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Real Hypersurfaces in Complex Two-Plane Grassmannians with Reeb Parallel Structure Jacobi Operator

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Abstract. In this paper we give a characterization of a real hypersurface of Type (A) in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, which means a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, by means of the Reeb parallel structure Jacobi operator $\nabla_\xi R_\xi = 0$.

1 Introduction

In the geometry of real hypersurfaces in complex space forms or in quaternionic space forms, there have been many characterizations of homogeneous hypersurfaces. For example, in complex projective space $\mathbb{C}P^m$ we call them real hypersurfaces of type (A_1), (A_2), (B), (C), (D), and (E); in complex hyperbolic space $\mathbb{C}P^m$, of type (A_0), (A_1), (A_2), and (B); in quaternionic projective space $\mathbb{H}P^m$, of type (A_1), (A_2), and (B); and in quaternionic hyperbolic space $\mathbb{H}H^m$, of type (A_0), (A_1), (A_2), and (B). They are completely classified by Kimura [12], Berndt [2, 3], and Martinez and Pérez [15].

Now let us consider a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, which consists of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J (see Berndt and Suh [5, 6]). Accordingly, in $G_2(\mathbb{C}^{m+2})$ we have two natural conditions for a real hypersurface M so that $[\xi] = \operatorname{Span}\{\xi\}$ or $\mathfrak{D}^\perp = \operatorname{Span}\{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. Here $\xi = -JN$, $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$, and N is a local unit normal vector field on M.

Using the two invariant conditions mentioned above, Berndt and Suh proved the following theorem.

Theorem 1.1 (Berndt and Suh [5]) *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, where $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape*

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operator of M if and only if M is one of the following types:

Type (A) *M is an open part of a tube around a totally geodesic Grassmannian $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Type (B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic quaternionic projective space $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

Furthermore, the Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The 1-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the formulas in Section 3 it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf. In such a case, the Reeb flow of ξ on M is said to be *geodesic*, and we say M is a real hypersurface with *geodesic Reeb flow*.

Remark 1.2 Related to a geodesic Reeb flow, we give an example of a ruled real hypersurface M in $G_2(\mathbb{C}^{m+2})$ that is not Hopf. It is foliated by complex hypersurfaces that include a maximal totally geodesic submanifold $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ (see Choi and Suh [7]). Its integrable distribution is given by $T_0(x) = \{X{\in}T_xM|X{\perp}\xi\}$, and the expression of the shape operator A of M is given by

$$A\xi = \alpha\xi + \beta U, \quad AU = \beta\xi, \quad \text{and} \quad AX = 0$$

for any X orthogonal to ξ and U. By virtue of the expression of the shape operator, we know that the distribution $T_0(x)$ is integrable. Then the shape operator never commutes with the structure tensor ϕ. Usually, the function $\alpha = g(A\xi, \xi)$ is not constant along the direction of ξ, because $\xi\alpha = g((\nabla_\xi A)\xi, \xi)$ cannot vanish in general. Of course, the Reeb vector field for a ruled hypersurface M in $G_2(\mathbb{C}^{m+2})$ does not have a geodesic Reeb flow; that is, M is not Hopf.

The Reeb vector field ξ on M is called *Killing* if the Reeb flow on M in $G_2(\mathbb{C}^{m+2})$ is *isometric*. It is denoted by $\mathcal{L}_\xi g = 0$, where $\mathcal{L}$ (resp. g) denotes the Lie derivative (resp. the induced Riemannian metric) of M in the direction of the Reeb vector field ξ. This means that the metric tensor g is *invariant* under the Reeb flow of ξ on M.

In [6], Berndt and Suh have given a characterization of real hypersurfaces of Type (A) in Theorem 1.1 when the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor ϕ. This is equivalent to the condition that the Reeb flow on M is isometric.

By using such a notion, Berndt and Suh [6] gave the following characterization of Type (A) in $G_2(\mathbb{C}^{m+2})$.

Theorem 1.3 *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

On the other hand, for real hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$, Lee and Suh [14] recently proved the following theorem.

Theorem 1.4 *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

Now we introduce the notion of structure Jacobi operator R_ξ defined by

$$R_\xi(X) = R(X, \xi)\xi,$$

where $R(X, Y)Z$ denotes the curvature tensor of M in $G_2(\mathbb{C}^{m+2})$ for any tangent vector fields X, Y, and Z on M. Then the structure Jacobi operator R_ξ for the Reeb vector ξ is said to be *parallel* if the covariant derivative of the structure Jacobi operator R_ξ vanishes, that is, if $\nabla_X R_\xi = 0$ for any vector field X on M.

Related to such a structure Jacobi operator R_ξ, many authors have studied some geometric properties for real hypersurfaces in complex space form $M_n(c)$. In [11], Ki, Pérez, Santos, and Suh investigated the covariant derivative $\nabla_\xi S = 0$ for the Ricci tensor S and the parallel structure Jacobi operator $\nabla_\xi R_\xi = 0$ along the direction of ξ. In [19], Pérez, Santos, and Suh classified real hypersurfaces in $\mathbb{C}P^m$ with a ξ-invariant structure Jacobi operator, that is, $\mathcal{L}_\xi R_\xi = 0$. Also, they proved the non-existence of any real hypersurfaces in $\mathbb{C}P^m$ with a $\mathfrak{D}$-parallel structure Jacobi operator $\nabla_X R_\xi = 0$ for any $X \in \mathfrak{D}$, where the distribution $\mathfrak{D}$ is defined by the subspace $\mathfrak{D}_x = \{\, X \in T_xM \mid X \perp \xi \,\}$, $x \in M$. So the distribution $\mathfrak{D}$ becomes an orthogonal complement of the Reeb vector field ξ on real hypersurfaces in $\mathbb{C}P^m$ (see [20]).

Moreover, Pérez, and Suh [17] classified real hypersurfaces in quaternionic projective space $\mathbb{H}P^m$ whose curvature tensor is parallel in the direction of the distribution $\mathfrak{D}^\perp$, that is, $\nabla_{\xi_i} R = 0$, $i = 1, 2, 3$. In such a case they are congruent to a tube of radius $\frac{\pi}{4}$ over a totally geodesic $\mathbb{H}P^k$ in $\mathbb{H}P^m$, $2 \leq k \leq m-2$.

But in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, if we consider these properties, the situation is quite different from that of $\mathbb{C}P^m$ and $\mathbb{H}P^m$.

Recently, Jeong, Pérez, and Suh [10] proved that there does not exist a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with parallel structure Jacobi operator. Also, Jeong, Machado, Pérez, and Suh [9] obtained the non-existence for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}^\perp$-parallel structure Jacobi operator $\nabla_X R_\xi = 0$ for any X belonging to the distribution $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$.

Motivated by such a notion of parallel structure Jacobi operators, in this paper, we consider the parallelism of R_ξ on M in $G_2(\mathbb{C}^{m+2})$ in the direction of the Reeb vector field ξ.

We note here that the Reeb parallel structure Jacobi operator $\nabla_\xi R_\xi = 0$ is weaker than the parallel structure Jacobi operator $\nabla_X R_\xi = 0$ for any tangent vector field X on M in $G_2(\mathbb{C}^{m+2})$.

In such a case we say that M has a *Reeb parallel* structure Jacobi operator. We can give a characterization of Type (A) hypersurfaces in Theorem 1.1 as follows.

Theorem 1.5 (Main Theorem) *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel structure Jacobi operator. If the principal curvature of the Reeb vector field ξ on M is non-vanishing and constant along the direction of the Reeb vector field ξ, then M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius $r \in (0, \frac{\pi}{4\sqrt{2}}) \cup (\frac{\pi}{4\sqrt{2}}, \frac{\pi}{\sqrt{8}})$.*

Remark 1.6 When the function $\alpha = g(A\xi, \xi)$ vanishes identically, we know that the ruled hypersurface M in $G_2(\mathbb{C}^{m+2})$ in Remark 1.2 becomes a minimal ruled real hypersurface in $G_2(\mathbb{C}^{m+2})$ like in Kimura [13]and Ahn, Lee, and Suh [1] for real hypersurfaces in complex projective space $\mathbb{C}P^m$ and complex hyperbolic space $\mathbb{C}H^m$, respectively. In this case, the shape operator becomes

$$A\xi = \beta U, \quad AU = \beta\xi, \quad \text{and} \quad AX = 0$$

for any X orthogonal to ξ and U (see [8]). Then the Reeb vector field cannot be Hopf, so we know that the structure Jacobi operator cannot be Reeb parallel.

2 Riemannian Geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details refer to [4–6]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan–Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $\mathrm{Ad}(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $\mathrm{Ad}(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $\mathrm{Spin}(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$. In this paper, we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ denotes the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kaehler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kaehler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\mathrm{tr}(JJ_1) = 0$.

A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\overline{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\overline{\nabla}_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\overline{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}
\overline{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\
& - g(JX,Z)JY - 2g(JX,Y)JZ \\
& + \sum_{\nu=1}^{3} \big\{ g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z \big\} \\
& + \sum_{\nu=1}^{3} \big\{ g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY \big\},
\end{aligned} \tag{2.1}$$

where $\{J_1, J_2, J_3\}$ is any canonical local basis of $\mathfrak{J}$ (see [4]).

3 Some Fundamental Formulas in $G_2(\mathbb{C}^{m+2})$

In this section we derive some basic formulae and the equation of Codazzi and Gauss for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [5,6]).

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M is denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal vector field of M and let A denote the shape operator of M with respect to N.

The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ on M induces an almost contact metric structure (ϕ, ξ, η, g). More explicitly, we can define a tensor field ϕ of type (1,1), a vector field ξ and its dual 1-form η on M by $g(\phi X, Y) = g(JX, Y)$ and $\eta(X) = g(\xi, X)$ for any tangent vector fields X and Y on M. Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \text{and} \quad \eta(\xi) = 1$$

for any tangent vector field X on M. Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M in such a way that a tensor field ϕ_ν of type (1,1), a vector field ξ_ν and its dual 1-form η_ν on M are defined by $g(\phi_\nu X, Y) = g(J_\nu X, Y)$ and $\eta_\nu(X) = g(\xi_\nu, X)$ for any tangent vector fields X and Y on M respectively. Then they also satisfy the following:

$$\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \phi_\nu\xi_\nu = 0, \quad \eta_\nu(\phi_\nu X) = 0, \quad \text{and} \quad \eta_\nu(\xi_\nu) = 1$$

for any tangent vector field X on M and $\nu = 1, 2, 3$.

Using the above expression (2.1) for the curvature tensor $\overline{R}$ of $G_2(\mathbb{C}^{m+2})$, the equations of Gauss and Codazzi are respectively given by

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y \\ & + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ & + \sum_{\nu=1}^{3}\left\{ g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y)\phi_\nu Z\right\} \\ & + \sum_{\nu=1}^{3}\left\{ g(\phi_\nu\phi Y,Z)\phi_\nu\phi X - g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\right\} \\ & - \sum_{\nu=1}^{3}\left\{ \eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\right\} \\ & - \sum_{\nu=1}^{3}\left\{ \eta(X)g(\phi_\nu\phi Y,Z) - \eta(Y)g(\phi_\nu\phi X,Z)\right\}\xi_\nu \\ & + g(AY,Z)AX - g(AX,Z)AY \end{aligned}$$

and

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X = {} & \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \\ & + \sum_{\nu=1}^{3}\left\{ \eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X,Y)\xi_\nu\right\} \\ & + \sum_{\nu=1}^{3}\left\{ \eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\right\} \\ & + \sum_{\nu=1}^{3}\left\{ \eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\right\}\xi_\nu, \end{aligned}$$

where R denotes the curvature tensor of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

for any tangent vector field X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$.

Then the following identities can be proved in a straightforward way and will be used frequently in subsequent calculations:

$$\begin{gathered} \phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \quad \phi\xi_\nu = \phi_\nu\xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X), \\ \phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \quad \phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}. \end{gathered}$$

From this and the above formulae we have

$$
\begin{aligned}
&(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX,\\
&\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX,\\
&(\nabla_X\phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX,Y)\xi_\nu.
\end{aligned}
\tag{3.1}
$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu.$$

On the other hand, by using the fact of $A\xi = \alpha\xi$, $\alpha = g(A\xi,\xi)$, and the Codazzi equation, we have

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y)$$

for any tangent vector field Y on M in $G_2(\mathbb{C}^{m+2})$.

Now let us recall a lemma due to Berndt and Suh [6].

Lemma 3.1 *If M is a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$ with geodesic Reeb flow, then*

$$
\begin{aligned}
&\alpha g\big((A\phi + \phi A)X, Y\big) - 2g(A\phi AX, Y) + 2g(\phi X, Y)\\
&\quad = 2\sum_{\nu=1}^{3}\big\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\\
&\qquad\qquad - 2\eta(X)\eta_\nu(\phi Y)\eta_\nu(\xi) + 2\eta(Y)\eta_\nu(\phi X)\eta_\nu(\xi)\big\}
\end{aligned}
$$

for all vector fields X and Y on M.

On the other hand, we introduce the following lemma due to Jeong, Machado, Pérez, and Suh [9].

Lemma 3.2 *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$. If the principal curvature α is constant along the direction of ξ, then the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$ component of the structure vector field ξ is invariant by the shape operator.*

4 The Reeb Parallel Structure Jacobi Operator

In this section we give some lemmas which will be useful in the proof of Theorem 1.5.

Now we put the structure vector $\xi = -JN$ into the curvature tensor R of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then for any tangent vector field X on M in $G_2(\mathbb{C}^{m+2})$ we calculate the structure

Jacobi operator R_ξ in such a way that

$$
\begin{aligned}
R_\xi X = R(X,\xi)\xi &= X - \eta(X)\xi \\
&\quad - \sum_{\nu=1}^{3}\left\{ \left(\eta_\nu(X) - \eta(X)\eta_\nu(\xi)\right)\xi_\nu + 3\eta_\nu(\phi X)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X\right\} \\
&\quad + \alpha AX - \eta(AX)A\xi,
\end{aligned}
\tag{4.1}
$$

where α denotes the function defined by $g(A\xi,\xi)$.

Let us assume that the structure Jacobi operator R_ξ on a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ satisfies the Reeb parallelism $(\nabla_\xi R_\xi)X = 0$ for any tangent vector field X on M. By differentiating (4.1), we have

$$
\begin{aligned}
0 &= (\nabla_X R_\xi)Y \\
&= \nabla_X(R_\xi Y) - R_\xi \nabla_X Y \\
&= -g(\phi AX, Y)\xi - \eta(Y)\phi AX \\
&\quad - \sum_{\nu=1}^{3}\Big[g(\phi_\nu AX, Y)\xi_\nu - 2\eta(Y)\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\
&\qquad\quad + 3\left\{ g(\phi_\nu AX, \phi Y)\phi_\nu\xi + \eta(Y)\eta_\nu(AX)\phi_\nu\xi \right. \\
&\qquad\qquad \left. - \eta_\nu(\phi Y)\eta(AX)\xi_\nu + \eta_\nu(\phi Y)\phi_\nu\phi AX \right\} \\
&\qquad\quad + 4\eta_\nu(\xi)\{\eta_\nu(\phi Y)AX - g(AX,Y)\phi_\nu\xi\} + 2\eta_\nu(\phi AX)\phi_\nu\phi Y \Big] \\
&\quad + \eta\left((\nabla_X A)\xi\right)AY + \alpha(\nabla_X A)Y - \alpha\eta\left((\nabla_X A)Y\right)\xi \\
&\quad - \alpha g(AY, \phi AX)\xi - \alpha\eta(Y)(\nabla_X A)\xi - \alpha\eta(Y)A\phi AX
\end{aligned}
\tag{4.2}
$$

for any tangent vector fields X and Y on M.

If we put $X = \xi$ and $Y = X$ in (4.2), then we have

$$
\begin{aligned}
0 &= (\nabla_\xi R_\xi)X \\
&= 4\alpha\sum_{\nu=1}^{3}\left\{ \eta_\nu(\phi X)\xi_\nu - \eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\eta_\nu(\phi X)\xi + \eta_\nu(\xi)\eta(X)\phi_\nu\xi \right\} \\
&\quad + (\xi\alpha)AX + \alpha(\nabla_\xi A)X - 2\alpha(\xi\alpha)\eta(X)\xi
\end{aligned}
\tag{4.3}
$$

for any tangent vector field X on M.

Remark 4.1 When the function α vanishes, the above equation gives that the structure Jacobi operator is Reeb parallel $\nabla_\xi R_\xi = 0$. Moreover, from Pérez and Suh [18], we know that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.

Lemma 4.2 *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel structure Jacobi operator. If the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$ component of the Reeb vector field ξ is invariant under the shape operator, then ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Proof In order to prove this lemma, let us put $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit vector $X_0 \in \mathfrak{D}$ and non-zero functions $\eta(X_0)$ and $\eta(\xi_1)$. By putting $X = X_0$ into (4.3) we have

$$0 = 4\alpha\eta_1(\xi)\eta(X_0)\phi_1\xi + (\xi\alpha)AX_0 + \alpha(\nabla_\xi A)X_0 - 2\alpha(\xi\alpha)\eta(X_0)\xi.$$

Using a method similar to that in [10, Lemma 3.1], we obtain $\phi X_0 = 0$. This gives a contradiction, which completes the proof of our lemma. ∎

By Lemmas 3.2 and 4.2, we have the following lemma.

Lemma 4.3 *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel structure Jacobi operator. If the principal curvature α is constant along the direction of ξ, then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

5 Proof of Theorem 1.5

In this section, we assume that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with Reeb parallel structure Jacobi operator. Then by Lemma 4.2 we assume that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.

First, let us investigate the case that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$. Then we have the following lemma, which will be useful in the proof of Theorem 5.3.

Lemma 5.1 *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel structure Jacobi operator. If the principal curvature of the Reeb vector field ξ is non-vanishing and ξ belongs to the distribution $\mathfrak{D}^\perp$, then the shape operator A commutes with the structure tensor field ϕ.*

Proof In order to prove this lemma, we may put $\xi = \xi_1$, because $\xi \in \mathfrak{D}^\perp$. From (4.3), we have $\alpha(\nabla_\xi A)X = 0$ for any tangent vector field X on M.

Since the geodesic Reeb flow α is *non-vanishing*, we have $(\nabla_\xi A)X = 0$. By using the Codazzi equation, we have

$$\begin{aligned} 0 &= (\nabla_\xi A)X \\ &= -A\phi AX + (X\alpha)\xi + \alpha\phi AX + \phi X + \phi_1 X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3. \end{aligned}$$

From this, by taking an inner product with ξ, it follows that $X\alpha = 0$ for any tangent vector field X on M.

This gives that the principal curvature α is constant. Then we have

$$A\phi AX = \alpha\phi AX + \phi X + \phi_1 X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3. \tag{5.1}$$

From Lemma 3.1, we have

$$2A\phi AX = \alpha A\phi X + \alpha\phi AX + 2\phi X + 2\phi_1 X + 4\eta_3(X)\xi_2 - 4\eta_2(X)\xi_3 \tag{5.2}$$

for any tangent vector field X on M. Using (5.1) and (5.2), we know that $A\phi = \phi A$. Thus we complete the proof of our lemma. ■

By Theorem 1.3, we assert that a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with the assumption in Lemma 5.1 is a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. In other words, M is locally congruent to a real hypersurface of Type (A) in Theorem 1.1.

Conversely, let us check whether real hypersufaces of Type (A) satisfy the Reeb parallel structure Jacobi operator $\nabla_\xi R_\xi = 0$.

We recall a proposition given by Berndt and Suh [5].

Proposition 5.2 *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m-2 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1, \\ T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3, \\ T_\lambda &= \{X | X \perp \mathbb{H}\xi, JX = J_1 X\}, \\ T_\mu &= \{X | X \perp \mathbb{H}\xi,\ JX = -J_1 X\}, \end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$, and $\mathbb{H}\xi$ respectively denote the real, complex, and quaternionic spans of the structure vector ξ and $\mathbb{C}^\perp\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Now let us check case by case whether real hypersurfaces of Type (A) satisfy formula (4.3).

Case A-1 $X \in T_\alpha$

By using the conditions of $\xi \in \mathfrak{D}^\perp$ and $\xi\alpha = 0$ in (4.3), we assert formula (5.1) (see [10]). Then it can be easily checked by putting $X = \xi$ in (5.1).

Case A-2 $X \in T_\beta$

We put $A\xi_2 = \beta\xi_2$, $A\xi_3 = \beta\xi_3$, where $\beta = \sqrt{2}\cot(\sqrt{2}r)$. By putting $X = \xi_2$ in (5.1), we have

$$\begin{aligned} (\nabla_\xi A)\xi_2 &= -A\phi A\xi_2 + \alpha\phi A\xi_2 + \phi\xi_2 + \phi_1\xi_2 + 2\eta_3(\xi_2)\xi_2 - 2\eta_2(\xi_2)\xi_3 \\ &= -\beta A\phi\xi_2 + \alpha\beta\phi\xi_2 - 2\xi_3 = \beta^2\xi_3 - \alpha\beta\xi_3 - 2\xi_3 \\ &= (\beta^2 - \alpha\beta - 2)\xi_3 = 0. \end{aligned}$$

Similarly, by putting $X = \xi_3$ in (5.1), we obtain

$$(\nabla_\xi A)\xi_3 = -(\beta^2 - \alpha\beta - 2)\xi_2 = 0.$$

Case A-3 $X \in T_\lambda = \{X \mid X \perp \mathbb{H}\xi,\ \phi X = \phi_1 X\}$
For any tangent vector field $X \in T_\lambda$, $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$ we get

$$\begin{aligned}(\nabla_\xi A)X &= -A\phi AX + \alpha\phi AX + \phi X + \phi_1 X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3 \\ &= -\lambda A\phi X + \alpha\lambda\phi X + \phi X + \phi_1 X = -\lambda^2\phi X + \alpha\lambda\phi X + 2\phi X \\ &= -(\lambda^2 - \alpha\lambda - 2)\phi X = 0.\end{aligned}$$

Case A-4 $X \in T_\mu = \{X \mid X \perp \mathbb{H}\xi,\ \phi X = -\phi_1 X\}$
For any tangent vector field $X \in T_\mu$, $\mu = 0$ we get

$$\begin{aligned}(\nabla_\xi A)X &= -A\phi AX + \alpha\phi AX + \phi X + \phi_1 X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3 \\ &= -\mu A\phi X + \alpha\mu\phi X + \phi X + \phi_1 X = 0.\end{aligned}$$

Summing up all cases, we have formula (4.3). Thus we can assert the following theorem.

Theorem 5.3 *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel structure Jacobi operator. If the principal curvature of the Reeb vector field ξ is non-vanishing and $\xi \in \mathfrak{D}^\perp$, then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius $r \in (0, \frac{\pi}{4\sqrt{2}}) \cup (\frac{\pi}{4\sqrt{2}}, \frac{\pi}{\sqrt{8}})$.*

Next we consider the case that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$. By Theorem 1.4, we see that a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with ξ-parallel structure Jacobi operator is of Type (B) in Theorem 1.1. In order to complete the proof of our main theorem let us recall a proposition due to Berndt and Suh [5].

Proposition 5.4 *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r)\,, \quad \beta = 2\cot(2r)\,, \quad \gamma = 0\,, \quad \lambda = \cot(r)\,, \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu),$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi, \quad T_\beta = \mathfrak{J}J\xi, \quad T_\gamma = \mathfrak{J}\xi, \quad T_\lambda\,, \quad T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

Then for $\xi \in \mathfrak{D}$ and $\xi\alpha = 0$ in (4.3), we have

$$0 = 4\alpha \sum_{\nu=1}^{3} \{ \eta_\nu(\phi X)\xi_\nu - \eta_\nu(X)\phi_\nu\xi \} + \alpha(\nabla_\xi A)X.$$

From this, by putting $X = \xi_2$ we have $0 = -4\alpha\phi_2\xi + \alpha(\nabla_\xi A)\xi_2$. By taking the inner product with $\phi_2\xi$ and using (3.1), we have $-4\alpha + \alpha^2\beta = 0$.

Since the principal curvature α is non-zero, it follows that $\alpha\beta = 4$. This gives a contradiction. Then we assert that the structure Jacobi operator R_ξ of real hypersurfaces of Type (B) in Theorem 1.1 does not satisfy $\nabla_\xi R_\xi = 0$. Then from this fact, we assert the following theorem.

Theorem 5.5 *There does not exist any connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel structure Jacobi operator if the principal curvature of the Reeb vector field ξ is non-vanishing and $\xi \in \mathfrak{D}$.*

Combining Theorems 5.3 and 5.5, we complete the proof of Theorem 1.5. ■

Remark 5.6 Recently, we have been informed that the Reeb invariant structure Jacobi operator $\mathcal{L}_\xi R_\xi = 0$ for the Lie derivative $\mathcal{L}_\xi$ along the Reeb vector field ξ was studied by Machado and Pérez [16]. But usually the Reeb parallel structure Jacobi operator $\nabla_\xi R_\xi = 0$ for the covariant derivative ∇_ξ along the direction of ξ becomes a condition weaker than the Reeb invariant $\mathcal{L}_\xi R_\xi = 0$.

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