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Real hypersurfaces in the complex quadric with Reeb invariant shape operator ☆

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ABSTRACT

First we introduce the notion of Reeb invariant shape operator for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. Next we give a complete classification of real hypersurfaces in $Q^m = SO_{m+2}/SO_mSO_2$ with invariant shape operator.

1. Introduction

When we consider some Hermitian symmetric spaces of rank 2, usually we can give examples of Riemannian symmetric spaces $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians respectively (see [1,2,12,13]). Those are said to be Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$ on $SU_{2,m}/S(U_2U_m)$. The rank of $SU_{2,m}/S(U_2U_m)$ is 2 and there are exactly two types of singular tangent vectors X of $SU_{2,m}/S(U_2U_m)$ which are characterized by the geometric properties $JX \in \mathfrak{J}X$ and $JX \perp \mathfrak{J}X$ respectively.

As another kind of Hermitian symmetric space with rank 2 of compact type different from the above ones, we can give an example of complex quadric $Q^m = SO_{m+2}/SO_mSO_2$, which is a complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$ (see Berndt and Suh [3], and Smyth [10]). The complex quadric also can be regarded as a kind of real Grassmann manifolds of compact type with rank 2 (see Kobayashi and Nomizu [6]). Accordingly, the complex quadric admits both a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple (Q^m, J, g)

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is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [5] and Reckziegel [9]).

For the complex projective space $\mathbb{C}P^m$ a full classification was obtained by Okumura in [7]. He proved that the Reeb flow on a real hypersurface in $\mathbb{C}P^m = SU_{m+1}/S(U_mU_1)$ is isometric if and only if M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^m$ for some $k \in \{0, \dots, m-1\}$. For the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_mU_2)$ the classification was obtained by Berndt and Suh in [1]. The Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1}) \subset G_2(\mathbb{C}^{m+2})$. But, when we consider an isometric Reeb flow for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$, the result is quite different from $\mathbb{C}P^m$ and $G_2(\mathbb{C}^{m+2})$. In view of the previous two results in $\mathbb{C}P^m$ and $G_2(\mathbb{C}^{m+2})$ a natural expectation might be that the classification involves at least the totally geodesic $Q^{m-1} \subset Q^m$. But, surprisingly, in [2] Berndt and Suh have proved the following result:

Theorem 1.1. *Let M be a real hypersurface of the complex quadric Q^m, $m \geq 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*

In a paper due to Pérez, Santos and Suh [8], we have considered a notion of Lie ξ-parallel structure Jacobi operator, $\mathcal{L}_\xi R_\xi = 0$, for real hypersurfaces in complex projective space $\mathbb{C}P^m$, and in a paper [11], Suh has given a characterization of a tube of radius r, $0 < r < \frac{\pi}{2}$, over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ in terms of Lie ξ-parallel shape operator.

In this paper we consider a notion of *Lie parallel shape operator* S for real hypersurfaces in complex quadric Q^m along the direction of the Reeb vector field ξ, that is, $\mathcal{L}_\xi S = 0$. In this case the shape operator S of M in Q^m is said to be *Reeb invariant*. Motivated by the results mentioned above and using the notion of *isometric Reeb flow* in Theorem 1.1, we give a new characterization of real hypersurfaces in complex quadric Q^m with Reeb *invariant* shape operator as follows:

Main Theorem. *Let M be a real hypersurface in the complex quadric Q^m, $m \geq 3$ with Reeb invariant shape operator. Then $m = 2k$, and M is locally congruent to a tube over a totally geodesic complex projective space $\mathbb{C}P^k$ in Q^{2k}.*

2. The complex quadric

For more details in this section we refer to [3–6,9]. The complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_1^2 + \dots + z_{m+2}^2 = 0$, where $z_1, \dots, z_{m+2}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric which is induced from the Fubini Study metric on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J, g) on the complex quadric. For each $z \in Q^m$ we identify $T_z\mathbb{C}P^{m+1}$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus \mathbb{C}z$ of $\mathbb{C}z$ in $\mathbb{C}^{m+2}$ (see Kobayashi and Nomizu [6]). The tangent space T_zQ^m can then be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus (\mathbb{C}z \oplus \mathbb{C}N)$ of $\mathbb{C}z \oplus \mathbb{C}N$ in $\mathbb{C}^{m+2}$, where $N \in \nu_zQ^m$ is a normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point z.

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group SU_{m+2}, namely $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. We denote by $o = [0, \dots, 0, 1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing o is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of Q^m as a Hermitian symmetric space of rank 2. The complex quadric Q^1 is isometric to a sphere S^2 with constant curvature, and Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m \geq 3$ from now on.

For a unit normal vector N of Q^m at a point $z \in Q^m$ we denote by $A = A_N$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to N. The shape operator is an involution on the tangent space T_zQ^m and

$$T_zQ^m = V(A) \oplus JV(A),$$

where $V(A)$ is the $+1$-eigenspace and $JV(A)$ is the (-1)-eigenspace of A_N. Geometrically this means that the shape operator A_N defines a real structure on the complex vector space T_zQ^m, or equivalently, is a complex conjugation on T_zQ^m. Since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, this induces an S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$ consisting of complex conjugations.

There is a geometric interpretation of these conjugations. The complex quadric Q^m can be viewed as the complexification of the m-dimensional sphere S^m. Through each point $z \in Q^m$ there exists a one-parameter family of real forms of Q^m which are isometric to the sphere S^m. These real forms are congruent to each other under action of the center SO_2 of the isotropy subgroup of SO_{m+2} at z. The isometric reflection of Q^m in such a real form S^m is an isometry, and the differential at z of such a reflection is a conjugation on T_zQ^m. In this way the family $\mathfrak{A}$ of conjugations on T_zQ^m corresponds to the family of real forms S^m of Q^m containing z, and the subspaces $V(A) \subset T_zQ^m$ correspond to the tangent spaces T_zS^m of the real forms S^m of Q^m.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor R of Q^m can be described in terms of the complex structure J and the complex conjugation $A \in \mathfrak{A}$:

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY. \end{aligned}$$

Note that J and each complex conjugation A anti-commute, that is, $AJ = -JA$ for each $A \in \mathfrak{A}$.

Recall that a nonzero tangent vector $W \in T_zQ^m$ is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m:

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

For every unit tangent vector $W \in T_zQ^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. If $0 < t < \pi/4$ then the unique maximal flat containing W is $\mathbb{R}X \oplus \mathbb{R}JY$. Later we will need the eigenvalues and eigenspaces of the Jacobi operator $R_W = R(\cdot, W)W$ for a singular unit tangent vector W.

1. If W is an $\mathfrak{A}$-principal singular unit tangent vector with respect to $A \in \mathfrak{A}$, then the eigenvalues of R_W are 0 and 2 and the corresponding eigenspaces are $\mathbb{R}W \oplus J(V(A) \ominus \mathbb{R}W)$ and $(V(A) \ominus \mathbb{R}W) \oplus \mathbb{R}JW$, respectively.

2. If W is an $\mathfrak{A}$-isotropic singular unit tangent vector with respect to $A \in \mathfrak{A}$ and $X, Y \in V(A)$, then the eigenvalues of R_W are 0, 1 and 4 and the corresponding eigenspaces are $\mathbb{R}W \oplus \mathbb{C}(JX+Y)$, $T_zQ^m \ominus (\mathbb{C}X \oplus \mathbb{C}Y)$ and $\mathbb{R}JW$, respectively.

3. The totally geodesic $\mathbb{C}P^k \subset Q^{2k}$

We now assume that m is even, say $m = 2k$. The map

$$\mathbb{C}P^k \to Q^{2k} \subset \mathbb{C}P^{2k+1}, \quad [z_1, \dots, z_{k+1}] \mapsto [z_1, \dots, z_{k+1}, iz_1, \dots, iz_{k+1}]$$

provides an embedding of $\mathbb{C}P^k$ into Q^{2k} as a totally geodesic complex submanifold. We define a complex structure j on $\mathbb{C}^{2k+2}$ by

$$j(z_1, \dots, z_{k+1}, z_{k+2}, \dots, z_{2k+2}) = (-z_{k+2}, \dots, -z_{2k+2}, z_1, \dots, z_{k+1}).$$

Note that $ij = ji$. We can then identify $\mathbb{C}^{2k+2}$ with $\mathbb{C}^{k+1} \oplus j\mathbb{C}^{k+1}$ and get

$$T_z\mathbb{C}P^k = \{X + ijX \mid X \in \mathbb{C}^{k+1} \ominus \mathbb{C}z\}.$$

Now consider the standard embedding of U_{k+1} into SO_{2k+2} which is determined by the Lie algebra embedding

$$\mathfrak{u}_{k+1} \to \mathfrak{so}_{2k+2}, \quad C + iD \mapsto \begin{pmatrix} C & -D \\ D & C \end{pmatrix},$$

where $C, D \in M_{k+1,k+1}(\mathbb{R})$. The action of U_{k+1} on Q^{2k} is of cohomogeneity one and $\mathbb{C}P^k$ is the orbit of this action containing the point $z = [1, 0, \dots, 0, i, 0, \dots, 0] \in Q^{2k}$, where the i sits in the $(k+2)$-nd component. We now fix a unit normal vector N of Q^{2k} at z and denote the corresponding complex conjugation $A_N \in \mathfrak{A}$ by A. Then we can write alternatively

$$T_z\mathbb{C}P^k = \{X + ijX \mid X \in V(A)\}.$$

Note that the complex structure i on $\mathbb{C}^{2m+2}$ corresponds to the complex structure on T_zQ^{2m} via the obvious identifications.

We are now going to calculate the principal curvatures and principal curvature spaces of the tube with radius r around $\mathbb{C}P^k$ in Q^{2k}. For this we use the standard Jacobi field method as described in Section 8.2 of [4]. Let $N = (X + JY)/\sqrt{2}$ be a unit normal vector of $\mathbb{C}P^k$ in Q^{2k}, where $X, Y \in V(A)$ are orthonormal. The normal Jacobi operator R_N leaves the tangent space $T_z\mathbb{C}P^k$ and the normal space $\nu_z\mathbb{C}P^k$ invariant. When restricted to $T_z\mathbb{C}P^k$, the eigenvalues of R_N are 0 and 1 with corresponding eigenspaces $\mathbb{C}(JX+Y)$ and $T_z\mathbb{C}P^k \ominus \mathbb{C}(JX+Y)$. The corresponding principal curvatures on the tube of radius r are 0 and $\tan(r)$, and the corresponding principal curvature spaces are the parallel translates of $\mathbb{C}(JX+Y)$ and $T_z\mathbb{C}P^k \ominus \mathbb{C}(JX+Y)$ along the geodesic γ in Q^{2k} with $\gamma(0) = z$ and $\dot{\gamma}(0) = N$ from $\gamma(0)$ to $\gamma(r)$. We denote the latter parallel translate by W_1. When restricted to $\nu_z\mathbb{C}P^k \ominus \mathbb{R}N$, the eigenvalues of R_N are 1 and 4 with corresponding eigenspaces $\nu_z\mathbb{C}P^k \ominus \mathbb{C}N$ and $\mathbb{R}JN$. We denote the first parallel translate by W_2. The corresponding principal curvatures on the tube of radius r are $-\cot(r)$ and $-2\cot(2r)$, and the corresponding principal curvature spaces are the parallel translates of $\nu_z\mathbb{C}P^k \ominus \mathbb{C}N$ and $\mathbb{R}JN$ along γ from $\gamma(0)$ to $\gamma(r)$. This shows in particular that the tube is a Hopf hypersurface.

For $0 < r < \pi/2$ this process leads to real hypersurfaces in Q^{2k}, whereas for $r = \pi/2$ we get another totally geodesic $\mathbb{C}P^k \subset Q^{2k}$. The two totally geodesic complex projective spaces are precisely the two singular orbits of the U_{k+1}-action on Q^{2k}, and the tubes of radius $0 < r < \pi/2$ are the principal orbits of this action. Using the homogeneity of the tubes we can now conclude that the tube M of radius $0 < r < \pi/2$ has four distinct constant principal curvatures and the property that the shape operator leaves invariant the maximal complex subbundle $\mathcal{C}$ of TM. Moreover, all principal curvature spaces in $\mathcal{C}$ are J-invariant. Summing up all the properties mentioned above, we have the following

Proposition 3.1. *(See [3].) Let M be the tube of radius $0 < r < \pi/2$ around the totally geodesic $\mathbb{C}P^k$ in Q^{2k}. Then the following statements hold:*

1. *M is a Hopf hypersurface.*
2. *Every unit normal vector N of M is $\mathfrak{A}$-isotropic and therefore can be written in the form $N = (X + JY)/\sqrt{2}$ with some orthonormal vectors $X, Y \in V(A)$ and $A \in \mathfrak{A}$.*
3. *The principal curvatures and corresponding principal curvature spaces of M are*

principal curvature	*eigenspace*	*multiplicity*
0	$\mathbb{C}(JX + Y)$	2
$\tan(r)$	W_1	$2k - 2$
$-\cot(r)$	W_2	$2k - 2$
$-2\cot(2r)$	$\mathbb{R}JN$	1

4. *Each of the two focal sets of M is a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*
5. *The Reeb flow of M is an isometric flow.*
6. *The shape operator S and the structure tensor field ϕ satisfy $S\phi = \phi S$.*
7. *M is a homogeneous hypersurface of Q^{2k}. More precisely, it is an orbit of the U_{k+1}-action on Q^{2k} isomorphic to $U_{k+1}/U_{k-1}U_1$, an S^{2k-1}-bundle over $\mathbb{C}P^k$.*

4. Some general equations

Let M be a real hypersurface in Q^m and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

We now assume that M is a Hopf hypersurface. Then we have

$$S\xi = \alpha\xi$$

with the smooth function $\alpha = g(S\xi, \xi)$ on M. When we consider a transform JX of the Kähler structure J on Q^m for any vector field X on M in Q^m, we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal N to M. Then we now consider the Codazzi equation

$$\begin{aligned} g\big((\nabla_X S)Y - (\nabla_Y S)X, Z\big) = {} & \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ & + g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) \\ & + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \end{aligned}$$

Putting $Z = \xi$ we get

$$g\big((\nabla_X S)Y - (\nabla_Y S)X, \xi\big) = -2g(\phi X, Y) \\ + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi).$$

On the other hand, we have

$$g\big((\nabla_X S)Y - (\nabla_Y S)X, \xi\big) \\ = g\big((\nabla_X S)\xi, Y\big) - g\big((\nabla_Y S)\xi, X\big) \\ = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g\big((S\phi + \phi S)X, Y\big) - 2g(S\phi SX, Y).$$

Comparing the previous two equations and putting $X = \xi$ yields

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi).$$

Reinserting this into the previous equation yields

$$g\big((\nabla_X S)Y - (\nabla_Y S)X, \xi\big) \\ = -2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ + 2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X) \\ + \alpha g\big((\phi S + S\phi)X, Y\big) - 2g(S\phi SX, Y).$$

Altogether this implies

$$0 = 2g(S\phi SX, Y) - \alpha g\big((\phi S + S\phi)X, Y\big) - 2g(\phi X, Y) \\ + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ + 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ - 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X).$$

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \le t \le \frac{\pi}{4}$ (see Proposition 3 in [9]). Note that t is a function on M. First of all, since $\xi = -JN$, we have

$$N = \cos(t)Z_1 + \sin(t)JZ_2, \\ AN = \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi = \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi = \sin(t)Z_2 + \cos(t)JZ_1.$$

This implies $g(\xi, AN) = 0$ and hence

$$\begin{aligned} 0 &= 2g(S\phi SX, Y) - \alpha g\big((\phi S + S\phi)X, Y\big) - 2g(\phi X, Y) \\ &\quad + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ &\quad - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ &\quad - 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned}$$

5. Proof of Main Theorem

Before going to prove our Main Theorem, first let us show that the shape operator of M which becomes the tube of radius r over a complex projective space CP^k in Q^{2k} is Reeb invariant or not; that is, $\mathcal{L}_\xi S = 0$ or not. In fact, the Lie derivative vanishing along the Reeb vector field is given as follows:

$$\begin{aligned} (\mathcal{L}_\xi S)X &= \mathcal{L}_\xi(SX) - S\mathcal{L}_\xi X \\ &= (\nabla_\xi S)X - \phi S^2 X + S\phi SX \\ &= 0 \end{aligned}$$

for any vector field X on M in Q^{2k}. Then this is equivalent to the following

$$(\nabla_\xi S)X = \phi S^2 X - S\phi SX.$$

In order to do this, let us mention that the shape operator S of the tube over CP^k commutes with the structure tensor ϕ, that is, $S\phi = \phi S$ as in Proposition 3.1. So the right side of the above equation vanishes. Now let us check whether the left side $\nabla_\xi S = 0$ or not. Then by a paper due to Berndt and Suh (see [2], page 1350050-14), the expression of covariant derivative for the shape operator of M in complex quadric $\mathbb{Q}^m$ becomes

$$\begin{aligned} (\nabla_X S)Y &= \big\{d\alpha(X)\eta(Y) + g\big((\alpha S\phi - S^2\phi)X, Y\big) + \delta\eta(Y)\rho(X) \\ &\quad + \delta g(BX, \phi Y) + \eta(BX)\rho(Y)\big\}\xi \\ &\quad + \big\{\eta(Y)\rho(X) + g(BX, \phi Y)\big\}B\xi + g(BX, Y)\phi B\xi \\ &\quad - \rho(Y)BX - \eta(Y)\phi X - \eta(BY)\phi BX \end{aligned}$$

for any vector fields X and Y on M in Q^m, where we have put

$$AY = BY + \rho(Y)N, \quad \rho(Y) = g(AY, N)$$

for a complex conjugation $A \in \mathfrak{A}$. Putting $X = \xi$ and using α constant and $\rho(\xi) = 0$ for the $\mathfrak{A}$-isotropic unit normal vector field N of M in Q^{2k}, we have

$$\begin{aligned} (\nabla_\xi S)Y &= \delta g(B\xi, \phi Y) + \eta(B\xi)\rho(Y)\}\xi \\ &\quad + \big\{\eta(Y)\rho(\xi) + g(B\xi, \phi Y)\big\}B\xi + g(B\xi, Y)\phi B\xi \\ &\quad - \rho(Y)B\xi - \eta(Y)\phi\xi - \eta(BY)\phi B\xi \\ &= \big\{g(B\xi, \phi Y) - \rho(Y)\big\}B\xi \\ &= \big\{-g(\phi B\xi, Y) + g(Y, \phi B\xi)\big\}B\xi = 0, \end{aligned}$$

where in the third equality we have used

$$\begin{aligned}\rho(Y) &= g(AY, N) = g(Y, AN)\\ &= g(Y, AJ\xi)\\ &= -g(Y, JA\xi) = -g(Y, JB\xi)\\ &= -g(Y, \phi B\xi).\end{aligned}$$

As mentioned above, a real hypersurface M in $\mathbb{Q}^{2k}$ with commuting shape operator, that is, $S\phi = \phi S$, has a parallel shape operator along the Reeb direction $\nabla_\xi S = 0$. Accordingly, we know that the shape operator of the tube over CP^k is Reeb invariant, that is, $\mathcal{L}_\xi S = 0$.

Now conversely, let us prove our Main Theorem in the introduction. Let us assume $\mathcal{L}_\xi S = 0$.

On the other hand, the equation of Codazzi in Section 4 can be written as follows (see Berndt and Suh [2]):

$$\begin{aligned}(\nabla_X S)Y - (\nabla_Y S)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\\ &\quad + \rho(X)BY - \rho(Y)BX\\ &\quad + \eta(BX)\phi BY - \eta(BX)\rho(Y)\xi\\ &\quad - \eta(BY)\phi BX + \eta(BY)\rho(X)\xi,\end{aligned}$$

where we have put $AY = BY + \rho(Y)N$, and $\rho(X) = g(AX, N) = g(JX, A\xi)$ for a unit normal N of M in Q^m. Then we assert the following

Proposition 5.1. *Let M be a real hypersurface in complex quadrics Q^m, $m \geq 3$. If the Reeb flow on M satisfies $\mathcal{L}_\xi S = 0$, then the shape operator S commutes with the structure tensor ϕ, that is, $S\phi = \phi S$.*

Proof. First note that

$$\begin{aligned}(\mathcal{L}_\xi S)X &= \mathcal{L}_\xi(SX) - A\mathcal{L}_\xi X\\ &= \nabla_\xi(SX) - \nabla_{SX}\xi - S(\nabla_\xi X - \nabla_X \xi)\\ &= (\nabla_\xi S)X - \nabla_{SX}\xi + S\nabla_X\xi\\ &= (\nabla_\xi S)X - \phi S^2 X + S\phi SX\end{aligned}$$

for any vector field X on M. Then the assumption $\mathcal{L}_\xi S = 0$ holds if and only if $(\nabla_\xi S)X = \phi S^2 X - S\phi SX$. Now, let us take an orthonormal basis $\{e_1, e_2, \cdots, e_{2m-1}\}$ for the tangent space $T_x M$, $x \in M$, of a real hypersurface M in Q^m. Then by the equation of Codazzi we may put

$$\begin{aligned}(\nabla_{e_i} S)Y - (\nabla_Y S)e_i &= \eta(e_i)\phi Y - \eta(Y)\phi e_i - 2g(\phi e_i, Y)\xi\\ &\quad + \rho(e_i)BY - \rho(Y)Be_i + \eta(Be_i)\phi BY - \eta(BY)\rho(e_i)\xi\\ &\quad - \eta(BY)\phi Be_i + \eta(BY)\rho(e_i)\xi, \qquad (5.1)\end{aligned}$$

from which, together with the fundamental formulas which is introduced in Section 2 it follows that

$$\begin{aligned}\sum_{i=1}^{2m-1} g\big((\nabla_{e_i}S)Y,\phi e_i\big) = -(2m-2)\eta(Y) \\ &+ \sum_{i=1}^{2m-1}\rho(e_i)g(BY,\phi e_i) - \sum_{i=1}^{2m-1}\rho(Y)g(Be_i,\phi e_i) \\ &+ \sum_{i=1}^{2m-1}\eta(Be_i)g(\phi BY,\phi e_i) - \sum_{i=1}^{2m-1}\eta(BY)\rho(e_i)g(\xi,\phi e_i) \\ &- \eta(BY)\sum_{i=1}^{2m-1} g(\phi Be_i,\phi e_i). \end{aligned} \tag{5.2}$$

Now let us denote by U the vector $\nabla_\xi\xi = \phi S\xi$. Then using the equation $(\nabla_X\phi)Y = \eta(Y)AX - g(SX,Y)\xi$, its derivative can be given by

$$\begin{aligned}\nabla_{e_i}U &= (\nabla_{e_i}\phi)S\xi + \phi(\nabla_{e_i}S)\xi + \phi S\nabla_{e_i}\xi \\ &= \eta(S\xi)Se_i - g(Se_i,S\xi)\xi + \phi(\nabla_{e_i}S)\xi + \phi S\phi Se_i.\end{aligned}$$

Then its divergence is given by

$$\begin{aligned}\operatorname{div} U &= \sum_{i=1}^{2m-1} g(\nabla_{e_i}U,e_i) \\ &= h\eta(S\xi) - \eta\big(S^2\xi\big) - \sum_{i=1}^{2m-1} g\big((\nabla_{e_i}S)\xi,\phi e_i\big) - \sum_{i=1}^{2m-1} g(\phi Se_i,S\phi e_i),\end{aligned} \tag{5.3}$$

where h denotes the trace of the shape operator S of M in Q^m.

Now we calculate the squared norm of the tensor $S\phi - \phi S$ as follows:

$$\begin{aligned}\|\phi S - S\phi\|^2 &= \sum_i g\big((\phi S - S\phi)e_i,(\phi S - S\phi)e_i\big) \\ &= \sum_{i,j} g\big((\phi S - S\phi)e_i,e_j\big)g\big((\phi S - S\phi)e_i,e_j\big) \\ &= \sum_{i,j}\big\{g(\phi Se_j,e_i) + g(\phi Se_i,e_j)\big\}\big\{g(\phi Se_j,e_i) + g(\phi Se_i,e_j)\big\} \\ &= 2\sum_{i,j} g(\phi Se_j,e_i)g(\phi Se_j,e_i) + 2\textstyle\sum_{i,j} g(\phi Se_j,e_i)g(\phi Se_i,e_j) \\ &= 2\sum_j g(\phi Se_j,\phi Se_j) - 2\sum_j g(\phi Se_j,S\phi e_j) \\ &= -2\sum_j g\big(Se_j,\phi^2 Se_j\big) - 2\sum_j g(\phi Se_j,S\phi e_j) \\ &= 2\operatorname{Tr} S^2 - 2\eta\big(S^2\xi\big) + 2\operatorname{div} U - 2\eta(S\xi)\operatorname{Tr} S \\ &\quad + 2\eta\big(S^2\xi\big) + 2\sum_j g\big((\nabla_{e_j}S)\xi,\phi e_j\big),\end{aligned} \tag{5.4}$$

where $\sum_i$ (respectively, $\sum_{i,j}$) denotes the summation from $i = 1$ to $2m-1$ (respectively, from $i,j = 1$ to $2m-1$) and in the final equality we have used (5.3).

From this, together with the formula (5.2), it follows that

$$\begin{aligned}\operatorname{div} U = \frac{1}{2} \parallel \phi S - S\phi \parallel^2 - \operatorname{Tr} S^2 \\ + h\eta(S\xi) - \sum_i g\big((\nabla_{e_i} S)\xi, \phi e_i\big).\end{aligned} \tag{5.5}$$

By (5.5) and the assumption of $\mathcal{L}_\xi S = 0$, let us show that the structure tensor ϕ commutes with the shape operator S, that is, $\phi S = S\phi$.

On the other hand, we know that $\mathcal{L}_\xi S = 0$ is equivalent to

$$(\nabla_\xi S)X = \phi S^2 X - S\phi S X.$$

Then by the equation of Codazzi, we know that

$$\begin{aligned}(\nabla_X S)\xi &= (\nabla_\xi S)X - \phi X + \rho(X)B\xi - \rho(\xi)BX + \eta(BX)\phi B\xi \\ &\quad - \eta(BX)\rho(\xi)\xi - \eta(B\xi)\phi BX + \eta(B\xi)\rho(X)\xi \\ &= \phi S^2 X - S\phi S X - \phi X + \rho(X)B\xi \\ &\quad + \eta(BX)\phi B\xi - \eta(B\xi)\phi BX + \eta(B\xi)\rho(X)\xi.\end{aligned} \tag{5.6}$$

Now let us take an inner product (5.6) with the Reeb vector field ξ. Then we have

$$\begin{aligned}g\big((\nabla_X S)\xi, \xi\big) &= -g(S\phi S X, \xi) + \rho(X)g(B\xi, \xi) + \eta(B\xi)\rho(X) \\ &= g(SX, U) + 2g(B\xi, \xi)\rho(X).\end{aligned} \tag{5.7}$$

On the other hand, by the almost contact structure ϕ we have

$$\phi U = \phi^2 S\xi = -S\xi + \eta(S\xi)\xi = -S\xi + \alpha\xi,$$

where the function α denotes $\eta(S\xi)$. From this, differentiating and using the formula $(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi$ gives

$$\begin{aligned}-g(SX, U)\xi + \phi\nabla_X U &= (\nabla_X \phi)U + \phi\nabla_X U \\ &= -(\nabla_X S)\xi - S\nabla_X \xi + (X\alpha)\xi + \alpha\nabla_X \xi.\end{aligned} \tag{5.8}$$

From this, taking an inner product with ξ, it follows that

$$-g(SX, U) = -g\big((\nabla_X S)\xi, \xi\big) + g(SX, U) + X\alpha.$$

Then, together with (5.7), it follows that

$$g(SX, U) - 2g(B\xi, \xi)\rho(X) + X\alpha = 0. \tag{5.9}$$

Substituting (5.6) and (5.9) into (5.8), we have the following

$$\begin{aligned}\phi\nabla_X U &= \phi X - \phi S^2 X + \alpha\phi S X - \rho(X)B\xi \\ &\quad - \eta(BX)\phi B\xi + \eta(B\xi)\phi BX - \eta(B\xi)\rho(X)\xi.\end{aligned}$$

Now summing up with respect to $1, \cdots, m-1$ for an orthonormal basis of T_xM, $x \in M$, we have

$$\begin{aligned}\sum_i g(\phi\nabla_{e_i}U, \phi e_i) = (2m-2) - \{\operatorname{Tr} S^2 - \eta(S^2\xi)\} + \alpha\{\operatorname{Tr} S - \alpha\} \\ - \sum_i \rho(e_i)g(B\xi, \phi e_i) - \sum_i \eta(Be_i)g(\phi B\xi, \phi e_i) \\ + \sum_i \eta(B\xi)g(\phi Be_i, \phi e_i). \end{aligned} \tag{5.10}$$

On the other hand, we know that

$$\begin{aligned} g(\phi\nabla_{e_i}U, \phi e_i) &= -g(\nabla_{e_i}U, -e_i + \eta(e_i)\xi) \\ &= \sum_i g(\nabla_{e_i}U, e_i) - g(\nabla_\xi U, \xi) \\ &= \operatorname{div} U + \|U\|^2 \\ &= \operatorname{div} U + \eta(S^2\xi) - \eta(S\xi)^2, \end{aligned} \tag{5.11}$$

where $\|U\|^2 = g(\phi S\xi, \phi S\xi) = \eta(S^2\xi) - \eta(S\xi)^2$.

Summing up (5.10) with (5.11), we get the following

$$\begin{aligned}\operatorname{div} U = (2m-2) - \operatorname{Tr} S^2 + \alpha \operatorname{Tr} S \\ - \sum_i \rho(e_i)g(B\xi, \phi e_i) - \sum_i \eta(Be_i)g(\phi B\xi, \phi e_i) \\ + \sum_i \eta(B\xi)g(\phi Be_i, \phi e_i), \end{aligned} \tag{5.12}$$

where we have denoted the function $\eta(S\xi)$ by $\alpha = \eta(S\xi)$.

On the other hand, (5.2) for $Y = \xi$ gives the following

$$\begin{aligned}\sum_i g((\nabla_{e_i}S)\xi, \phi e_i) = -(2m-2) \\ + \sum_i \rho(e_i)g(B\xi, \phi e_i) \\ + \sum_i \eta(Be_i)g(\phi B\xi, \phi e_i) - \eta(B\xi)\sum_i g(\phi Be_i, \phi e_i), \end{aligned} \tag{5.13}$$

where we have used $\sum_i \rho(\xi)g(Be_i, \phi e_i) = 0$ and $\sum_i \eta(B\xi)\rho(e_i)g(\xi, \phi e_i) = 0$. From this, together with (5.5), (5.10) and (5.13), it follows that

$$\begin{aligned}\operatorname{div} U &= \frac{1}{2}\parallel \phi S - S\phi \parallel^2 - \operatorname{Tr} S^2 + h\eta(S\xi) - \sum_i g((\nabla_{e_i}S)\xi, \phi e_i) \\ &= \frac{1}{2}\parallel \phi S - S\phi \parallel^2 - \operatorname{Tr} S^2 + h\eta(S\xi) \\ &\quad + (2m-2) - \sum_i \rho(e_i)g(B\xi, \phi e_i) - \sum_i \eta(Be_i)g(\phi B\xi, \phi e_i) \\ &\quad + \eta(B\xi)\sum_i g(\phi Be_i, \phi e_i) \end{aligned}$$

$$
\begin{aligned}
&= (2m-2) - \operatorname{Tr} S^2 + h\eta(S\xi) \\
&\quad - \sum_i \rho(e_i) g(B\xi, \phi e_i) - \sum_i \eta(Be_i) g(\phi B\xi, \phi e_i) \\
&\quad + \sum_i \eta(B\xi) g(\phi Be_i, \phi e_i),
\end{aligned} \tag{5.14}
$$

where we have used (5.13) in the second equality and (5.12) in the third equality respectively. Then comparing the both sides in the third equality of (5.14) gives $\|\phi S - S\phi\|^2 = 0$, that is, the shape operator S commutes with the structure tensor ϕ. This means that the Reeb flow on M is an isometric flow, which gives a complete proof of our Proposition. □

By virtue of this proposition, together with Theorem 1.1, we give a complete proof of our Main Theorem in the introduction.

References

[1] J. Berndt, Y.J. Suh, Real hypersurfaces with isometric Reeb flow in complex two-plane Grassmannians, Monatshefte Math. 137 (2002) 87–98.
[2] J. Berndt, Y.J. Suh, Hypersurfaces in noncompact complex Grassmannians of rank two, Int. J. Math. 23 (2012) 1250103, 35 pp.
[3] J. Berndt, Y.J. Suh, Real hypersurfaces with isometric Reeb flow in complex quadrics, Int. J. Math. 24 (2013) 1350050, 18 pp.
[4] J. Berndt, S. Console, C. Olmos, Submanifolds and Holonomy, Chapman & Hall/CRC, Boca Raton, FL, 2003.
[5] S. Klein, Totally geodesic submanifolds in the complex quadric, Differ. Geom. Appl. 26 (2008) 79–96.
[6] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vol. II, A Wiley-Interscience Publ., Wiley Classics Library Ed., 1996.
[7] M. Okumura, On some real hypersurfaces of a complex projective space, Trans. Am. Math. Soc. 212 (1975) 355–364.
[8] J.D. Pérez, F.G. Santos, Y.J. Suh, Real hypersurfaces in complex projective space whose structure Jacobi operator is Lie ξ-parallel, Differ. Geom. Appl. 22 (2005) 181–188.
[9] H. Reckziegel, On the geometry of the complex quadric, in: Geometry and Topology of Submanifolds VIII, Brussels/Nordfjordeid, 1995, World Sci. Publ., River Edge, NJ, 1995, pp. 302–315.
[10] B. Smyth, Differential geometry of complex hypersurfaces, Ann. Math. 85 (1967) 246–266.
[11] Y.J. Suh, Real hypersurfaces in complex two-plane Grassmannians with vanishing Lie derivatives, Can. Math. Bull. 49 (2006) 131–143.
[12] Y.J. Suh, Real hypersurfaces with isometric Reeb flow in complex hyperbolic two-plane Grassmannians, Adv. Appl. Math. 50 (2013) 645–659.
[13] Y.J. Suh, Real hypersurfaces in complex hyperbolic two-plane Grassmannians with Reeb vector field, Adv. Appl. Math. 55 (2014) 131–145.