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Addendum

Addendum to the paper "Real hypersurfaces in complex two-plane Grassmannians with ξ-invariant Ricci tensor"

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ABSTRACT

In this paper, we introduce some notions of invariancy for the Ricci tensor on real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, namely, $\mathcal{F}$-invariant and invariant Ricci tensor. Using these notions, we give non-existence theorem and characterization for the special case among the real hypersurfaces of Type (A) in $G_2(\mathbb{C}^{m+2})$, respectively. Here the distribution $\mathcal{F}$ is defined by $\mathcal{F} = [\xi] \cup \mathfrak{D}^\perp$ where $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ and $[\xi] = \text{Span}\{\xi\}$.

0. Introduction

In [1] Suh proved the following theorem:

Theorem A. *Let M be a connected orientable Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with ξ-invariant Ricci tensor. Then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

From this theorem we can obtain corollaries whose proof is not trivial from the theorem. It needs some calculations that we present in this addendum. Consider the distribution $\mathcal{F} = \text{Span}\{\xi\} \cup \mathfrak{D}^\perp$ on M. We will prove:

Corollary 1. *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then its Ricci tensor is $\mathcal{F}$-invariant if and only if M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ satisfying $\tan^2(\sqrt{2}r) = 2/(m-1)$ and whose one-form q_1 vanishes on the distribution $\mathfrak{D}$.*

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Moreover, as a consequence of this result we obtain:

Corollary 2. *There does not exist a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Lie-vanishing (or invariant) Ricci tensor.*

We use some references [2–4,1] to recall the Riemannian geometry of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and some fundamental formulas including the Codazzi and Gauss equations for a real hypersurface in $G_2(\mathbb{C}^{m+2})$. In particular, the formula for the Ricci tensor S and its covariant derivative ∇S was shown explicitly in [5–7]. In Sections 1 and 2 respectively we will give a proof of Corollaries 1 and 2.

1. The proof of Corollary 1

From [1] a new condition, namely the $\mathcal{F}$-invariant Ricci tensor, yields M is locally congruent to an open part of a type (A) real hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$. Now let us take $\xi = \xi_1$ (which is possible for a type (A) real hypersurface M_A). Then it follows that

$$q_2(Y) = 2g(AY, \xi_2), \qquad q_3(Y) = 2g(AY, \xi_3) \tag{1.1}$$

for any tangent vector field Y on M_A. From this, the covariant derivatives of ξ_2 and ξ_3 with respect to an arbitrary tangent vector field $Y \in TM_A$ are

$$\begin{aligned} \nabla_Y \xi_2 &= q_1(Y)\xi_3 - q_3(Y)\xi_1 + \phi_2 AY \\ &= q_1(Y)\xi_3 - 2g(AY, \xi_3)\xi_1 + \phi_2 AY \end{aligned} \tag{1.2-(i)}$$

and

$$\nabla_Y \xi_3 = 2g(AY, \xi_2)\xi_1 - q_1(Y)\xi_2 + \phi_3 AY, \text{ respectively.} \tag{1.2-(ii)}$$

On the other hand, from the equation of Ricci tensor S (see (3.12) in [7]) the Ricci tensor S of M_A is given as follows:

$$SY = \kappa Y \quad \text{where } \kappa = \begin{cases} 4m + h\alpha - \alpha^2 \ (:=\delta) & \text{if } Y \in T_\alpha \\ 4m + 6 + h\beta - \beta^2 \ (:=\rho) & \text{if } Y \in T_\beta \\ 4m + 6 + h\lambda - \lambda^2 \ (:=\sigma) & \text{if } Y \in T_\lambda \\ 4m + 8 \ (:=\tau) & \text{if } Y \in T_\mu, \end{cases} \tag{1.3}$$

where the tangent bundle TM_A is composed of four eigenspaces T_α, T_β, T_λ and T_μ, that is, $TM_A = T_\alpha \oplus T_\beta \oplus T_\lambda \oplus T_\mu$ (see Proposition B in [7]). Since κ is constant, we obtain

$$\begin{aligned} (\mathcal{L}_{\xi_2} S)Y &= \mathcal{L}_{\xi_2}(SY) - S(\mathcal{L}_{\xi_2} Y) \\ &= [\xi_2, SY] - S([\xi_2, Y]) \\ &= \nabla_{\xi_2}(SY) - \nabla_{SY}\xi_2 - S(\nabla_{\xi_2} Y) + S(\nabla_Y \xi_2) \\ &= \kappa(\nabla_{\xi_2} Y - \nabla_Y \xi_2) - S(\nabla_{\xi_2} Y - \nabla_Y \xi_2) \end{aligned} \tag{1.4}$$

for all $Y \in T_x M_A$ and $x \in M_A$. Using (1.3), let us calculate (1.4) with respect to the eigenspaces T_α, T_β, T_λ and T_μ on $T_x M_A$.

Case A-I. $Y \in T_\alpha$, that is, $Y = \xi (=\xi_1)$.

From (1.2-(i)) and (1.3), Eq. (1.4) becomes

$$\begin{aligned} (\mathcal{L}_{\xi_2} S)\xi &= \kappa(\nabla_{\xi_2}\xi - \nabla_\xi \xi_2) - S(\nabla_{\xi_2}\xi - \nabla_\xi \xi_2) \\ &= \delta\big(\phi A\xi_2 - (q_1(\xi) - \alpha)\xi_3\big) - S\big(\phi A\xi_2 - (q_1(\xi) - \alpha)\xi_3\big). \end{aligned}$$

Since $A\xi_2 = \beta\xi_2$ and $S\xi_3 = \rho\xi_3$, it follows that

$$\begin{aligned} (\mathcal{L}_{\xi_2} S)\xi &= -\delta\beta\xi_3 - \delta q_1(\xi)\xi_3 + \delta\alpha\xi_3 + \beta\rho\xi_2 + \rho q_1(\xi)\xi_3 - \alpha\rho\xi_3 \\ &= (\rho - \delta)(\beta + q_1(\xi) - \alpha)\xi_3. \end{aligned} \tag{1.5}$$

Case A-II. $Y \in T_\beta$, that is, $Y = \xi_2$ or $Y = \xi_3$.

If $Y = \xi_2$, then $[\xi_2, S\xi_2] = \rho[\xi_2, \xi_2] = 0$. Thus we have

$$(\mathcal{L}_{\xi_2} S)\xi_2 = 0. \tag{1.6-(i)}$$

Now, put $Y = \xi_3$. Since $\nabla_{\xi_2}\xi_3 - \nabla_{\xi_3}\xi_2 = 2\beta\xi_1 - q_1(\xi_2)\xi_2 - q_1(\xi_3)\xi_3$ from (1.2-(i)) to (1.2-(ii)), Eq. (1.4) can be written as

$$\begin{aligned} (\mathcal{L}_{\xi_2} S)\xi_3 &= \rho(\nabla_{\xi_2}\xi_3 - \nabla_{\xi_3}\xi_2) - S(\nabla_{\xi_2}\xi_3 - \nabla_{\xi_3}\xi_2) \\ &= \rho\big(2\beta\xi_1 - q_1(\xi_2)\xi_2 - q_1(\xi_3)\xi_3\big) - S\big(2\beta\xi_1 - q_1(\xi_2)\xi_2 - q_1(\xi_3)\xi_3\big) \\ &= 2\beta(\rho\xi_1 - S\xi_1) - q_1(\xi_2)(\rho\xi_2 - S\xi_2) - q_1(\xi_3)(\rho\xi_3 - S\xi_3) \\ &= 2\beta(\rho - \delta)\xi_1, \end{aligned} \tag{1.6-(ii)}$$

where $S\xi_1 = \delta\xi_1$, $\delta = (4m + h\alpha - \alpha^2)$ and $S\xi_\nu = \rho\xi_\nu$, $\rho = (4m + 6 + h\beta - \beta^2)$ for $\nu = 2, 3$.

Case A-III. $Y \in T_\lambda$, where $T_\lambda = \{Y \in \mathfrak{D} \mid \phi Y = \phi_1 Y\}$.

By (1.2-(i)) we obtain $\nabla_Y \xi_2 = q_1(Y)\xi_3 + \lambda\phi_2 Y$. Moreover, we see that if any vector field Y belongs to the eigenspace T_λ, then the vector field $\phi_2 Y$ belongs to T_μ, that is, it satisfies the properties $g(\phi_2 Y, \xi_\iota) = 0$ and $\phi\phi_2 Y = -\phi_1\phi_2 Y$ for $Y \in T_\lambda$ and $\iota = 1, 2, 3$. It implies that $S\phi_2 Y = \tau\phi_2 Y$ where $\tau = (4m + 8)$.

From these facts, we get

$$\begin{aligned}(\mathcal{L}_{\xi_2} S)Y &= \sigma(\nabla_{\xi_2} Y - \nabla_Y \xi_2) - S(\nabla_{\xi_2} Y - \nabla_Y \xi_2)\\ &= \sigma(\nabla_{\xi_2} Y - q_1(Y)\xi_3 - \lambda\phi_2 Y) - S(\nabla_{\xi_2} Y - q_1(Y)\xi_3 - \lambda\phi_2 Y)\end{aligned} \tag{1.7}$$

for any tangent vector $Y \in T_\lambda$.

On the other hand, for any $Y \in \mathfrak{D}$ the vector field $\nabla_{\xi_2} Y$ belongs to the distribution $\mathfrak{D}$, because

$$\begin{aligned}g(\nabla_{\xi_2} Y, \xi_\iota) &= -g(Y, \nabla_{\xi_2} \xi_\iota)\\ &= -g(Y, q_{\iota+2}(\xi_2)\xi_{\iota+1} - q_{\iota+1}(\xi_2)\xi_{\iota+2} + \phi_\iota A\xi_2)\\ &= -g(Y, q_{\iota+2}(\xi_2)\xi_{\iota+1} - q_{\iota+1}(\xi_2)\xi_{\iota+2} + \beta\phi_\iota \xi_2)\\ &= 0\end{aligned}$$

for any $\iota = 1, 2, 3$. By virtue of Proposition B given in [7], we know that the distribution $\mathfrak{D}$ of $T_x M_A$ at $x \in M_A$ is composed of two eigenspaces T_λ and T_μ, that is, $\mathfrak{D} = T_\lambda \oplus T_\mu$. Thus there exist unique $U \in T_\lambda$ and $W \in T_\mu$ such that

$$\begin{aligned}\nabla_{\xi_2} Y &= U + W\\ &= \sum_{i=1}^{2m-2} a_i^\lambda e_i^\lambda + \sum_{j=1}^{2m-2} a_j^\mu e_j^\mu\end{aligned}$$

for an orthonormal basis $\{e_i^\lambda,\ e_j^\mu \mid i, j = 1, 2, \dots, 2m-2\}$ of $\mathfrak{D}$. In general, from this we can consider the following three subcases:

- **Subcase 1.** $\nabla_{\xi_2} Y \in T_\lambda$, that is, $a_j^\mu = 0$ for any $j = 1, 2, \dots, 2m-2$,
- **Subcase 2.** $\nabla_{\xi_2} Y \in T_\mu$, that is, $a_i^\mu = 0$ for any $i = 1, 2, \dots, 2m-2$,
- **Subcase 3.** $\nabla_{\xi_2} Y \in T_\lambda + T_\mu$, that is, there exist some non-vanishing components a_s^λ and a_t^μ for some $s, t = 1, 2, \dots, 2m-2$, given as $\nabla_{\xi_2} Y = a_s^\lambda e_s^\lambda + a_t^\mu e_t^\mu$.

By the way, if $Y \in T_\lambda$, then $\phi Y = \phi_1 Y$. Differentiating this equation along the direction ξ_2, we have

$$\phi(\nabla_{\xi_2} Y) = -2\beta\phi_3 Y + \phi_1(\nabla_{\xi_2} Y),$$

together with (1.1). For a model space of Type (A) in $G_2(\mathbb{C}^{m+2})$ the eigenvalue β is non-vanishing. Thus we see that $\nabla_{\xi_2} Y$ does not belong to T_λ, and hence we must consider the following two subcases in the above ones:

Firstly, let us consider that $\nabla_{\xi_2} Y \in T_\mu$. That is, we can put $\nabla_{\xi_2} Y = \sum_{j=1}^{2m-2} a_j^\mu e_j^\mu$ for an orthonormal basis $\{e_j^\mu\}$ of T_μ. It follows that

$$\begin{aligned}S(\nabla_{\xi_2} Y) &= S\left(\sum_{j=1}^{2m-2} a_j^\mu e_j^\mu\right) = \sum_{j=1}^{2m-2} a_j^\mu (S e_j^\mu)\\ &= \sum_{j=1}^{2m-2} a_j^\mu (\tau e_j^\mu) = \tau \sum_{j=1}^{2m-2} a_j^\mu e_j^\mu = \tau\nabla_{\xi_2} Y\end{aligned}$$

where $\tau = (4m + 8)$. Hence for the case $Y \in T_\lambda$ and $\nabla_{\xi_2} Y \in T_\mu$ Eq. (1.7) becomes

$$\begin{aligned}(\mathcal{L}_{\xi_2} S)Y &= \sigma(\nabla_{\xi_2} Y - q_1(Y)\xi_3 - \lambda\phi_2 Y) - S(\nabla_{\xi_2} Y - q_1(Y)\xi_3 - \lambda\phi_2 Y)\\ &= (\sigma - \tau)\nabla_{\xi_2} Y + q_1(Y)(\rho - \sigma)\xi_3 + \lambda(\tau - \sigma)\phi_2 Y\end{aligned} \tag{1.8-(i)}$$

where $\sigma = (4m + 6 + h\lambda - \lambda^2)$, $\rho = (4m + 6 + h\beta - \beta^2)$ and $\tau = (4m + 8)$.

Secondly, let us find the formula related to $(\mathcal{L}_{\xi_2} S)Y$ with respect to $Y \in T_\lambda$ and $\nabla_{\xi_2} Y \in T_\lambda + T_\mu$. For the sake of convenience we may put $\nabla_{\xi_2} Y = a_s^\lambda e_s^\lambda + a_t^\mu e_t^\mu$ satisfying $a_s^\lambda a_t^\mu \neq 0$ for some $s, t = 1, 2, \dots, 2m-2$. From this notation, (1.7) can be changed into

$$\begin{aligned}(\mathcal{L}_{\xi_2} S)Y &= \sigma(\nabla_{\xi_2} Y - q_1(Y)\xi_3 - \lambda\phi_2 Y) - S(\nabla_{\xi_2} Y - q_1(Y)\xi_3 - \lambda\phi_2 Y)\\ &= \sigma(a_s^\lambda e_s^\lambda + a_t^\mu e_t^\mu - q_1(Y)\xi_3 - \lambda\phi_2 Y)\\ &\quad - S(a_s^\lambda e_s^\lambda + a_t^\mu e_t^\mu - q_1(Y)\xi_3 - \lambda\phi_2 Y)\\ &= (\sigma - \tau)a_t^\mu e_t^\mu - q_1(Y)(\sigma - \rho)\xi_3 - \lambda(\sigma - \tau)\phi_2 Y,\end{aligned} \tag{1.8-(ii)}$$

because $S e_s^\lambda = \sigma e_s^\lambda$ and $S e_t^\mu = \tau e_t^\mu$.

Case A-IV. $Y \in T_\mu$, where $T_\mu = \{Y \in \mathfrak{D} \mid \phi Y = -\phi_1 Y\}$.
Since $\mu = 0$, Eq. (1.2-(i)) implies that $\nabla_Y \xi_2 = q_1(Y)\xi_3$. From (1.3) and (1.4), we have

$$\begin{aligned}(\mathcal{L}_{\xi_2}S)Y &= \tau(\nabla_{\xi_2}Y - \nabla_Y\xi_2) - S(\nabla_{\xi_2}Y - \nabla_Y\xi_2)\\ &= \tau(\nabla_{\xi_2}Y - q_1(Y)\xi_3) - S(\nabla_{\xi_2}Y - q_1(Y)\xi_3)\\ &= \tau(\nabla_{\xi_2}Y) - q_1(Y)(\tau-\rho)\xi_3 - S(\nabla_{\xi_2}Y)\end{aligned} \tag{1.9}$$

where $\rho = (4m + 6 + h\beta - \beta^2)$ and $\tau = (4m + 8)$.

As mentioned in the Case A-III, we know that $\nabla_{\xi_2}Y$ belongs to the distribution $\mathfrak{D} = T_\lambda \oplus T_\mu$ for $Y \in \mathfrak{D}$. In addition, since $Y \in T_\mu$, we get $\phi(\nabla_{\xi_2}Y) = 2\beta\phi_3 Y - \phi_1(\nabla_{\xi_2}Y)$ from the property $\phi Y = -\phi_1 Y$. It means that the vector field $\nabla_{\xi_2}Y$ does not belong to the eigenspace T_μ. Thus we have only two subcases given by $\nabla_{\xi_2}Y \in T_\lambda$ and $\nabla_{\xi_2}Y \in T_\lambda + T_\mu$. For such subcases Eq. (1.9) becomes respectively

$$(\mathcal{L}_{\xi_2}S)Y = (\tau-\sigma)(\nabla_{\xi_2}Y) - q_1(Y)(\tau-\rho)\xi_3 \quad \text{for } \nabla_{\xi_2}Y \in T_\lambda \tag{1.10-(i)}$$

and

$$(\mathcal{L}_{\xi_2}S)Y = (\tau-\sigma)a_s^\lambda e_s^\lambda - q_1(Y)(\tau-\rho)\xi_3 \quad \text{for } \nabla_{\xi_2}Y \in T_\lambda + T_\mu \tag{1.10-(ii)}$$

where we put $\nabla_{\xi_2}Y = a_s^\lambda e_s^\lambda + a_t^\mu e_t^\mu \in T_\lambda + T_\mu$ satisfying $a_s^\lambda a_t^\mu \neq 0$ for some $s, t = 1, 2, \ldots, 2m-2$.

Summing up these calculations, we have the following equation for $X = \xi_2$.

$$(\mathcal{L}_{\xi_2}S)Y = \begin{cases}(\rho-\delta)(\beta + q_1(\xi) - \alpha)\xi_3 & \text{if } Y = \xi \in T_\alpha\\ 0 & \text{if } Y = \xi_2 \in T_\beta\\ 2\beta(\rho-\delta)\xi_1 & \text{if } Y = \xi_3 \in T_\beta\\ (\sigma-\tau)(\nabla_{\xi_2}Y) - q_1(Y)(\sigma-\rho)\xi_3 - \lambda(\sigma-\tau)\phi_2 Y & \text{if } Y \in T_\lambda,\ \nabla_{\xi_2}Y \in T_\mu\\ (\sigma-\tau)a_t^\mu e_t^\mu - q_1(Y)(\sigma-\rho)\xi_3 - \lambda(\sigma-\tau)\phi_2 Y & \text{if } Y \in T_\lambda,\ \nabla_{\xi_2}Y \in T_\lambda + T_\mu\\ -(\sigma-\tau)(\nabla_{\xi_2}Y) - q_1(Y)(\tau-\rho)\xi_3 & \text{if } Y \in T_\mu,\ \nabla_{\xi_2}Y \in T_\lambda\\ (\tau-\sigma)a_s^\lambda e_s^\lambda - q_1(Y)(\tau-\rho)\xi_3 & \text{if } \nabla_{\xi_2}Y \in T_\lambda + T_\mu.\end{cases} \tag{1.11}$$

Moreover, by the similar consideration for $X = \xi_3$ we obtain also:

$$(\mathcal{L}_{\xi_3}S)Y = \begin{cases}-(\rho-\delta)(\beta + q_1(\xi) - \alpha)\xi_2 & \text{if } Y = \xi \in T_\alpha\\ -2\beta(\rho-\delta)\xi_1 & \text{if } Y = \xi_2 \in T_\beta\\ 0 & \text{if } Y = \xi_3 \in T_\beta\\ (\sigma-\tau)(\nabla_{\xi_3}Y) + q_1(Y)(\sigma-\rho)\xi_2 - \lambda(\sigma-\tau)\phi_3 Y & \text{if } Y \in T_\lambda,\ \nabla_{\xi_3}Y \in T_\mu\\ (\sigma-\tau)a_t^\mu e_t^\mu + q_1(Y)(\sigma-\rho)\xi_2 - \lambda(\sigma-\tau)\phi_3 Y & \text{if } Y \in T_\lambda,\ \nabla_{\xi_3}Y \in T_\lambda + T_\mu\\ -(\sigma-\tau)(\nabla_{\xi_3}Y) + q_1(Y)(\tau-\rho)\xi_2 & \text{if } Y \in T_\mu,\ \nabla_{\xi_3}Y \in T_\lambda\\ -(\sigma-\tau)a_s^\lambda e_s^\lambda + q_1(Y)(\tau-\rho)\xi_2 & \text{if } Y \in T_\mu,\ \nabla_{\xi_3}Y \in T_\lambda + T_\mu.\end{cases} \tag{1.12}$$

For the Ricci tensor S of M_A, the definition of $\mathcal{F}$-invariant Ricci tensor gives that the vector fields $(\mathcal{L}_{\xi_\nu}S)Y$ $(\nu = 1, 2, 3)$ must vanish for each case mentioned above. So, from (1.6-(ii)), we have

$$\rho - \delta = 0, \tag{1.13}$$

together with $\beta = \sqrt{2}\cot(\sqrt{2}r) \neq 0$ for the radius $r \in (0, \pi/\sqrt{8})$. Moreover, it follows

$$\tan^2(\sqrt{2}r) = \frac{2}{m-1}, \tag{1.14}$$

because $h = \mathrm{Tr}A = \alpha + 2\beta + (2m-2)\lambda$ and $m \geq 3$. In addition, by using (1.3) it follows that

$$\sigma - \tau = -2 + h\lambda - \lambda^2 = -8 + 4(m-1)\tan^2(\sqrt{2}r) = 0$$

and

$$\sigma - \rho = 2 - h\beta + \beta^2 = 4m - 4\cot^2(\sqrt{2}r) = 2(m+1) = \tau - \rho.$$

From these and Eqs. (1.11) and (1.12), we obtain

$$\begin{cases}(\mathcal{L}_{\xi_2}S)Y = -2(m+1)q_1(Y)\xi_3,\\ (\mathcal{L}_{\xi_3}S)Y = 2(m+1)q_1(Y)\xi_2\end{cases} \tag{1.15}$$

for any tangent vector field $Y \in \mathfrak{D}$. Since $m \geq 3$, the one form q_1 must vanish for $Y \in \mathfrak{D}$, that is, $q_1(Y) = 0$.

It gives us a complete proof of our Corollary 1 in the introduction. □

2. The proof of Corollary 2

We only have to check if real hypersurfaces appearing in our Corollary 1 do or do not satisfy the condition, $(\mathcal{L}_X S)Y = 0$ for any tangent $X, Y \in TM$. In order to do this, putting $X \in T_\mu$ and $Y = \xi_3 \in T_\beta$, it implies

$$(\mathcal{L}_X S)\xi_3 = \rho(\nabla_X \xi_3 - \nabla_{\xi_3} X) - S(\nabla_X \xi_3 - \nabla_{\xi_3} X) = 0$$

by Eq. (1.3) in Section 1. On the other hand, from (1.1), (1.2-(ii)) and $q_1(X) = 0$ for any $X \in \mathfrak{D}$ it follows that the vector field $\nabla_X \xi_3 = \phi_3 AX = 0$ for any $X \in T_\mu$, $\mu = 0$. Thus we obtain

$$(\mathcal{L}_X S)\xi_3 = -\rho \nabla_{\xi_3} X + S(\nabla_{\xi_3} X) = 0. \tag{2.1}$$

On the other hand, since $g(\nabla_{\xi_3} X, \xi_\iota) = 0$ for any $X \in T_\mu$ and $\iota = 1, 2, 3$, we see that $\nabla_{\xi_3} X \in \mathfrak{D} = T_\lambda \oplus T_\mu$. It is well known that $\phi X = -\phi_1 X$ for $X \in T_\mu$. Taking the covariant derivative along the direction ξ_3, we obtain

$$\phi(\nabla_{\xi_3} X) = -2\beta\phi_2 X - \phi_1(\nabla_{\xi_3} X),$$

together with the basic formulas (see Section 2 in [1]) and (1.1). It follows that for any $X \in T_\mu$ the vector field $\nabla_{\xi_3} X$ does not belong to T_μ, because β is non-vanishing. Therefore for the vector field $\nabla_{\xi_3} X$ we have the following two subcases:

- **Subcase 1.** $\nabla_{\xi_3} Y \in T_\lambda$, that is, $\nabla_{\xi_3} Y = \sum_{i=1}^{2m-2} a_i^\lambda e_i^\lambda$ for an orthonormal basis $\{e_i^\lambda\}$ of T_λ,
- **Subcase 2.** $\nabla_{\xi_3} Y \in T_\lambda + T_\mu$, that is, there exist some non-vanishing components a_s^λ and a_t^μ for some $s, t = 1, 2, \ldots, 2m-2$, given as follows:

$$\nabla_{\xi_3} Y = a_s^\lambda e_s^\lambda + a_t^\mu e_t^\mu.$$

From this, we first assume that $\nabla_{\xi_3} Y \in T_\lambda$. Then Eq. (2.1) becomes

$$-\rho \sum_{i=1}^{2m-2} a_i^\lambda e_i^\lambda + S\Big(\sum_{i=1}^{2m-2} a_i^\lambda e_i^\lambda\Big) = (\sigma - \rho) \sum_{i=1}^{2m-2} a_i^\lambda e_i^\lambda = (\sigma - \rho)\nabla_{\xi_3} X = 0, \tag{2.2}$$

where $Se_i^\lambda = \sigma e_i^\lambda$ for $i = 1, 2, \ldots, 2m-2$. By our assumption that M_A has the invariant Ricci tensor, we already knew that $\tan^2(\sqrt{2}r) = 2/(m-1)$. From this and $m \geq 3$, $(\sigma - \rho)$ is non-vanishing. So, Eq. (2.2) implies that

$$\nabla_{\xi_3} X = 0. \tag{2.3}$$

Since $AX = 0$ for any $X \in T_\mu$, we get $(\nabla_{\xi_3} A)X = 0$, together with (2.3). From this, the Codazzi equation given in [1] becomes

$$-(\nabla_X A)\xi_3 = 2\phi_3 X,$$

where we have used $\phi X = -\phi_1 X$ for any $X \in T_\mu$. On the other hand, for $X \in T_\mu$ we get $(\nabla_X A)\xi_3 = 0$ by using $A\xi_3 = \beta\xi_3$ and $\nabla_X \xi_3 = 0$. Hence we have $\phi_3 X = 0$ from the previous two equations. It follows that $X = 0$ for any $X \in T_\mu$, that is, $\dim T_\mu = 0$. It gives us a contradiction. In fact, the dimension of the eigenspace T_μ is $2m-2$.

Next we consider the Subcase 2 mentioned above. For the sake of convenience we may put $\nabla_{\xi_3} Y = a_s^\lambda e_s^\lambda + a_t^\mu e_t^\mu$ such that $a_s^\lambda a_t^\mu \neq 0$ for some $s, t = 1, 2, \ldots, 2m-2$.

Then Eq. (2.1) becomes

$$a_s^\lambda(\sigma - \rho)e_s^\lambda + (\tau - \rho)a_t^\mu e_t^\mu = 0.$$

Taking the inner product with $e_s^\lambda \in T_\lambda$ to this equation, we have $(\sigma - \rho) = 0$, together with $a_s^\lambda \neq 0$ for some $s = 1, 2, \ldots, 2m-2$. It gives us a contradiction with $m \geq 3$. In fact, by virtue of Proposition B in [7] we get $(\sigma-\rho) = 2(m+1)$ where $h = \mathrm{Tr}A = \alpha + 2\beta + (2m-2)\lambda$.

Hence this gives the complete proof of Corollary 2 in the introduction. □

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