \alpha_i K^m_{jkl} + \alpha_j K^m_{kil} + \alpha_k K^m_{ijl}. \tag{14}$$

Proof. From Eq. (13) we easily obtain the following expression:

$$\begin{aligned}
&(\nabla_i K^m_{jkl} - \alpha_i K^m_{jkl}) dx^i \wedge dx^j \wedge dx^k \\
&\quad = \frac{1}{3!} (\nabla_i K^m_{jkl} - \alpha_i K^m_{jkl}) \delta^{ijk}_{rst} dx^r \wedge dx^s \wedge dx^t \\
&\quad = \sum_{r<s<t} (\nabla_i K^m_{jkl} - \alpha_i K^m_{jkl}) \delta^{ijk}_{rst} dx^r \wedge dx^s \wedge dx^t = 0.
\end{aligned} \tag{15}$$

The above condition is fulfilled if and only if $(\nabla_i K^m_{jkl} - \alpha_i K^m_{jkl}) \delta^{ijk}_{rst} = 0$ from which Eq. (14) follows immediately. Obviously if the manifold is K-recurrent $\nabla_i K^m_{jkl} = \alpha_i K^m_{jkl}$, then condition (14) is satisfied. □

Now we focus on the notion of recurrence for the generalized Ricci 1-form $\Lambda_{(K)l} = K_{kl} dx^k$ where $K_{kl} = -K^m_{mkl}$.

Definition 5. Let M be an n-dimensional pseudo-Riemannian manifold. The generalized Ricci 1-form $\Lambda_{(K)l} = K_{kl} dx^k$ is said to be recurrent if there exist a nonzero scalar 1-form β for which:

$$D\Lambda_{(K)l} = \beta \wedge \Lambda_{(K)l}, \tag{16}$$

being $\beta = \beta_i dx^i$ the associated 1-form.

In local components the previous equation may be written in the form:

$$(\nabla_i K_{kl} - \beta_i K_{kl}) dx^i \wedge dx^k = 0. \tag{17}$$

If $\beta = 0$ the closedness of the generalized Ricci 1-form is recovered. The following theorem explains the meaning of this kind of recurrence [44].

Theorem 6. *Let M be an n-dimensional pseudo-Riemannian manifold. The generalized Ricci* 1*-form $\Lambda_{(K)l} = K_{kl} dx^l$ satisfies condition* (16) *if and only if*:

$$\nabla_i K_{kl} - \nabla_k K_{il} = \beta_i K_{kl} - \beta_k K_{il}. \tag{18}$$

Proof. From Eq. (16) we easily obtain the following expression:

$$\begin{aligned}(\nabla_i K_{kl} - \beta_i K_{kl}) dx^k \wedge dx^k &= \frac{1}{2!}(\nabla_i K_{kl} - \beta_i K_{kl})\delta^{ik}_{rs} dx^r \wedge dx^s \\ &= \sum_{r<s}(\nabla_i K_{kl} - \beta_i K_{kl})\delta^{ik}_{rs} dx^r \wedge dx^s = 0. \end{aligned} \tag{19}$$

The above condition is fulfilled if and only if $(\nabla_i K_{kl} - \alpha_i K_{kl})\delta^{ik}_{rs} = 0$ from which one concludes immediately. □

Remark 7. The previous Theorem 6 written for the Ricci 1-form $\Lambda_l = R_{kl} dx^k$ becomes $\nabla_i R_{kl} - \nabla_k R_{il} = \beta_i R_{kl} - \beta_k R_{il}$. It is easy to see that if we consider a Ricci recurrent manifold, i.e. the condition $\nabla_i R_{kl} = \beta_i R_{kl}$ [53], then on such manifold the Ricci 1-form is recurrent. Ricci recurrent space-times were also studied in detail by Hall [28]. Moreover other differential structures satisfy Theorem 6 for the Ricci 1-form. Tamassy and Binh [69] introduced and studied a pseudo-Riemannian manifold whose Ricci tensor satisfies the equation:

$$\nabla_k R_{jl} = A_k R_{jl} + B_j R_{kl} + D_l R_{kj},$$

being A, B, D nonzero 1-forms. From this we infer easily $\nabla_k R_{jl} - \nabla_j R_{kl} = (A_k - B_k) R_{jl} - (A_j - B_j) R_{kl}$ and the Ricci 1-form is recurrent. Finally, it should be mentioned that also pseudo-Ricci symmetric manifolds defined by the condition $\nabla_k R_{jl} = 2A_k R_{jl} + A_j R_{kl} + A_l R_{kj}$ and introduced by Chaki [6] obviously satisfy (18).

Some other useful relations may be obtained from Eq. (14). A contraction with g^{kl} gives immediately:

$$\nabla_m K^m_{jkl} + \nabla_j K_{kl} - \nabla_k K_{jl} = \alpha_m K^m_{jkl} + \alpha_j K_{kl} - \alpha_k K_{jl}. \tag{20}$$

A further contraction gives:

$$2\nabla^l K_{jl} - \nabla_j K = 2\alpha^l K_{jl} - \alpha_j K. \tag{21}$$

A further interesting theorem may be stated for recurrent 2-form

$$\Omega^m_{(K)l} = K^m_{jkl} dx^j \wedge dx^k.$$

Theorem 8. *Let M be an n-dimensional pseudo-Riemannian manifold with recurrent 2-form $\Omega^m_{(K)l} = K^m_{jkl} dx^j \wedge dx^k$. Then the following relation is fulfilled*:

$$\nabla_i K^m_{jkl} + \nabla_j K^m_{kil} + \nabla_k K^m_{lji} + \nabla_l K^m_{ikj} = \alpha_i K^m_{jkl} + \alpha_j K^m_{kil} + \alpha_k K^m_{lji} + \alpha_l K^m_{ikj}. \quad (22)$$

Proof. We write four versions of Eq. (14) with cyclically permuted indices i, j, k, l and sum up; then use the first Bianchi identity for the tensor K to simplify. □

Hereafter, we concentrate on recurrent conformal 2-forms on pseudo-Riemannian manifolds. We recall that a manifold is conformally recurrent [1, 36] when the conformal curvature tensor satisfies the relation $\nabla_i C^m_{jkl} = \alpha_i C^m_{jkl}$ where α_i is a nonzero covector. In [68] Suh, Kwon and Yang studied conformally recurrent pseudo-Riemannian manifolds with harmonic conformal curvature tensor, i.e. with $\nabla_m C^m_{jkl} = 0$ [5]. In the Riemannian case they stated the following theorem:

Theorem 9 ([68, Remark 3.3]). *Let M be an $n(n \geq 4)$-dimensional Riemannian manifold with Riemaniann connection ∇. Assume that M is conformally recurrent and has the harmonic conformal curvature tensor. Then M is conformally symmetric.*

Now if we consider the recurrent conformal 2-form $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$ on a pseudo-Riemannian manifold the general Eq. (14) may be written as [44]:

$$\nabla_i C^m_{jkl} + \nabla_j C^m_{kil} + \nabla_k C^m_{ijl} = \alpha_i C^m_{jkl} + \alpha_j C^m_{kil} + \alpha_k C^m_{ijl} = B^m_{ijkl}. \quad (23)$$

Obviously Eq. (23) is satisfied when $\nabla_i C^m_{jkl} = \alpha_i C^m_{jkl}$. As a consequence of the previous condition if we take $i = m$ we get:

$$\nabla_m C^m_{jkl} = \alpha_m C^m_{jkl}. \quad (24)$$

We remark that a $(0, 4)$ version of Eq. (23) may be written, i.e.:

$$\nabla_i C_{jklm} + \nabla_j C_{kilm} + \nabla_k C_{ijlm} = \alpha_i C_{jklm} + \alpha_j C_{kilm} + \alpha_k C_{ijlm}, \quad (25)$$

where $C_{jklm} = g_{pm} C^p_{jkl}$. It is well-known (see [1, Eq. (3.7)]) that the source term for the second Bianchi identity for the conformal curvature tensor may be written in the following form:

$$\begin{aligned} &\nabla_i C^m_{jkl} + \nabla_j C^m_{kil} + \nabla_k C^m_{ijl} \\ &\quad = \frac{1}{n-3}[\delta^m_j \nabla_p C^p_{kil} + \delta^m_k \nabla_p C^p_{ijl} + \delta^m_i \nabla_p C^p_{jkl} \\ &\qquad + g_{kl} \nabla_p C^{mp}_{ji} + g_{il} \nabla_p C^{mp}_{kj} + g_{jl} \nabla_p C^{mp}_{ik}]. \end{aligned} \quad (26)$$

Note that $C^{mp}_{kj} = g^{pr} C^m_{kjr} = g^{pr} g^{ms} C_{kjrs}$. Considering Eqs. (23) and (26) we may easily state the following proposition:

Proposition 10. *Let M be an $n(\geq 4)$-dimensional non-conformally flat pseudo-Riemannian manifold with recurrent conformal 2-form*: *then $\alpha_i C^m_{jkl} + \alpha_j C^m_{kil} + \alpha_k C^m_{ijl} = 0$ if and only if $\nabla_m C^m_{jkl} = 0$.*

Proof. If $\nabla_m C^m_{jkl} = 0$ from (26) we have $\nabla_i C^m_{jkl} + \nabla_j C^m_{kil} + \nabla_k C^m_{ijl} = 0$ and thus $\alpha_i C^m_{jkl} + \alpha_j C^m_{kil} + \alpha_k C^m_{ijl} = 0$; on the other hand if the right-hand side of (23) vanishes, then taking $i = m$ we get the result. □

The condition $\alpha_i C^m_{jkl} + \alpha_j C^m_{kil} + \alpha_k C^m_{ijl} = 0$ on a pseudo-Riemannian manifold has been extensively studied in the geometric literature (see for example [13, 20, 22, 23]).

Here, we prove the following lemma.

Lemma 11. *Let M be an n-dimensional non-conformally flat pseudo-Riemannian manifold with recurrent conformal 2-form: if $\alpha_i C^m_{jkl} + \alpha_j C^m_{kil} + \alpha_k C^m_{ijl} = 0$ then:*

$$(1)\quad \alpha^i \alpha_i = 0,$$

$$(2)\quad C^m_{jkl} C^{jkl}_m = 0,$$

$$(3)\quad C^k_{lmj} C^j_{pqk} = 0.$$

Proof. From $\alpha_i C^m_{jkl} + \alpha_j C^m_{kil} + \alpha_k C^m_{ijl} = 0$ we have also $\alpha_m C^m_{jkl} = 0$ and thus contracting with α^i one obtains $\alpha^i \alpha_i C^m_{jkl} = 0$ from which we infer (1); on the other hand contracting with C^{jkl}_m one obtains $\alpha_i C^m_{jkl} C^{jkl}_m = 0$ from which we infer (2). Finally contracting $\alpha_i C_{jklm} + \alpha_j C_{kilm} + \alpha_k C_{ijlm} = 0$ with C^{kj}_{pq} and using $\alpha_m C^m_{jkl} = 0$ we get $\alpha_i C_{jklm} C^{kj}_{pq} = 0$ from which (3) follows immediately. □

Following the same trick of [22] we are in a position to prove the following:

Proposition 12. *Let M be an n-dimensional non-conformally flat pseudo-Riemannian manifold with recurrent conformal 2-form: then $\alpha_i C_{jklm} + \alpha_j C_{kilm} + \alpha_k C_{ijlm} = 0$ (i.e. $\nabla_m C^m_{jkl} = 0$) if and only if there exist a symmetric $(0, 2)$ tensor A_{kl} for which*

$$C_{jklm} = \alpha_j \alpha_m A_{kl} - \alpha_j \alpha_l A_{mk} - \alpha_k \alpha_m A_{jl} + \alpha_k \alpha_l A_{jm}, \tag{27}$$

being $\alpha^i \alpha_i = 0$.

Proof. If the Weyl tensor is of the form (27), then after straightforward calculations it is easy to see that $\alpha_i C_{jklm} + \alpha_j C_{kilm} + \alpha_k C_{ijlm} = 0$ holds and thus the covector α_i is null.

On the other hand, let θ^i be a unit vector such that $\theta^i \alpha_i = 1$: then contracting the condition $\alpha_i C_{jklm} + \alpha_j C_{kilm} + \alpha_k C_{ijlm} = 0$ with θ^i a first time we infer

$$C_{jklm} = \alpha_j \theta^i C_{iklm} - \alpha_k \theta^i C_{ijlm}. \tag{28}$$

Contracting again the previous result with θ^m we get:

$$\theta^m C_{mlkj} = \alpha_j A_{kl} - \alpha_k A_{jl},$$

being $A_{kl} = \theta^i \theta^m C_{iklm}$ a symmetric $(0, 2)$ tensor. Inserting back in Eq. (28) we get the result. □

Remark 13. Let us consider conformal recurrent 2-form that are also Ricci flat, i.e. with $R_{kl} = 0$: in this case we have $R^m_{jkl} = C^m_{jkl}$ from Eq. (23) recalling the second Bianchi identity for the Riemann tensor we immediately have

$$\alpha_i R^m_{jkl} + \alpha_j R^m_{kil} + \alpha_k R^m_{ijl} = 0. \tag{29}$$

Such a condition describes the so-called *B-space* introduced and studied by Venzi [70]. Moreover, the previous condition is fulfilled by any four-dimensional space-time of Petrov type T_2 (see [55, p. 111] or [13]).

Now, we will state a fundamental theorem involving an algebraic relation which is satisfied in the case of recurrent conformal 2-form $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$ with closed recurrence parameter (see [44]).

Theorem 14. *Let M be an $n(\geq 4)$-dimensional pseudo-Riemannian manifold with recurrent conformal 2-form, i.e. with $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$: then if α_k is a closed 1-form the following algebraic relation is fulfilled*:

$$R_{im} R^m_{jkl} + R_{jm} R^m_{kil} + R_{km} R^m_{ijl} = 0. \tag{30}$$

The proof of the previous theorem need some auxiliary lemmas. The first one is known as *Lovelock's differential identity* and may be found for example in [35, 36].

Lemma 15 (Lovelock's differential identity). *Let M be an n-dimensional pseudo-Riemannian manifold*: *then the following identity is fulfilled*:

$$\nabla_i \nabla_m R^m_{jkl} + \nabla_j \nabla_m R^m_{kil} + \nabla_k \nabla_m R^m_{ijl} = -R_{im} R^m_{jkl} - R_{jm} R^m_{kil} - R_{km} R^m_{ijl}. \tag{31}$$

Lovelock's identity is thus written for the conformal curvature tensor (see [36, 44]):

$$\begin{aligned} &\nabla_i \nabla_m C^m_{jkl} + \nabla_j \nabla_m C^m_{kil} + \nabla_k \nabla_m C^m_{ijl} \\ &\quad = -\frac{n-3}{n-2}(R_{im} R^m_{jkl} + R_{jm} R^m_{kil} + R_{km} R^m_{ijl}). \end{aligned} \tag{32}$$

Now, we recall that $\nabla_m C^m_{jkl} = \alpha_m C^m_{jkl}$ and $\nabla_i C^m_{jkl} + \nabla_j C^m_{kil} + \nabla_k C^m_{ijl} = \alpha_i C^m_{jkl} + \alpha_j C^m_{kil} + \alpha_k C^m_{ijl}$ thus the left-hand side of previous equation may be written in the form:

$$(\nabla_i \alpha_m) C^m_{jkl} + (\nabla_j \alpha_m) C^m_{kil} + (\nabla_k \alpha_m) C^m_{ijl} + \alpha_m (\alpha_i C^m_{jkl} + \alpha_j C^m_{kil} + \alpha_k C^m_{ijl}). \tag{33}$$

Now the divergence of $\alpha_i C^m_{jkl} + \alpha_j C^m_{kil} + \alpha_k C^m_{ijl} = B^m_{ijkl}$ is taken to give straightforwardly:

$$\begin{aligned} &(\nabla_m \alpha_i) C^m_{jkl} + (\nabla_m \alpha_j) C^m_{kil} + (\nabla_m \alpha_k) C^m_{ijl} \\ &\quad + \alpha_m (\alpha_i C^m_{jkl} + \alpha_j C^m_{kil} + \alpha_k C^m_{ijl}) = \nabla_m B^m_{ijkl}. \end{aligned} \tag{34}$$

If the closeness of the recurrence parameter is taken into account we can write finally:

$$\nabla_m B^m_{ijkl} = -\frac{n-3}{n-2}(R_{im}R^m_{jkl} + R_{jm}R^m_{kil} + R_{km}R^m_{ijl}). \tag{35}$$

We have thus proved the following result:

Lemma 16. *Let M be an n-dimensional pseudo-Riemannian manifold with the recurrent conformal curvature 2-form $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$: if the recurrence parameter is closed then* (35) *holds.*

Now the following lemma about the source term of the second Bianchi identity for the conformal curvature tensor is stated [44]:

Lemma 17. *The divergence of the source term in the second Bianchi identity for the conformal curvature tensor takes the form*:

$$\nabla_m B^m_{ijkl} = -\frac{1}{n-2}(R_{im}R^m_{jkl} + R_{jm}R^m_{kil} + R_{km}R^m_{ijl}). \tag{36}$$

Proof. We recall that in the case of conformal curvature tensor the source term B takes the form:

$$\begin{aligned} B^m_{ijkl} = {} & \frac{1}{n-2}[\delta^m_j(\nabla_i R_{kl} - \nabla_k R_{il}) + \delta^m_i(\nabla_k R_{jl} - \nabla_j R_{kl}) + \delta^m_k(\nabla_j R_{il} - \nabla_i R_{jl}) \\ & + g_{il}(\nabla_j R^m_k - \nabla_k R^m_j) + g_{jl}(\nabla_k R^m_i - \nabla_i R^m_k) + g_{kl}(\nabla_i R^m_j - \nabla_j R^m_i)] \\ & - \frac{1}{(n-1)(n-2)}[\delta^m_j(\nabla_i R g_{kl} - \nabla_k R g_{il}) \\ & + \delta^m_i(\nabla_k R g_{jl} - \nabla_j R g_{kl}) + \delta^m_k(\nabla_j R g_{il} - \nabla_i R g_{jl})]. \end{aligned} \tag{37}$$

Taking the covariant derivative ∇_m of the previous equation and recalling that $\nabla_l \nabla_m R^{lm}_{jk} = 0$ (see [35, 36, 44]) we obtain:

$$\nabla_m B^m_{ijkl} = -\frac{1}{n-2}(\nabla_i \nabla_m R^m_{jkl} + \nabla_j \nabla_m R^m_{kil} + \nabla_k \nabla_m R^m_{ijl}). \tag{38}$$

Now Lovelock's identity is used to conclude. □

Proof of Theorem 14. If we take into account both Lemmas 16 and 17 we simply infer $R_{im}R^m_{jkl} + R_{jm}R^m_{kil} + R_{km}R^m_{ijl} = 0$ and the theorem is proven. □

Now the left-hand side of Eq. (32) is simply the exterior covariant derivative of the vector-valued 1-form associated with the divergence of the conformal curvature tensor, i.e. $\Pi_{(C)l} = \nabla_m C^m_{jkl} dx^j \wedge dx^k$: thus in view of the previous theorem the following result is obtained

Theorem 18. *Let M be an n-dimensional pseudo-Riemannian manifold with recurrent conformal curvature 2-form, i.e. $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$: if the recurrence parameter is closed then $D\Pi_{(C)l} = 0$.*

If the Ricci tensor satisfies Eq. (30) it is named Riemann-compatible (see [14, 19, 38–40, 57]). Geometric and topological consequences of this condition were extensively studied in [39]. If we insert in relation the local form of the Weyl tensor (4) (see [56]) we obtain:

$$R_{im}C^m_{jkl} + R_{jm}C^m_{kil} + R_{km}C^m_{ijl} = 0. \tag{39}$$

The Ricci tensor is thus *Weyl-compatible*. In recent works Weyl compatibility has been extensively investigated in the Riemannian case [39] and in the pseudo-Riemannian case [40]. We note that results on semi-Riemannian manifolds satisfying (30) and (39) are also given in some papers published earlier than [38–40] (see [14, 19, 57]). In Sec. 4, we will give a deep account of its consequences. If we use Einstein's equations in (39) we infer an analogous condition for the stress-energy tensor, namely:

$$T_{im}C^m_{jkl} + T_{jm}C^m_{kil} + T_{km}C^m_{ijl} = 0. \tag{40}$$

From the above discussion we may state the following:

Theorem 19. *Let M be an n-dimensional pseudo-Riemannian manifold with recurrent conformal curvature 2-form, i.e. $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$: if the recurrence parameter is closed then the Ricci and the stress-energy tensors are Weyl-compatible.*

Now, we provide some examples of recurrent conformal 2-form as follows:

Example 1. De and Bandyopadhyay in [10] (see also [9] for a detailed compendium) introduced the notion of *weakly conformally symmetric* manifolds and proved its existence in the Riemannian case by an example. The same kind of Riemannian manifolds were also investigated in [64]. Here, we extend this notion to manifolds endowed with metric of arbitrary signature. A non-conformally flat pseudo-Riemannian manifold is said to be *weakly conformally symmetric* $(\mathrm{WCS})_n$ if the conformal curvature tensor satisfies the condition:

$$\nabla_i C_{jklm} = A_i C_{jklm} + B_j C_{iklm} + C_k C_{jilm} + D_l C_{jkim} + E_m C_{jkli}, \tag{41}$$

where A, B, C, D, E are called associated 1-forms. By permuting cyclically the indices i, j, k and summing the resulting equations we infer simply:

$$\begin{aligned} &\nabla_i C_{jklm} + \nabla_j C_{kilm} + \nabla_k C_{ijlm} \\ &\quad = (A_i - B_i - C_i)C_{jklm} + (A_j - B_j - C_j)C_{kilm} + (A_k - B_k - C_k)C_{ijlm}. \end{aligned} \tag{42}$$

Example 2 (See [67]). For any integer $p(\geq 2)$ and any complex number c such that $|c| \geq 1$ we define complex Euclidean space C^{2n+1}_n of index $2n$ is defined as follows:

Let $\{z^j, z^{j^*}, z^{2n+1}\} = \{z^1, \ldots, z^{2n+1}\}$ be a complex coordinate of C_n^{2n+1}. Then $M = M(p, c)$ is an indefinite complete complex hypersurface of index $2n$ defined by

$$z^{2n+1} = \sum_j h_j(z^j + cz^{j^*}), \quad h_j(z) = z^p,$$

where c is any complex number such that $|c| \geq 1$. Then the component of the curvature tensor of M is given by

$$K_{\bar{i}jk\bar{m}} = -h_{jk}\bar{h}_{im} = -p^2(p-1)^2\delta_{jk}\delta_{im}|z|^{2(p-2)},$$
$$K_{\bar{i}jk\bar{m}^*} = -h_{jk}\bar{h}_{im^*} = -cp^2(p-1)^2\delta_{jk}\delta_{im}|z|^{2(p-2)}.$$

From this M is not necessarily flat. Moreover, we get

$$\begin{aligned} K_{\bar{i}jk\bar{m}n} &= -h_{jkn}\bar{h}_{im} = -p^2(p-1)^2(p-2)\delta_{jm}\delta_{ik}|z|^{2(p-2)}z^{-1} \\ &= (p-2)\delta_{jn}z^{-1}K_{i\bar{j}k\bar{m}} = \alpha_j\delta_{jn}K_{\bar{i}jk\bar{m}}, \end{aligned}$$

where $\alpha_j = d\beta_j$ and the smooth function β_j is defined by

$$\beta_j = \log\frac{h_j(z)}{z^2} = \log z^{p-2}, \quad p \geq 3.$$

So, we know here that the recurrent parameter 1-form α becomes *coclosed*, that is, $\nabla_i\alpha_j = \nabla_j\alpha_i$. Then the derivative of the component of Riemannian curvature tensor $R_{\alpha\beta\gamma\delta} = g(R(E_\alpha, E_\beta)E_\gamma, E_\delta)$, $\alpha, \beta, \ldots = 1, \ldots, 4n$, is given as follows:

$$R_{ABCDE} = 2\alpha_E R_{ABCD},$$

where the indices $A, B, \ldots = 1, \ldots, 2n$. In such a case we have known that the Ricci tensor is flat if $|c| = 1$ and the complex hypersurface M of index $2n$ in a $(2n+1)$-dimensional indefinite complex Euclidean space C_n^{2n+1} of index $2n$ defined above is *conformally recurrent*. So, it naturally composes a subclass of recurrent conformal curvature 2-form which is non-weakly conformal symmetric satisfying (25). Also this gives an example of Theorem 6, because M is Ricci flat but not conformally flat.

We have thus proven the following.

Theorem 20. *On an n-dimensional weakly conformally symmetric pseudo-Riemannian manifold the conformal curvature 2-form is recurrent, i.e.* $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$ *with* $\alpha_i = A_i - B_i - C_i$.

Moreover, if the covector $\alpha_i = A_i - B_i - C_i$ is a closed 1-form from the above discussion we have the following theorem.

Theorem 21. *Let M be an n-dimensional weakly conformally symmetric pseudo-Riemannian manifold. If the covector $\alpha_i = A_i - B_i - C_i$ is closed, then the Ricci tensor is Riemann-compatible, i.e. the relation* $R_{im}C^m_{jkl} + R_{jm}C^m_{kil} + R_{km}C^m_{ijl} = 0$ *is fulfilled.*

Example 3. De and Gazi introduced and studied in [11] the notion of almost pseudo-conformally symmetric manifold and proved its existence by a suitable example. A non-conformally flat pseudo-Riemannian manifold is said to be almost pseudo-conformally symmetric manifold $(\mathrm{APCS})_n$ if the conformal curvature tensor satisfies the condition:

$$\nabla_i C_{jklm} = (A_i + B_i)C_{jklm} + A_j C_{iklm} + A_k C_{jilm} + A_l C_{jkim} + A_m C_{jkli},$$

where A, B are non-null 1-forms. By permuting cyclically the indices i, j, k and summing the resulting equations we obtain:

$$\begin{aligned}&\nabla_i C_{jklm} + \nabla_j C_{iklm} + \nabla_k C_{ijlm}\\ &\quad = (B_i - A_i)C_{jklm} + (B_j - A_j)C_{kilm} + (B_k - A_k)C_{ijlm}.\end{aligned}$$

Moreover it can be shown ([11, Theorem 2.1]) that $\nabla_m C^m_{jkl} = B_m C^m_{jkl}$. This gives another example of non-weakly conformal symmetric satisfying (25).

We have thus proven the following:

Theorem 22. *On an n-dimensional almost pseudo-conformally symmetric pseudo-Riemannian manifold the conformal 2-form is recurrent, i.e. $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$ with $\alpha_i = B_i - A_i$; moreover it is $\nabla_m C^m_{jkl} = 0$ if and only if $B_m C^m_{jkl} = 0$.*

Example 4. We consider again the B-space introduced by Venzi [70], i.e. $\alpha_i R_{jklm} + \alpha_j R_{kilm} + \alpha_k R_{ijlm} = 0$ and suppose that it is also Ricci recurrent with the same recurrence parameter, i.e. $\nabla_i R_{kl} = \alpha_i R_{kl}$. From the definition of conformal curvature tensor it is then $\nabla_i C_{jklm} = \nabla_i R_{jklm} + \alpha_i (C_{jklm} - R_{jklm})$ and thus the conformal 2-form is recurrent satisfying (25) by the second Bianchi identity. This kind of B-space also becomes another subclass which is not weakly conformal symmetric.

Theorem 23. *Let M be an n-dimensional B-space described by the condition $\alpha_i R_{jklm} + \alpha_j R_{kilm} + \alpha_k R_{ijlm} = 0$: if the Ricci tensor is recurrent with the same recurrence parameter, then the conformal 2-form is recurrent, i.e. $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$.*

3. Recurrent Conformal 2-Forms and Conformal Transformations

In this section, we study the behavior of recurrent conformal 2-forms under conformal transformations of the metric tensor. Let M be an n-dimensional pseudo-Riemannian manifold with the metric tensor g_{kl}. The metric transformation $\bar{g}_{kl} = e^{2\sigma} g_{kl}$ being σ a scalar function defines a *conformal transformation* which leaves the angle between two vectors unchanged (see [71, 72; 56, p. 238]); if the scalar function is constant the transformation is called homothetic (see [66]). It is well-known that under conformal transformations the Christoffel symbols change as follows:

$$\bar{\Gamma}^m_{ij} = \Gamma^m_{ij} + \delta^m_j X_i + \delta^m_i X_j - g_{ij} X^m, \tag{43}$$

being $X_i = \nabla_i \sigma$ a closed 1-form. It is a matter of fact that under a conformal transformation the Weyl tensor remains unchanged (see [56, p. 241]), i.e.:

$$\bar{C}^m_{jkl} = C^m_{jkl}. \tag{44}$$

We take a covariant derivative $\bar{\nabla}$ of the tensor $\bar{C}_{jklm}$ obtaining after straightforward calculations recalling that we have $\bar{C}_{jklm} = e^{2\sigma} C_{jklm}$ (see [56, p. 241] and [71]):

$$\begin{aligned}\bar{\nabla}_i C_{jklm} &= \nabla_i C_{jklm} - 4X_i C_{jklm} - X_j C_{iklm} - X_l C_{jkim} - X_m C_{jkli} \\ &\quad + g_{ji} X^p C_{pklm} + g_{ki} X^p C_{jplm} + g_{li} X^p C_{jkpm} + g_{mi} X^p C_{jklp}.\end{aligned} \tag{45}$$

Now three versions of the previous equation with cyclically permuted indices i, j, k are written and summed up to obtain:

$$\begin{aligned}\bar{\nabla}_i C_{jklm} + \bar{\nabla}_j C_{kilm} + \bar{\nabla}_k C_{ijlm} &= \nabla_i C_{jklm} + \nabla_j C_{kilm} + \nabla_k C_{ijlm} \\ &\quad - 2X_i C_{jklm} - 2X_j C_{iklm} - 2X_k C_{jilm} \\ &\quad + g_{li} X^p C_{jkpm} + g_{lj} X^p C_{kipm} \\ &\quad + g_{lk} X^p C_{ijpm} + g_{mi} X^p C_{jklp} \\ &\quad + g_{mj} X^p C_{kilp} + g_{mk} X^p C_{ijlp}.\end{aligned} \tag{46}$$

We may thus state the following result.

Theorem 24. *Let M be an n-dimensional pseudo-Riemannian manifold with recurrent conformal curvature 2-form, i.e. $D\Omega^m_{(C)l} = X \wedge \Omega^m_{(C)l}$ being X closed. If $\nabla_m C^m_{jkl} = 0$ then a conformal change may be chosen such that $\bar{\nabla}_m C^m_{jkl} = 0$.*

Proof. Recalling that X is closed, take the conformal change such that $X_i = \nabla_i \sigma$. Now from the definition of recurrent conformal 2-form $\nabla_i C_{jklm} + \nabla_j C_{kilm} + \nabla_k C_{ijlm} = X_i C_{jklm} + X_j C_{kilm} + X_k C_{ijlm}$ with the condition $\nabla_m C^m_{jkl} = 0$ recalling Proposition 10 we have $X_i C_{jklm} + X_j C_{kilm} + X_k C_{ijlm} = 0$ and thus $X^m C_{jklm} = 0$. From Eq. (46) we infer simply $\bar{\nabla}_i C_{jklm} + \bar{\nabla}_j C_{kilm} + \bar{\nabla}_k C_{ijlm} = 0$: contracting with $\bar{g}^{im}$ we get $\bar{\nabla}^m C_{jklm} = 0$. □

4. Riemann- and Weyl-Compatible Tensors

In this section, we review the notions of Riemann- and Weyl-compatible tensors. These notions were introduced by one of the present authors and deeply investigated in [38–40]. Moreover applications of the extended Derdziński–Shen's theorem [38] to particular Weyl-compatible tensors give some interesting new results about the vanishing of particular scalars associated to the Weyl tensor. Moreover some new useful theorems will be pointed out.

Definition 25. A symmetric tensor b_{kl} is named *Riemann-compatible* (see [38, 39]) if it satisfies the following algebraic condition:

$$b_{im}R^m_{jkl} + b_{jm}R^m_{kil} + b_{km}R^m_{ijl} = 0. \tag{47}$$

This definition arises naturally even on a manifold endowed with a symmetric connection when considering the exterior covariant derivative of the vector-valued 1-form given by $B_l = b_{kl}dx^k$. A first covariant derivative brings:

$$DB_l = \frac{1}{2}\Theta_{jkl}dx^j \wedge dx^k, \tag{48}$$

being $\Theta_{jkl} =: \nabla_j b_{kl} - \nabla_k b_{jl}$ a $(0,3)$ tensor defined in [39] as the Codazzi deviation tensor. The following statement is well-known in literature:

Theorem 26 (See [17, 39]). *Let M be an n-dimensional manifold endowed with a symmetric connection: then $DB_l = 0$ if and only if b_{kl} is a Codazzi tensor.*

Whether $DB \neq 0$ or not a further covariant derivative brings [40]:

$$D^2B_l = \frac{1}{2\cdot 3!}\nabla_i\Theta_{jkl}\delta^{ijk}_{rst}dx^r \wedge dx^s \wedge dx^t. \tag{49}$$

With standard algebraic manipulations the previous expressions may be written in the form:

$$\begin{aligned} D^2B_l &= \frac{1}{2}\sum_{r<s<t} \nabla_i\Theta_{jkl}\delta^{ijk}_{rst}dx^r \wedge dx^s \wedge dx^t \\ &= \sum_{r<s<t}(\nabla_r\Theta_{stl} + \nabla_s\Theta_{trl} + \nabla_t\Theta_{rsl})dx^r \wedge dx^s \wedge dx^t. \end{aligned} \tag{50}$$

Now the following identity links the Codazzi deviation tensor to the Riemann tensor (see [39] and [40] for further details):

$$\nabla_i\Theta_{jkl} + \nabla_j\Theta_{kil} + \nabla_k\Theta_{ijl} = b_{im}R^m_{jkl} + b_{jm}R^m_{kil} + b_{km}R^m_{ijl}. \tag{51}$$

Thus the following result is easily inferred [40]:

Theorem 27. *Let M be an n-dimensional manifold endowed with a symmetric connection: then $D^2B_l = 0$ if and only if b_{kl} is Riemann-compatible.*

On the other hand, the notion of Weyl compatibility obviously needs the presence of a pseudo-Riemannian metric and we suppose also $\nabla_k g_{jl} = 0$. The next theorem follows immediately from [39, Theorem 7.6] (see also [40]).

Theorem 28. *Let M be an n-dimensional pseudo-Riemannian manifold. If a symmetric tensor b_{kl} is Riemann-compatible then it is also Weyl-compatible, i.e. it satisfies the condition:*

$$b_{im}C^m_{jkl} + b_{jm}C^m_{kil} + b_{km}C^m_{ijl} = 0. \tag{52}$$

The existence of Weyl-compatible tensors poses strong restrictions on the structure of the Weyl tensor. The following extension of Derdziński and Shen's theorem was proved in [38] by one of the present authors (see also [45] for an alternative proof).

Theorem 29 ([45]). *Let M be an n-dimensional pseudo-Riemannian manifold with a Weyl-compatible tensor b_{kl}. If X, Y, Z are eigenvectors of b with eigenvalues λ, μ, ν (i.e. $b_i^j X_j = \lambda X_i, b_i^j Y_j = \mu Y_i, b_i^j Z_j = \nu Z_i$) then*:

$$X^j Y^k Z^l C_{jklm} = 0, \quad \nu \neq \lambda, \mu. \tag{53}$$

It will be shown that the previous theorem may allow some particular scalars associated to the Weyl tensor to vanish. A simple example of application of this deep result is given in the following example.

Example 5. Consider the Weyl compatibility of the symmetric tensor:

$$b_{ab} = \lambda_1(X_a Y_b + X_b Y_a) + \lambda_2(Z_a W_b + Z_b W_a), \tag{54}$$

being X, Y, Z, W four null vectors with the properties $X^a Y_a = 1,\ Z^a W_a = 1$ while all others scalar products vanish. It easy to see that the four vectors are eigenvectors of b_{ab}, *i.e.*:

$$X^a b_{ab} = \lambda_1 X_b,$$

$$Y^a b_{ab} = \lambda_1 Y_b,$$

$$Z^a b_{ab} = \lambda_2 Z_b,$$

$$W^a b_{ab} = \lambda_2 W_b.$$

If $\lambda_1 \neq \lambda_2$ a simple consequence of the extended Derdziński–Shen theorem are the following sets of relations:

$$\begin{cases} X^a Y^b Z^c C_{abcd} = 0, \\ X^a Y^b W^c C_{abcd} = 0, \end{cases} \qquad \begin{cases} Z^a W^b C^c C_{abcd} = 0, \\ Z^a W^b C^c C_{abcd} = 0, \end{cases}$$

Thus the following sets of scalars vanish:

$$\begin{gathered} \mathrm{I} \begin{cases} X^a Y^b Z^c X^d C_{abcd} = 0, \\ X^a Y^b Z^c Y^d C_{abcd} = 0, \\ X^a Y^b Z^c W^d C_{abcd} = 0, \end{cases} \quad \mathrm{II} \begin{cases} X^a Y^b W^c X^d C_{abcd} = 0, \\ X^a Y^b W^c Y^d C_{abcd} = 0, \\ X^a Y^b W^c Z^d C_{abcd} = 0, \end{cases} \\ \mathrm{III} \begin{cases} Z^a W^b Y^c X^d C_{abcd} = 0, \\ Z^a W^b Y^c Z^d C_{abcd} = 0, \\ Z^a W^b Y^c W^d C_{abcd} = 0. \end{cases} \quad \mathrm{IV} \begin{cases} Z^a W^b X^c Y^d C_{abcd} = 0, \\ Z^a W^b X^c Z^d C_{abcd} = 0, \\ Z^a W^b X^c W^d C_{abcd} = 0. \end{cases} \end{gathered} \tag{55}$$

We have thus proven the following.

Theorem 30. *Let M be an n-dimensional pseudo-Riemannian manifold. If the tensor $b_{ab} = \lambda_1(X_aY_b + X_bY_a) + \lambda_2(Z_aW_b + Z_bW_a)$ (with the properties $X^aY_a = 1, Z^aW_a = 1$ while all other scalar products vanish) is Weyl-compatible and $\lambda_1 \neq \lambda_2$ then Eqs.* (55) *are fulfilled.*

In view of Theorems 19 and 30, we may assert.

Corollary 31. *Let M be an n-dimensional pseudo-Riemannian manifold with recurrent conformal 2-form, i.e. with $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$ being α closed. If the Ricci tensor (or the stress-energy tensor) is of the form $R_{ab} = \lambda_1(X_aY_b + X_bY_a) + \lambda_2(Z_aW_b + Z_bW_a)$ with $\lambda_1 \neq \lambda_2$ then Eqs.* (55) *are satisfied.*

Remark 32 (See [67]). In Example 2 in Sec. 2 we have introduced an indefinite complete complex hypersurface of index $2n$ in an indefinite complex Euclidean space C^{2n+1}_n of index $2n$ as follows:

$$z^{2n+1} = \sum\nolimits_j h_j(z^j + cz^{j^*}), \quad h_j(z) = z^p,$$

where c is any complex number such that $|c| \geq 1$. In such a case we have known that the Ricci tensor is flat if $|c| = 1$. The complex hypersurface M of index $2n$ in C^{2n+1}_n of index $2n$ defined above is *conformally recurrent*. Here, we have known that its conformal curvature tensor is coclosed, that is, harmonic $\nabla_M C^M_{ABC} = 0$, which is neither locally symmetric nor conformally flat if $p \geq 3$. From this, together with Proposition 10, it follows that the hypersurface M has a closed recurrent conformal 2-form, that is, $D\Omega^m_{(C)l} = \alpha\Omega^m_{(C)l} = 0$. Moreover, the recurrent parameter 1-form is coclosed, that is, $\nabla_i\alpha_j = \nabla_j\alpha_i$. This gives an example of Theorem 40 in Sec. 5.

On four-dimensional Lorentzian manifold a stress-energy tensor of the form (54) is typical of a non-null Maxwell field (see [66, p. 62]).

Some other properties involving particular classes of Weyl-compatible tensors may be pointed out without directly using Derdziński–Shen's theorem: they are based only on the definition of Weyl compatibility. Let us consider for example the following one:

$$b_{kl} = \lambda X_k X_l, \tag{56}$$

where we suppose that $X^kX_k = 0$. Weyl's compatibility reads in full:

$$X_iX^mC_{jklm} + X_jX^mC^m_{kil} + X_kX^mC^m_{ijl} = 0. \tag{57}$$

Now consider a vector Y orthogonal to X, i.e. $X^kY_k = 0$; the previous expression is thus multiplied by $X^iY^jY^l$ to get easily:

$$X_k(X^iX^jY^lX^mC_{ijlm}) = 0, \tag{58}$$

from which the following scalar vanishes:

$$X^iY^jX^mY^lC_{ijlm} = 0. \tag{59}$$

On the other hand transvecting Eq. (57) with $X^iY^jZ^l$ being Z any other arbitrary vector it is inferred:

$$X_k(X^iY^jZ^lX^mC_{ijlm}) = 0, \tag{60}$$

from which the following scalar vanishes

$$X^iY^jX^mZ^lC_{ijlm} = 0. \tag{61}$$

We have just proven the following.

Theorem 33. *Let M be an n-dimensional pseudo-Riemannian manifold. If the symmetric tensor $b_{kl} = \lambda X_kX_l$ (with $X^kX_k = 0$) results to be Weyl-compatible then $X^iY^jX^mY^lC_{ijlm} = 0$ and $X^iX^jX^mZ^lC_{ijlm} = 0$ for any vector Y orthogonal to X and for any other vector Z.*

In view of Theorems 19 and 33, we may assert.

Corollary 34. *Let M be an n-dimensional pseudo-Riemannian manifold with recurrent conformal 2-form, i.e. with $D\Omega^m_{(C)l} = \alpha\wedge\Omega^m_{(C)l}$ being α closed. If the Ricci tensor (or the stress-energy tensor) is of the form $R_{kl} = \lambda X_kX_l$ (with $X^kX_k = 0$) then $X^iY^jX^mY^lC_{ijlm} = 0$ and $X^iY^jX^mZ^lC_{ijlm} = 0$ for any vector Y orthogonal to X and for any other vector Z.*

Remark 35. On a four-dimensional Lorentzian manifold a stress-energy tensor of the form (56) is typical of a null Maxwell field or a pure dust field (see [66, p. 61]); in particular a Ricci tensor of the form $b_{kl} = \lambda X_kX_l$ in (56) emerges from a *pp*-wave metric (see [27, Eq. (8.11)] and Sec. 6).

On a four-dimensional Lorentzian manifold a stress-energy tensor of the form (56) is typical of a null Maxwell field or of a pure dust field (see [66, p. 61]).

Now we stress other consequences of Weyl compatibility. Let us consider a symmetric Weyl-compatible $(0,2)$ tensor written as follows:

$$b_{kl} = \lambda_l X_kX_l + \lambda_2Y_kY_l, \tag{62}$$

where $X^kX_k = 1$, $Y_kY^k = 1$, $X^kY_k = 0$. Let Z^k be a vector belonging to the null space of b, i.e. $Z^kb_{kl} = 0$. In this case Derdziński–Shen theorem gives:

$$X^mY^lZ^kC_{jklm} = 0, \tag{63}$$

if the condition $\lambda_1, \lambda_2 \neq 0$ is satisfied. Another useful consequence is obtained if the condition of Weyl compatibility is written in full:

$$\begin{aligned}\lambda_1X_iX_mC^m_{jkl} + \lambda_1X_jX_mC^m_{kil} + \lambda_1X_kX_mC^m_{ijl}\\ + \lambda_2Y_iY_mC^m_{jkl} + \lambda_2Y_jY_mC^m_{kil} + \lambda_2Y_kY_mC^m_{ijl} = 0.\end{aligned} \tag{64}$$

The previous expression (64) is thus multiplied by Y^l to give:

$$X_iX^mY^lC_{jklm} + X_jX^mY^lC_{kilm} + X_kX^mY^lC_{ijlm} = 0. \tag{65}$$

On the other hand on transvecting (64) with X^l brings:

$$Y_i Y^m X^l C_{jklm} + Y_j Y^m X^l C_{kilm} + Y_k Y^m X^l C_{ijlm} = 0. \tag{66}$$

Now (65) is multiplied by Y^i recalling that $X^k Y_k = 0$; we thus infer that:

$$X_j X^m Y^l Y^i C_{ikml} - X_k X^m Y^l Y^i C_{ijml} = 0. \tag{67}$$

Define the $(0,2)$ symmetric tensor $D_{km} = C_{imkl} Y^i Y^l$; then:

$$X_j X^m D_{km} = X_k X^m D_{jm}. \tag{68}$$

Thus $X^m D_{km} = X_k(X^m X^j D_{jm})$, and X turns out to be an eigenvector of the symmetric tensor D. On an $n = 4$ Lorentzian manifold and if Y is time-like, D identifies with the electric part of the Weyl tensor (see [4] and Sec. 6).

Theorem 36. *Let M be an n-dimensional pseudo-Riemannian manifold. If the symmetric tensor $b_{kl} = \lambda_1 X_k X_l + \lambda_2 Y_k Y_l$ results to be Weyl-compatible, then X turns out to be an eigenvector of the symmetric tensor $D_{km} = C_{imkl} Y^i Y^l$. Moreover if M is a four-dimensional Lorentzian manifold and if Y is time-like, then X turns out to be an eigenvector of the electric part of the Weyl tensor.*

Consider now the same Weyl-compatible tensor (62) with the properties $X^k X_k = 0$, $Y_k Y^k = 0$, $X^k Y_k = 1$. In this case, Eq. (65) is multiplied by $X^i Y^j Z^k$ being $Z^k X_k = 0$ to give easily:

$$X^i Z^k X^m Y^l C_{kilm} = 0. \tag{69}$$

On the other hand on multiplying Eq. (66) by $Y^i X^j Z^k$ being $Z^k Y_k = 0$ it is inferred that

$$Y^m X^l Z^k Y^i C_{kilm} = 0. \tag{70}$$

Thus, the following result holds:

Theorem 37. *Let M be an n-dimensional pseudo-Riemannian manifold. If the symmetric tensor $b_{kl} = \lambda_1 X_k X_l + \lambda_2 Y_k Y_l$, is Weyl-compatible (being $X^k X_k = 0$, $Y_k Y^k = 0$, $X^k Y_k = 1$), then $X^i Z^k X^m Y^l C_{kilm} = 0$ and $Y^m X^l Z^k Y^i C_{kilm} = 0$ for any vector Z being orthogonal both to X and Y.*

In view of Theorems 19 and 37, we may assert.

Corollary 38. *Let M be an n-dimensional pseudo-Riemannian manifold with recurrent conformal 2-form, i.e. with $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$ being α closed. If the Ricci tensor (or the stress-energy tensor) is of the form $R_{kl} = \lambda_1 X_k X_l + \lambda_2 Y_k Y_l$ (being $X^k X_k = 0$, $Y_k Y^k = 0$, $X^k Y_k = 1$) then $X^i Z^k X^m Y^l C_{kilm} = 0$ and $Y^m X^l Z^k Y^i C_{kilm} = 0$ for any vector Z being orthogonal both to vector fields X and Y on M.*

5. Recurrent Conformal 2-Forms and Pontryagin Forms

In this section, we prove some topological properties of n-dimensional pseudo-Riemannian manifolds equipped with recurrent conformal 2-form. First, we need some background material about Pontryagin forms. Consider the following $4k$ forms ω_k on an orthonormal basis of tangent vectors built with the Riemann tensor [16, 49, 39]:

$$\begin{aligned}
\omega_1(X_1\cdots X_4) &= R_{ija}^{b}R_{klb}^{a}(X_1^i\wedge X_2^j)(X_3^k\wedge X_4^l),\\
\omega_2(X_1\cdots X_8) &= R_{ija}^{b}R_{klb}^{c}R_{mnc}^{d}R_{pqd}^{a}(X_1^i\wedge X_2^j)\cdots(X_7^p\wedge X_8^q).\\
&\ \ \vdots
\end{aligned}\tag{71}$$

The *Pontryagin forms* (see [49, 39] and also [56, pp. 317–318]) P_k result from total antisymmetrization of ω_k : $P_k(X_1\cdots X_{4k}) = \sum_p(-1)^p\omega_k(X_1\cdots X_{4k})$ where P is the permutation taking $(1\cdots 4k)$ to $(i_1\cdots i_{4k})$.

In [16] the authors considered compact manifolds admitting indefinite metrics with $\nabla_i C_{jkl}^{m} = 0$: they showed that in such case all the Pontryagin forms vanish. We consider here first the topological consequences originating from a recurrent conformal 2-form with harmonic conformal curvature tensor, i.e. with $\nabla_m C_{jkl}^{m} = 0$: from Proposition 10 and Lemma 11 we have thus $C_{lmj}^{k}C_{pqk}^{j} = 0$. Now as shown by Avez [2] (see also [16]) in the definition of the forms ω_k one may replace the Riemann curvature tensor with the conformal curvature tensor, i.e. for example:

$$\omega_1(X_1\cdots X_4) = C_{ija}^{b}C_{klb}^{a}(X_1^i\wedge X_2^j)(X_3^k\wedge X_4^l).\tag{72}$$

In the case of recurrent conformal 2-forms with harmonic conformal curvature tensor it is thus $\omega_1 = 0$ and the following theorem is stated.

Theorem 39. *Let M be an n-dimensional pseudo-Riemannian manifold with recurrent conformal 2-form, i.e. with $D\Omega_{(C)l}^{m} = \alpha\wedge\Omega_{(C)l}^{m}$: if $\nabla_m C_{jkl}^{m} = 0$ the first Pontryagin form vanishes, i.e. $P_1 = 0$.*

Let us now consider a compact orientable four-dimensional pseudo-Riemannian manifold. The vanishing of the first Pontryagin form has a deep topological consequence. In fact the Hirzebruch's *signature theorem* (see [31] and [56, pp. 229–230]) can be written as follows:

$$3\tau(M) = \int_M P_1.\tag{73}$$

In the previous expression $\tau(M)$ is the Hirzebruch's signature: it is a topological invariant that coincides with the usual topological signature. We conclude the following theorem.

Theorem 40. *Let M be a compact orientable n-dimensional pseudo-Riemannian manifold with recurrent conformal 2-form, i.e. with $D\Omega_{(C)l}^{m} = \alpha\wedge\Omega_{(C)l}^{m}$: if $\nabla_m C_{jkl}^{m} = 0$ the Hirzebruch's signature is null.*

Further in [39] the authors pointed out that in the Riemannian case the R-compatibility of a real symmetric tensor b_{kl} equipped with n distinct eigenvalues implies all the Pontryagin forms to vanish ([39, Theorem 5.3]). We recall that in the pseudo-Riemannian case (or in the Lorentzian case) the eigenvectors and the eigenvalues are often complex (see [63] or Hall [27, p. 202]). We take into consideration a symmetric R-compatible tensor b_{kl} that may be written in the form:

$$b_{kl} = \lambda_1 X_k(1)X_l(1) + \lambda_2 X_k(2)X_l(2) + \cdots + \lambda_n X_k(n)X_l(n), \tag{74}$$

where $X^j(a)X_j(b) = \varepsilon\delta_{ab}$ is an orthonormal basis of the pseudo-Riemannian manifold and $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$. In view of the generalized Derdziński–Shen's theorem [39] it is:

$$R_{ij}^{kl} X(a)^i \wedge X(b)^j X(c)_k = 0, \quad a \neq b \neq c. \tag{75}$$

It follows that all column vector of the matrix $M(a,b)^{kl} = R_{ij}^{kl} X(a)^i \wedge X(b)$ are orthogonal to $X(c)$, i.e. they belong to the subspace spanned by $X(a)$ and $X(b)$. Because of the antisymmetry of indices k, l, it is $M(a,b)^{kl} = \lambda_{ab} X(a)^i \wedge X(b)$ and thus it is inferred that:

$$R_{ij}^{kl} X(a)^i \wedge X(b)^j = \lambda_{ab} X(a)^k \wedge X(b)^l. \tag{76}$$

Now, it is easy to see that the generic Pontryagin form vanishes: for example we have:

$$\begin{aligned} \omega_1(X_1 \cdots X_4) &= R_{ija}^{b} R_{klb}^{a} (X_1^i \wedge X_2^j)(X_3^k \wedge X_4^l) \\ &= \lambda_{12}\lambda_{34}(X_1^a \wedge X_{2b})(X_{3a} \wedge X_4^b) \\ &= 0 \end{aligned}$$

and so on. It is thus proven that the following statement holds.

Theorem 41. *Let M be an n-dimensional pseudo-Riemannian manifold with a symmetric Riemann-compatible tensor b_{kl} of the form* (74) *with n distinct eigenvalues: then all Pontryagin form vanish.*

Moreover considering Hirzebruch's signature theorem we may prove, in four dimensions.

Corollary 42. *Let M be a four-dimensional compact orientable pseudo-Riemannian manifold with a symmetric Riemann-compatible tensor b_{kl} of the form* (74) *with four distinct eigenvalues: then the Hirzebruch's signature vanishes.*

Remark 43. As an example for the previous Corollary 42 we may choose a $(0,2)$ symmetric Codazzi tensor $\nabla_k b_{jl} = \nabla_j b_{kl}$ [17] with four distinct eigenvalues: in this case such tensor is Riemann- and Weyl-compatible and the Hirzebruch's signature vanishes.

6. Recurrent Conformal 2-Forms on Lorentzian Manifolds

In this section, we study the properties of recurrent conformal 2-forms on Lorentzian manifolds (space-times). Previous theorems stated on a pseudo-Riemannian manifold of arbitrary signature are then interpreted in the light of the classification of space-times in four or in higher dimensions. In particular the applications of extended Derdziński–Shen theorem and the consequence of Weyl's compatibility offer the backbone for the vanishing of some scalars involving the Weyl tensor. This gives rise to some constrictions on the Petrov types of such space-times.

We begin with a review of Petrov classification of four-dimensional space-times (see [27, 54, 55, 66]). This is the algebraic classification of the dual part of the Weyl tensor in terms of its eigenvalues and eigenvectors (see [27, 66]). It turns out that the eigenvalues of the Weyl tensor satisfy a fourth-order equation. The eigenvalue multiplicity classifies five different types of space-times. Thus for Petrov type I space-time the quartic roots are all distinct, for type II one double root is present, for type D there are two double roots, for type III one triple root is found, and finally for type N there is a four-fold root.

The completely degenerate case of conformally flat space-time forms the sixth type (named O). To be more concrete on a four-dimensional Lorentzian manifold (a manifold with metric signature +2) let us consider the null tetrad given by $k, l, m, \bar{m}$ being k and l two real null vectors and m, $\bar{m}$ two complex conjugate null vectors such that $k^a l_a = -1, m^a \bar{m}_a = 1$ and all remaining scalar products vanishing. The following bivectors are then defined:

$$\begin{aligned} U_{ab} &= -l_a \bar{m}_b + l_b \bar{m}_a, \\ V_{ab} &= k_a m_b - k_b m_a, \\ W_{ab} &= m_a \bar{m}_b - m_b \bar{m}_a - k_a l_b + k_b l_a. \end{aligned} \tag{77}$$

Then the Weyl tensor may be expanded as follows (see [66, p. 38] or [27, p. 191]):

$$\begin{aligned} C_{abcd} = {} & \psi_0 U_{ab} U_{cd} + \psi_1 (U_{ab} W_{cd} + W_{ab} U_{cd}) \\ & + \psi_2 (V_{ab} U_{cd} + U_{ab} V_{cd} + W_{ab} W_{cd}) \\ & + \psi_3 (V_{ab} W_{cd} + W_{ab} V_{cd}) + \psi_4 V_{ab} V_{cd}. \end{aligned} \tag{78}$$

In this expression the five complex coefficients $\psi_0, \psi_1, \psi_2, \psi_3, \psi_4$ are named Weyl's scalars [65, 66]. One obtains:

$$\begin{aligned} \psi_0 &= C_{abcd} k^a m^b k^c m^d, \\ \psi_1 &= C_{abcd} k^a l^b k^c m^d, \\ \psi_3 &= C_{abcd} k^a l^b \bar{m}^c l^d, \\ \psi_4 &= C_{abcd} \bar{m}^a l^b \bar{m}^c l^d, \\ \psi_2 &= C_{abcd} k^a m^b \bar{m}^c l^d. \end{aligned} \tag{79}$$

Thus it turns out that for Petrov type I it is $\psi_0 = 0$, for Petrov type II or D it is $\psi_0 = \psi_1 = 0$, for Petrov type III we have $\psi_0 = \psi_1 = \psi_2 = 0$, for Petrov type N it is $\psi_0 = \psi_1 = \psi_2 = \psi_3 = 0$ and finally for Petrov type O all Weyl's scalar vanish (see [65, Table 32.1, p. 276]).

Hereafter let M be a four-dimensional Lorentzian manifold on which the conformal 2-form is recurrent with closed recurrence parameter: from Theorem 19 any stress-energy tensor T_{ab} results to be Weyl-compatible. Choose $T_{ab} = \Phi^2 k_a k_b$ where Φ is a complex function: it represents the stress-energy tensor of a pure dust model (see [66, p. 61]) or a null radiation field. In view of Theorem 33 and Corollary 34 choosing $X = k$, $Y = m$, $Z = l$ we have thus $k^a m^b k^c m^d C_{abcd} = \psi_0 = 0$ and $k^a l^b k^c m^d C_{abcd} = \psi_1 = 0$. We have thus proved the following.

Theorem 44. *Let M be a four-dimensional Lorentzian manifold endowed with a recurrent conformal 2-form, i.e. with $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$ being α closed. If the stress-energy tensor is of null dust type, i.e. $T_{ab} = \Phi^2 k_a k_b$, then the Weyl's scalars ψ_0 and ψ_1 vanish and the Petrov types of such space-time is II or D.*

Consider now a stress-energy tensor T_{ab} of the form given in Example 2:

$$T_{ab} = \lambda_1(k_a l_b + k_b l_a) + \lambda_2(m_a \bar{m}_b + m_b \bar{m}_a). \tag{80}$$

We have simply $k^a T_{ab} = -\lambda_1 k_b$, $l^a T_{ab} = -\lambda_1 l_b$, $m^a T_{ab} = \lambda_2 m_b$, $\bar{m}^a T_{ab} = \lambda_2 \bar{m}_b$ and thus in view of Theorem 30 and Corollary 31 if in this case $\lambda_1 \neq -\lambda_2$ we have for example $k^a l^b m^c k^d C_{abcd} = \psi_1 = 0$ and $k^a l^b \bar{m}^c l^d C_{abcd} = \psi_3 = 0$. We may state the following.

Theorem 45. *Let M be a four-dimensional Lorentzian manifold endowed with a recurrent conformal 2-form, i.e. with $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$ being α closed. If the stress-energy tensor is of the form $T_{ab} = \lambda_1(k_a l_b + k_b l_a) + \lambda_2(m_a \bar{m}_b + m_b \bar{m}_a)$ with $\lambda_1 \neq -\lambda_2$ then the Weyl's scalars ψ_1 and ψ_3 vanish.*

We here note that the stress-energy tensor of a *non-null electromagnetic field* may be written in the form (see [66, p. 62])

$$T_{ab} = 2\Phi_1 \Phi_2 (k_a l_b + k_b l_a + m_a \bar{m}_b + m_b \bar{m}_a)$$

and the hypothesis of the previous theorem are satisfied.

An equivalent classification of four-dimensional Lorentzian manifolds arises from Bel and Debever criteria (see [3, 12]) which are based on null vectors k satisfying increasingly restricted conditions as follows:

$$\begin{aligned}
&\text{(a)} \quad \text{type}\, I \quad k_{[b} C_{a]rs[q} k_{n]} k^r k^s = 0,\\
&\text{(b)} \quad \text{type}\, II, D \quad k_{[b} C_{a]rsq} k^r k^s = 0,\\
&\text{(c)} \quad \text{type}\, III \quad k_{[b} C_{a]rsq} k^r = 0,\\
&\text{(d)} \quad \text{type}\, N \quad C_{arsq} k^r = 0,\\
&\text{(e)} \quad \text{type}\, O \quad C_{arsq} = 0.
\end{aligned} \tag{81}$$

When k satisfies condition (b) the Weyl tensor is named *algebraically special* (see [62, 65, 66]).

Consider now a four-dimensional Lorentzian manifold endowed with a recurrent conformal 2-form satisfying the further condition $\nabla_m C^m_{jkl} = 0$: in view of Proposition 10 we have $\alpha_i C^m_{jkl} + \alpha_j C^m_{kil} + \alpha_k C^m_{ijl} = 0$ and thus $\alpha_m C^m_{jkl} = 0$ being α a null vector. Choosing $k = \alpha$ in the null tetrad formalism we obtain a type N space-time. We may state the following.

Theorem 46. *Let M be a four-dimensional Lorentzian manifold endowed with a recurrent conformal 2-form, i.e. with $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$. If the condition $\nabla_m C^m_{jkl} = 0$ is fulfilled then the space-time is of Petrov type N with respect to the null vector α.*

Remark 47. A space-time admitting a covariantly constant null vector field k, i.e. $\nabla_b k_a = 0$ is named plane-fronted gravitational waves with parallel rays (*pp*-waves) (see [27, p. 248] and [66, pp. 383–384]). A metric of the form ([27, Eq. (8.11)])

$$ds^2 = H(x, y, u)du^2 + 2dudv + dx^2 + dy^2,$$

satisfies this condition: these space-times are either of Petrov type N or conformally flat; moreover the *pp*-waves are complex recurrent with closed recurrence parameter and thus satisfy Eq. (25) and the conformal 2-form results to be recurrent. The Ricci tensor satisfies ([27, Eq. (8.12)]):

$$R_{ab} = \left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right) k_a k_b,$$

being $k_a = \nabla_a u$ results to be Weyl-compatible. This gives an example of a space-time of the Petrov type N which satisfies (2.18).

On a four-dimensional Lorentzian manifold it is possible to define the *electric* and magnetic part of the Weyl tensor (see [4, 66]). Precisely given a time-like velocity vector u^i (i.e. $u_i u^i = -1$) the following $(0, 2)$ tensors are defined:

$$\begin{aligned} E_{kl} &= u^j u^m C_{jklm}, \\ H_{kl} &= \frac{1}{4} u^j u^m (\varepsilon_{\alpha\beta jk} C^{\alpha\beta}_{lm} + \varepsilon_{\alpha\beta jl} C^{\alpha\beta}_{km}), \end{aligned} \tag{82}$$

where ε_{ijkl} is the completely skew-symmetric Levi-Civita symbol. The tensor E_{kl} is named *electric part* of the Weyl tensor, while the tensor H_{kl} is named *magnetic part* of the Weyl tensor; elementary properties are found to be [4]:

$$\begin{aligned} g^{kl} E_{kl} &= g^{kl} H_{kl} = 0, \\ u^k E_{kl} &= u^k H_{kl} = 0. \end{aligned}$$

Moreover, the Weyl tensor is uniquely decomposed in its electric and magnetic parts. In [4] it was specified that Eq. (82) is valid also in the case $u_i u^i = 1$. A fundamental property of the magnetic part of the Weyl tensor satisfying condition

(30) was stated: we stress it again for completeness. We focus on stress-energy tensors of the perfect fluid form $T_{kl} = au_k u_l + bg_{kl}$ (see [66, p. 61]) with normalized covectors u_j: in this case such covector permits the decomposition of the Weyl tensor. Equation (40) takes the form:

$$u^i u^m C^{jk}_{lm} + u^j u^m C^{ki}_{lm} + u^k u^m C^{ij}_{lm} = 0. \tag{83}$$

The previous equation is thus multiplied by ε_{ijkp} to get:

$$\varepsilon_{ijkp} u^i u^m C^{jk}_{lm} + \varepsilon_{ijkp} u^j u^m C^{ki}_{lm} + \varepsilon_{ijkp} u^k u^m C^{ij}_{lm} = 0.$$

Recalling the skew-symmetric properties of the Levi-Civita symbol we simply have:

$$\varepsilon_{ijkp} u^i u^m C^{jk}_{lm} = \varepsilon_{kijp} u^k u^m C^{ij}_{lm} = \varepsilon_{ijkp} u^k u^m C^{ij}_{lm},$$
$$\varepsilon_{ijkp} u^j u^m C^{ki}_{lm} = \varepsilon_{jkip} u^j u^m C^{ki}_{lm} = \varepsilon_{ijkp} u^k u^m C^{ij}_{lm}.$$

Thus we infer that $3\varepsilon_{ijkp} u^k u^m C^{ij}_{lm} = 0$ and so the magnetic part of the Weyl tensor vanishes.

Theorem 48. *Let M be a space-time manifold having a Weyl-compatible stress-energy tensor of the perfect fluid form $T_{kl} = au_k u_l + bg_{kl}$. Then the magnetic part of the Weyl tensor vanishes.*

Remark 49. Gödel solution of Einstein's equation ([26]) is known to be:

$$ds^2 = a^2 \left(-(dx^1)^2 + \frac{1}{2} e^{2x^1} (dx^2)^2 - (dx^3)^2 + (dx^4)^2 + 2e^{2x^1} dx^2 dx^4 \right),$$

where $a^2 = \frac{1}{\omega^2}$, ω is a non-null real constant. In [21] the authors pointed out several curvature properties of Gödel space-time: among others they proved ([21, Eq. (19) and Theorem 2]) that the Ricci tensor of Gödel space-time is Riemann- and Weyl-compatible, and results to be of rank 1, namely $R_{kl} = \kappa \omega_k \omega_l$ being $\kappa = \frac{1}{a^2}$ and ω_j the 1-form defined as $\omega = (0, ae^{x^1}, 0, a)$. Moreover, the form ω_j is Riemann- and Weyl-compatible (see [40]). We have then $\omega_i \omega_m C^m_{jkl} + \omega_j \omega_m C^m_{kil} + \omega_k \omega_m C^m_{kil} = 0$ and the Gödel space-time results to be purely electric.

Space-times in which $H_{kl} = 0$ are named purely electric (see [4, 66]) space-times, while the condition $E_{kl} = 0$ defines purely magnetic space-times [4, 66]. It is well-known that purely electric space-times are of Petrov type I, D or O [66]. We have thus the following theorem.

Theorem 50. *Let M be a four-dimensional Lorentzian manifold endowed with a recurrent conformal 2-form, i.e. with $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$ being α closed. If the stress-energy tensor of the perfect fluid form $T_{kl} = au_k u_l + bg_{kl}$ then $H_{kl} = 0$ and the Petrov types are I or D.*

Hereafter we investigate n-dimensional Lorentzian manifolds on which the conformal curvature 2-form is recurrent. The Bel–Debever classification was recently extended to the n-dimensional case by some authors and it is exposed in several references such as [7, 8, 30, 48, 51]. Here, we refer to the classification given in [51, Table 1]. The supplementary condition $\nabla_m C^m_{jkl} = 0$ gives again $\alpha_i C^m_{jkl} + \alpha_j C^m_{kil} + \alpha_k C^m_{ijl} = 0$: in this case this matches with the n-dimensional space-time of type N (see [51, Table 1]). We have inferred the following result.

Theorem 51. *Let M be an n-dimensional Lorentzian manifold endowed with a recurrent conformal 2-form, i.e. with $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$. If the condition $\nabla_m C^m_{jkl} = 0$ is fulfilled then the space-time is of Petrov type N with respect to the null vector α.*

Again consider an n-dimensional Lorentzian manifold endowed with recurrent conformal 2-form and with the stress-energy tensor of a pure dust model, i.e. $T_{ab} = \Phi^2 k_a k_b$; the Weyl compatibility condition reads:

$$k_a k^m C_{bclm} + k_b k^m C_{calm} + k_c k^m C_{ablm} = 0. \tag{84}$$

It is readily seen that this condition matches with type II_d space-time of [51, Table 1]. We have thus the following theorem.

Theorem 52. *Let M be an n-dimensional Lorentzian manifold endowed with a recurrent conformal 2-form, i.e. with $D\Omega^m_{(C)l} = \alpha \wedge \Omega^m_{(C)l}$ being α closed. If the stress-energy tensor is of null dust type, i.e. $T_{ab} = \Phi^2 k_a k_b$, then the Weyl tensor is of type II_d with respect to k.*

In [30] the notions of purely electric and purely magnetic Weyl tensor was extended to n-dimensional Lorentzian manifolds. In particular it was stated (see [30], Proposition 3.5) that the Weyl tensor is purely electric with respect to the time-like vector u if and only if Eq. (83) holds. Thus the results of Theorem 50 are readily extended to n-dimensional Lorentzian manifolds.

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