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REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH CERTAIN COMMUTING CONDITION II

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Abstract. Lee, Kim and Suh (2012) gave a characterization for real hypersurfaces M of Type (A) in complex two plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with a commuting condition between the shape operator A and the structure tensors φ and φ_1 for M in $G_2(\mathbb{C}^{m+2})$. Motivated by this geometrical notion, in this paper we consider a new commuting condition in relation to the shape operator A and a new operator $\varphi\varphi_1$ induced by two structure tensors φ and φ_1. That is, this commuting shape operator is given by $\varphi\varphi_1 A = A\varphi\varphi_1$. Using this condition, we prove that M is locally congruent to a tube of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Keywords: complex two-plane Grassmannians; Hopf hypersurface; $\mathfrak{D}^{\perp}$-invariant hypersurface; commuting shape operator; Reeb vector field

MSC 2010: 53C40, 53C15

Introduction

The study of real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ was initiated by Berndt and Suh [3]. Let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This set can be identified with the homogeneous space $SU(m+2)/S(U(2)\times U(m))$. From this, we know that $G_2(\mathbb{C}^{m+2})$ becomes the unique compact irreducible Riemannian manifold being equipped with both a Kaehler structure J and a quaternionic Kaehler structure $\mathfrak{J}$ commuting with J. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kaehler, quaternionic Kaehler manifold which is not a hyper-Kaehler manifold ([3], [6], [7]).

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In $G_2(\mathbb{C}^{m+2})$ we have the following two natural geometric conditions for real hypersurfaces M: the 1-dimensional distribution $[\xi] = \text{Span}\{\xi\}$ and the 3-dimensional distribution $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator A of M. Here the almost contact structure vector field ξ defined by $\xi = -JN$ is said to be a *Reeb* vector field, where N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. The *almost contact* 3*-structure* vector fields $\{\xi_1, \xi_2, \xi_3\}$ for the 3-dimensional distribution $\mathfrak{D}^\perp$ of M in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ $(\nu = 1, 2, 3)$, where J_ν denotes a canonical local basis of a quaternionic Kaehler structure $\mathfrak{J}$, such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

By using these two invariant conditions and the result in Alekseevskij [1], Berndt and Suh [3] proved the following:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

The Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The 1-dimensional foliation of M by the integral curves of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the almost contact metric structure (φ, ξ, η, g) and the formula $\nabla_X \xi = \varphi AX$ for any $X \in TM$, it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf. And when the distribution $\mathfrak{D}^\perp$ of a hypersurface M in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator, that is, $A\mathfrak{D}^\perp \subset \mathfrak{D}^\perp$, we call M a *$\mathfrak{D}^\perp$-invariant hypersurface*. Note that $A\mathfrak{D}^\perp \subset \mathfrak{D}^\perp$ implies $A\mathfrak{D} \subset \mathfrak{D}$, and vice versa.

Here, we say that the Reeb flow on M in $G_2(\mathbb{C}^{m+2})$ is *isometric*, when the Reeb vector field ξ is Killing. In [4], Berndt and Suh gave some equivalent conditions on the isometric Reeb flow. Among these conditions, the authors paid their attention to the following: *the Reeb flow on M is isometric if and only if the shape operator A commutes with the structure tensor field φ, that is, $A\varphi = \varphi A$.* Using this notion, they gave a characterization for real hypersurfaces of Type (A) in Theorem A.

In [13] Suh considered a commuting condition that the shape operator A commutes with three structure tensor fields φ_ν for $\nu = 1, 2, 3$, that is, $A\varphi_\nu = \varphi_\nu A$ for any $\nu = 1, 2, 3$, and gave a characterization of Hopf hypersurfaces of Type (B) in Theorem A while in [18] he gave another characterization of Type (B) in terms of contact hypersurfaces, that is, $A\varphi + \varphi A = k\varphi$ where $k \neq 0$ (for the case $k = 0$, Jeong, Lee and Suh gave the non-existence theorem in [5]). Moreover, with the normal Jacobi operator or the Ricci tensor, Pérez, Jeong, Suh and Watanabe considered such commuting problems for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ (see [10], [11] and [12]).

In particular, focused on the Ricci tensor, Suh gave some classifications for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with a parallel, ξ-invariant or Reeb-parallel Ricci tensor (see [16], [15], and [17]). More generally, in [14] he has given a complete classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with harmonic curvature or Weyl harmonic curvature.

From the geometric structure of $G_2(\mathbb{C}^{m+2})$ we have a natural commuting condition $JJ_\nu = J_\nu J$, $\nu = 1, 2, 3$, between the Kaehler structure J and the quaternionic Kaehler structure $\mathfrak{J}$ having a canonical local basis J_ν, $\nu = 1, 2, 3$. In the case $\nu = 1$, from this commuting condition $JJ_1 = J_1 J$ we have a relation between two structure tensors φ and φ_1 in such a way that

$$\varphi\varphi_1 X = \varphi_1\varphi X - \eta(X)\xi_1 + \eta_1(X)\xi$$

for any tangent vector field X on M. Using this operator $\varphi\varphi_1$, recently Lee, Kim and Suh [8] considered a commuting condition between the shape operator A and two structure tensors φ and φ_1 as follows:

Theorem B. *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then the shape operator A satisfies the commuting condition $\varphi\varphi_1 A = A\varphi_1\varphi$ if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Motivated by this results, naturally we consider another new commuting condition $(*)$ with the operator $\varphi\varphi_1$ on the distribution $\mathfrak{D}^\perp$, that is, the shape operator A commutes with the new operator $\varphi\varphi_1$ composed by two structure tensors φ and φ_1. Then in this paper we assert the following:

Main Theorem. *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$, with a commuting shape operator, that is,*

$$(*) \qquad \varphi\varphi_1 AX = A\varphi\varphi_1 X$$

for any tangent vector field $X \in \mathfrak{D}^\perp$. If the integral curve of the $\mathfrak{D}$-component of the Reeb vector field ξ is geodesic, then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

In order to give a complete proof of our Main Theorem, in Section 1 we recall the Riemannian geometry of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. In Section 2 some fundamental formulas for real hypersurfaces are also recalled and the information for a model space in $G_2(\mathbb{C}^{m+2})$ is given in detail.

In Lemma 3.2 of Section 3 we give some detailed information when the $\mathfrak{D}$ (or the $\mathfrak{D}^{\perp}$)-component of the Reeb vector field is principal. Moreover, in Lemma 3.3 of Section 3, by virtue of Lemma 3.2 we prove that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^{\perp}$ under the assumption that the integral curve of the $\mathfrak{D}$-component is geodesic. In Section 4 we give a complete proof of our Main Theorem according to the geodesic Reeb flow satisfying $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^{\perp}$.

1. Preliminaries

Before going to give our assertions, let us summarize the basic material about complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$; for details we refer to [2], [3], [4], [6] and [7].

By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2)\times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $\mathrm{Ad}(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restriction to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $\mathrm{Ad}(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g so that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. In addition, when $m = 2$, we note that the isomorphism $\mathrm{Spin}(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^{+}(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. From such a point of view, we consider complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition

$$\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R},$$

where $\mathfrak{R}$ denotes the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kaehler structure J and the $\mathfrak{su}(2)$-part of a quaternionic Kaehler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_ν is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and JJ_ν is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\mathrm{tr}(JJ_\nu) = 0$ for $\nu = 1, 2, 3$.

A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\overline{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\overline{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

Moreover, the Riemannian curvature tensor $\overline{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\overline{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ &\quad - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\ &\quad + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned} \tag{1.2}$$

where $\{J_1, J_2, J_3\}$ denotes a canonical local basis of $\mathfrak{J}$.

2. Some fundamental formulas and previous results

In this section we derive some basic formulas for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [3], [4], [5], [9], [11], etc.). In addition, we introduce some previous results used in our proof as primary tools.

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Levi-Civita connection of (M, g). Let N be a local unit normal vector field of M and A the shape operator of M with respect to N. Now let us put

$$JX = \varphi X + \eta(X)N, \quad J_\nu X = \varphi_\nu X + \eta_\nu(X)N \tag{2.1}$$

for any tangent vector field X of M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. Owing to the Kaehler structure J of $G_2(\mathbb{C}^{m+2})$ there exists an almost contact metric structure (φ, ξ, η, g) induced on M in such a way that

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta(X) = g(X, \xi) \tag{2.2}$$

for any vector field X on M. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kaehler structure J_ν of $G_2(\mathbb{C}^{m+2})$, together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$ mentioned in Section 1, induces an almost contact metric 3-structure $(\varphi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$
\begin{aligned}
\varphi_\nu^2 X &= -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \varphi_\nu\xi_\nu = 0, \\
\varphi_{\nu+1}\xi_\nu &= -\xi_{\nu+2}, \quad \varphi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\
\varphi_\nu\varphi_{\nu+1}X &= \varphi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\
\varphi_{\nu+1}\varphi_\nu X &= -\varphi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}
\end{aligned}
\tag{2.3}
$$

for any vector field X tangent to M. Moreover, due to the commuting property of $J_\nu J = JJ_\nu$, $\nu = 1, 2, 3$ in Section 1 and (2.1), the relation between these two contact metric structures (φ, ξ, η, g) and $(\varphi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, can be given by

$$
\begin{aligned}
\varphi\varphi_\nu X &= \varphi_\nu\varphi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu, \\
\eta_\nu(\varphi X) &= \eta(\varphi_\nu X), \quad \varphi\xi_\nu = \varphi_\nu\xi.
\end{aligned}
\tag{2.4}
$$

On the other hand, from the Kaehler structure J, that is, $\overline{\nabla}J = 0$ and the quaternionic Kaehler structure J_ν (see (1.1)), together with Gauss and Weingarten formulas it follows that

$$(\nabla_X\varphi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \varphi AX, \tag{2.5}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \varphi_\nu AX, \tag{2.6}$$

$$(\nabla_X\varphi_\nu)Y = -q_{\nu+1}(X)\varphi_{\nu+2}Y + q_{\nu+2}(X)\varphi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX, Y)\xi_\nu. \tag{2.7}$$

As we have mentioned in the introduction, with two invariant conditions on the shape operator for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ Berndt and Suh [3] classified all real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ into two kinds of hypersurfaces which are said to be of Type (A) or of Type (B). For these model spaces, they gave some detailed information for eigenvalues, its corresponding eigenspaces and multiplicities and some geometric structures.

Now let us introduce a proposition concerned with a tube of Type (A) as follows:

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$*. The corresponding multiplicities are*

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned}
T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1 = \mathrm{Span}\{\xi\} = \mathrm{Span}\{\xi_1\},\\
T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3 = \mathrm{Span}\{\xi_2,\ \xi_3\},\\
T_\lambda &= \{X;\ X \perp \mathbb{H}\xi,\ JX = J_1X\},\\
T_\mu &= \{X;\ X \perp \mathbb{H}\xi,\ JX = -J_1X\}
\end{aligned}$$

where $\mathbb{R}\xi$*,* $\mathbb{C}\xi$ *and* $\mathbb{H}\xi$*, respectively, denote the real, complex and quaternionic span of the structure vector field* ξ *and* $\mathbb{C}^\perp\xi$ *denotes the orthogonal complement of* $\mathbb{C}\xi$ *in* $\mathbb{H}\xi$*.*

On the other hand, for a model space of Type (B) in $G_2(\mathbb{C}^{m+2})$ we have (see [3])

Proposition B. *Let* M *be a connected real hypersurface in* $G_2(\mathbb{C}^{m+2})$*. Suppose that* $A\mathfrak{D} \subset \mathfrak{D}$*,* $A\xi = \alpha\xi$*, and* ξ *is tangent to* $\mathfrak{D}$*. Then the quaternionic dimension* m *of* $G_2(\mathbb{C}^{m+2})$ *is even, say* $m = 2n$*, and* M *has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$*. The corresponding multiplicities are*

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$\begin{aligned}
T_\alpha &= \mathbb{R}\xi = \mathrm{Span}\{\xi\},\\
T_\beta &= \mathfrak{J}J\xi = \mathrm{Span}\{\xi_\nu;\ \nu = 1, 2, 3\},\\
T_\gamma &= \mathfrak{J}\xi = \mathrm{Span}\{\varphi_\nu\xi;\ \nu = 1, 2, 3\},\\
&T_\lambda, \quad T_\mu,
\end{aligned}$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

The distribution $(\mathbb{H}\mathbb{C}\xi)^\perp$ *is the orthogonal complement of* $\mathbb{H}\mathbb{C}\xi$ *where*

$$\mathbb{H}\mathbb{C}\xi = \mathbb{R}\xi \oplus \mathbb{R}J\xi \oplus \mathfrak{J}\xi \oplus \mathfrak{J}J\xi.$$

Next, let us introduce a lemma due to Berndt and Suh [4]. Using the fact $A\xi = \alpha\xi$ and the equation of Codazzi, they gave the following lemma:

Lemma A. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then the smooth function $\alpha = g(A\xi, \xi)$ on M satisfies*

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\varphi Y) \tag{2.8}$$

for any tangent vector field Y on M.

3. Key lemmas

Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with the commuting shape operator, that is, the shape operator A of M commutes with the new operator $\varphi\varphi_1$ composed by two structure tensors φ and φ_1 as follows:

$$\varphi\varphi_1 AX = A\varphi\varphi_1 X \tag{$*$}$$

for any tangent vector field $X \in \mathfrak{D}^\perp$.

In this section, our purpose is to show that the Reeb vector field ξ belongs either to the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$ under the assumption $(*)$. In order to do this, we suppose that the Reeb vector field ξ is given by

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1, \tag{$**$}$$

so that $\eta(X_0)\eta(\xi_1) \neq 0$ for some unit vector fields $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$.

Under this assumption, we first prove the following:

Lemma 3.1. *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. If the $\mathfrak{D}$-component of the Reeb vector field ξ is principal, then it has a corresponding principal curvature α and the $\mathfrak{D}^\perp$-component of ξ becomes a principal curvature with the same principal curvature α. Moreover, the converse also holds.*

P r o o f. To show this lemma, first we assume that the $\mathfrak{D}$-component of the Reeb vector field ξ is a principal curvature vector field with the corresponding principal curvature λ, that is, $AX_0 = \lambda X_0$ for some smooth function λ on M.

Since we have assumed that M is Hopf, it follows that

$$\begin{aligned} A\xi = \alpha\xi &\Leftrightarrow \eta(X_0)AX_0 + \eta(\xi_1)A\xi_1 = \alpha\eta(X_0)X_0 + \alpha\eta(\xi_1)\xi_1 \\ &\Leftrightarrow \lambda\eta(X_0)X_0 + \eta(\xi_1)A\xi_1 = \alpha\eta(X_0)X_0 + \alpha\eta(\xi_1)\xi_1. \end{aligned} \tag{3.1}$$

From this, if we take the inner product with X_0, we have $\lambda = \alpha$, because $\eta(X_0) \neq 0$. That is, we have $AX_0 = \alpha X_0$. Then it follows that $A\xi_1 = \alpha\xi_1$, because (3.1) and $\eta(\xi_1) \neq 0$. This means that the $\mathfrak{D}^\perp$-component of ξ becomes a principal curvature vector field with the corresponding principal curvature α.

Now let us show the converse. In fact, due to our assumption of the A-invariance to the $\mathfrak{D}^\perp$-component of ξ we may put $A\xi_1 = \mu\xi_1$. From this, we get

$$\eta(X_0)AX_0 + \mu\eta(\xi_1)\xi_1 = \alpha\eta(X_0)X_0 + \alpha\eta(\xi_1)\xi_1,$$

together with our assumption $A\xi = \alpha\xi$. If we take the inner product with ξ_1, then it follows that $\mu = \alpha$ and $AX_0 = \alpha X_0$ because $\eta(X_0)\eta(\xi_1) \neq 0$. □

By virtue of Lemma 3.1, we have:

Lemma 3.2. *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. If the $\mathfrak{D}$ (or $\mathfrak{D}^\perp$)-component of the Reeb vector field ξ is principal, then we have the following formulae:*

(i) $q_\nu(\xi) = 0$, $q_\nu(X_0) = 0$, $q_\nu(\xi_1) = 0$ *for* $\nu = 2, 3$,
(ii) $\nabla_{X_0}X_0 = -2\alpha(\eta(\xi_1)/\eta(X_0))\varphi_1 X_0 = (2\alpha/\eta(X_0))\varphi X_0$,
(iii) $\nabla_{X_0}\xi_1 = \alpha\varphi_1 X_0$, $\nabla_{\xi_1}\xi_1 = 0$, $\nabla_{\xi_1}X_0 = \alpha\varphi_1 X_0$.

P r o o f. By the above Lemma 3.1, under our assumptions we know that

$$AX_0 = \alpha X_0 \quad \text{and} \quad A\xi_1 = \alpha\xi_1.$$

From the equation (2.6), we obtain

$$\begin{cases} \nabla_\xi\xi_2 = q_1(\xi)\xi_3 - q_3(\xi)\xi_1 + \alpha\varphi_2\xi, \\ \nabla_\xi\xi_3 = q_2(\xi)\xi_1 - q_1(\xi)\xi_2 + \alpha\varphi_3\xi, \end{cases}$$

under our assumption $A\xi = \alpha\xi$. Taking the inner product with ξ in these equations, we have $\eta(\xi_1)q_\nu(\xi) = 0$ $(\nu = 2, 3)$, where $g(\nabla_\xi\xi_\nu, \xi) = -g(\xi_\nu, \nabla_\xi\xi) = -g(\xi_\nu, \varphi A\xi) = -\alpha g(\xi_\nu, \varphi\xi) = 0$ for $\nu = 2, 3$. Since $\eta(\xi_1) \neq 0$, it means that for $\nu = 2, 3$,

$$q_\nu(\xi) = 0.$$

Now $(**)$ implies that

$$\eta(X_0)q_2(X_0) = -\eta(\xi_1)q_2(\xi_1), \quad \eta(X_0)q_3(X_0) = -\eta(\xi_1)q_3(\xi_1). \tag{3.2}$$

Next, we consider the covariant derivative of ξ along the direction of X_0, that is, $\nabla_{X_0}\xi$. From (2.5) and our assumption $AX_0 = \alpha X_0$, it follows that $\nabla_{X_0}\xi = \varphi AX_0 = \alpha\varphi X_0$. In addition, the assumption $(**)$ gives

$$
\begin{aligned}
(3.3)\qquad \nabla_{X_0}\xi &= \nabla_{X_0}(\eta(X_0)X_0 + \eta(\xi_1)\xi_1) \\
&= g(\nabla_{X_0}X_0, \xi)X_0 + g(X_0, \nabla_{X_0}\xi)X_0 + \eta(X_0)\nabla_{X_0}X_0 \\
&\quad + g(\nabla_{X_0}\xi_1, \xi)\xi_1 + g(\xi_1, \nabla_{X_0}\xi)\xi_1 + \eta(\xi_1)\nabla_{X_0}\xi_1 \\
&= \eta(X_0)g(\nabla_{X_0}X_0, X_0)X_0 + \eta(\xi_1)g(\nabla_{X_0}X_0, \xi_1)X_0 + g(\varphi AX_0, X_0)X_0 \\
&\quad + \eta(X_0)\nabla_{X_0}X_0 + \eta(X_0)g(\nabla_{X_0}\xi_1, X_0)\xi_1 + \eta(\xi_1)g(\nabla_{X_0}\xi_1, \xi_1)\xi_1 \\
&\quad + g(\varphi AX_0, \xi_1)\xi_1 + \eta(\xi_1)\nabla_{X_0}\xi_1 \\
&= \eta(\xi_1)g(\nabla_{X_0}X_0, \xi_1)X_0 + g(\varphi AX_0, X_0)X_0 + \eta(X_0)\nabla_{X_0}X_0 \\
&\quad + \eta(X_0)g(\nabla_{X_0}\xi_1, X_0)\xi_1 + g(\varphi AX_0, \xi_1)\xi_1 + \eta(\xi_1)\nabla_{X_0}\xi_1.
\end{aligned}
$$

Summing up the above formulas, we have

$$
\begin{aligned}
\alpha\varphi X_0 = &-\eta(\xi_1)g(X_0, \nabla_{X_0}\xi_1)X_0 + \alpha g(\varphi X_0, X_0)X_0 + \eta(X_0)\nabla_{X_0}X_0 \\
&+ \eta(X_0)g(\nabla_{X_0}\xi_1, X_0)\xi_1 + \alpha g(\varphi X_0, \xi_1)\xi_1 + \eta(\xi_1)\nabla_{X_0}\xi_1 \\
= &\ \eta(X_0)\nabla_{X_0}X_0 + \eta(\xi_1)\{q_3(X_0)\xi_2 - q_2(X_0)\xi_3 + \alpha\varphi_1 X_0\},
\end{aligned}
$$

where we have used $\nabla_{X_0}\xi_1 = q_3(X_0)\xi_2 - q_2(X_0)\xi_3 + \alpha\varphi_1 X_0$ and $g(\varphi X_0, \xi_1) = g(\nabla_{X_0}\xi_1, X_0) = 0$. Moreover, it follows that

$$
(3.4)\qquad -2\alpha\eta(\xi_1)\varphi_1 X_0 = \eta(X_0)\nabla_{X_0}X_0 + \eta(\xi_1)q_3(X_0)\xi_2 - \eta(\xi_1)q_2(X_0)\xi_3,
$$

where we have used $\varphi X_0 = -\eta(\xi_1)\varphi_1 X_0$ which comes from $\varphi\xi = 0$ and the assumption $(**)$.

Since $\varphi_1\xi_2 = \xi_3$ and $g(\nabla_{X_0}X_0, \xi_2) = -g(X_0, \nabla_{X_0}\xi_2) = 0$, let us take the inner product with ξ_2 in (3.4). This yields $\eta(\xi_1)q_3(X_0)\xi_2 = 0$. Since $\eta(\xi_1) \neq 0$, we have

$$
q_3(X_0) = 0.
$$

Similarly, taking the inner product with ξ_3 in (3.4), we have

$$
q_2(X_0) = 0,
$$

because we know that $\varphi_1\xi_3 = -\xi_2$ and $\nabla_{X_0}\xi_3 = q_2(X_0)\xi_1 - q_1(X_0)\xi_2 + \alpha\varphi_3 X_0$.

By using these results, the equation (3.2) implies

$$
q_2(\xi_1) = 0, \quad q_3(\xi_1) = 0
$$

because $\eta(\xi_1) \neq 0$.

Moreover, since $q_2(X_0) = q_3(X_0) = 0$, the equation (3.4) becomes

$$-2\alpha\eta(\xi_1)\varphi_1 X_0 = \eta(X_0)\nabla_{X_0} X_0,$$

that is,

$$\nabla_{X_0} X_0 = -2\alpha\frac{\eta(\xi_1)}{\eta(X_0)}\varphi_1 X_0 = \frac{2\alpha}{\eta(X_0)}\varphi X_0,$$

because $\eta(X_0) \neq 0$ and $\varphi X_0 = -\eta(\xi_1)\varphi_1 X_0$.

On the other hand, the formula $\nabla_Y \xi_1 = q_3(Y)\xi_2 - q_2(Y)\xi_3 + \varphi_1 AY$ for any $Y \in TM$ gives that

$$\begin{aligned} \nabla_{X_0}\xi_1 &= q_3(X_0)\xi_2 - q_2(X_0)\xi_3 + \varphi_1 AX_0 = \alpha\varphi_1 X_0, \\ \nabla_{\xi_1}\xi_1 &= q_3(\xi_1)\xi_2 - q_2(\xi_1)\xi_3 + \varphi_1 A\xi_1 = \alpha\varphi_1\xi_1 = 0. \end{aligned} \tag{3.5}$$

To show the last equation in Lemma 3.2, we consider the covariant derivative of ξ along the ξ_1-direction, that is, $\nabla_{\xi_1}\xi$. From our assumption $(**)$, we obtain that

$$\begin{aligned} \nabla_{\xi_1}\xi &= \nabla_{\xi_1}(\eta(X_0)X_0 + \eta(\xi_1)\xi_1) \\ &= \eta(\xi_1)g(\nabla_{\xi_1}X_0, \xi_1)X_0 + g(X_0, \varphi A\xi_1)X_0 + \eta(X)\nabla_{\xi_1}X_0 \\ &\quad + \eta(X_0)g(\nabla_{\xi_1}\xi_1, X_0)\xi_1 + g(\xi_1, \varphi A\xi_1)\xi_1 + \eta(\xi_1)\nabla_{\xi_1}\xi_1. \end{aligned}$$

It follows that

$$\nabla_{\xi_1}\xi = \eta(X_0)\nabla_{\xi_1}X_0, \tag{3.6}$$

and from (3.5), $AX_0 = \alpha X_0$ and $A\xi_1 = \alpha\xi_1$. On the other hand, from (2.5) we get $\nabla_{\xi_1}\xi = \varphi A\xi_1 = \alpha\varphi\xi_1 = \alpha\eta(X_0)\varphi_1 X_0$, because $\varphi\xi_1 = \eta(X_0)\varphi_1 X_0$. From this and (3.6), we see that

$$\nabla_{\xi_1}X_0 = \alpha\varphi_1 X_0,$$

since $\eta(X_0) \neq 0$. □

Using Lemmas 3.1 and 3.2, we can assert the following:

Lemma 3.3. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with the commuting shape operator $(*)$. If the integral curve of the $\mathfrak{D}$-component of the Reeb vector field ξ is geodesic, then ξ belongs either to the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Proof. Putting $X = \xi_1$ into $(*)$, it becomes $\varphi\varphi_1 A\xi_1 = 0$, because $\varphi_1\xi_1 = 0$. From this, applying the structure tensor φ, we get

$$-\varphi_1 A\xi_1 + \eta(\varphi_1 A\xi_1)\xi = 0. \tag{3.7}$$

From this, if we take the inner product with ξ_1, it follows that $\eta(\varphi_1 A\xi_1) = 0$, because $\eta(\xi_1) \neq 0$. Thus from (3.7) we get $\varphi_1 A\xi_1 = 0$, from which applying the structure tensor φ_1, it follows that

$$A\xi_1 = \lambda\xi_1, \quad \text{where} \quad \lambda = \eta_1(A\xi_1), \tag{3.8}$$

that is, the $\mathfrak{D}^\perp$-component of ξ becomes a principal vector field. Hence, we have

$$g(A\xi_1, X_0) = g(A\xi_1, \xi_2) = g(A\xi_1, \xi_3) = 0. \tag{3.9}$$

Now, substituting $X = \xi_2$ in $(*)$, it follows that

$$\varphi\varphi_1 A\xi_2 = A\varphi\varphi_1\xi_2 = A\varphi\xi_3 = A\varphi_3\xi = \eta(X_0)A\varphi_3 X_0 + \eta(\xi_1)A\xi_2,$$

together with (2.3), (2.4) and $(**)$. Taking the inner product with ξ_1 of this equation, we have $g(\varphi\varphi_1 A\xi_2, \xi_1) = 0$ from (3.8). Thus we obtain

$$g(A\xi_2, X_0) = 0, \tag{3.10}$$

where $\varphi_1\varphi\xi_1 = \varphi_1^2\xi = -\xi + \eta(\xi_1)\xi_1 = -\eta(X_0)X_0$ and $\eta(X_0)\eta(\xi_1) \neq 0$.

Similarly, putting $X = \xi_3$ in $(*)$, we have

$$g(A\xi_3, X_0) = 0. \tag{3.11}$$

Combining (3.9), (3.10) and (3.11), we see that the $\mathfrak{D}$-component of ξ is invariant under the shape operator, that is, $AX_0 \in \mathfrak{D}$.

Next, in order to use Lemma 3.2, we first want to prove that

$$AX_0 = \alpha X_0, \quad \text{and} \quad A\xi_1 = \alpha\xi_1.$$

Since we have assumed that M is a Hopf hypersurface, the formula $A\xi = \alpha\xi$ can be rewritten as

$$\begin{aligned}\alpha\eta(X_0)X_0 + \alpha\eta(\xi_1)\xi_1 &= \eta(X_0)AX_0 + \eta(\xi_1)A\xi_1 \\ &= \eta(X_0)AX_0 + \lambda\eta(\xi_1)\xi_1,\end{aligned}$$

by using $(**)$ and (3.8). By comparing the $\mathfrak{D}$ and $\mathfrak{D}^\perp$ parts, we obtain $\alpha\eta(X_0)X_0 = \eta(X_0)AX_0$ and $\alpha\eta(\xi_1)\xi_1 = \lambda\eta(\xi_1)\xi_1$, because $AX_0 \in \mathfrak{D}$. This implies that $\lambda = \alpha$, from which together with (3.8) and $(**)$ we have

$$AX_0 = \alpha X_0, \quad A\xi_1 = \alpha\xi_1. \tag{3.12}$$

This formula gives that the $\mathfrak{D}$ and $\mathfrak{D}^\perp$-components of the Reeb vector field ξ become principal vectors. Then by Lemma 3.2 we have

$$\eta(X_0)\nabla_{X_0}X_0 = 2\alpha\varphi X_0.$$

Since we have assumed that the $\mathfrak{D}$-component of ξ is geodesic, it follows that

$$2\alpha\varphi X_0 = 0.$$

From this, taking the inner product with φX_0, we obtain that $2\alpha\eta^2(\xi_1) = 0$, because $g(\varphi X_0, \varphi X_0) = 1 - \eta^2(X_0) = \eta^2(\xi_1)$. By our assumption $\eta(\xi_1) \neq 0$, it means that the smooth function α must be vanishing, that is, $\alpha = 0$. Then by Lemma A in Section 2, we have

$$\sum_{\nu=1}^{3} \eta_\nu(\xi)\varphi_\nu\xi = 0,$$

for the case $\alpha = 0$. Together with $\eta(\xi_2) = \eta(\xi_3) = 0$, this yields $\eta(X_0)\eta_1(\xi)\varphi_1 X_0 = 0$. Since $\varphi_1 X_0$ is a unit vector field, we consequently have $\eta(X_0)\eta_1(\xi) = 0$. This makes a contradiction. It gives us a complete proof of our lemma. $\square$

4. Proof of Main Theorem

In this section, we give a proof of our Main Theorem asserted in the introduction. By virtue of Lemma 3.3, we can divide it into the following two cases:

Case I. The Reeb vector field ξ belongs to the distribution $\mathfrak{D}$,

Case II. The Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$,

provided M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting shape operator $(*)$ and the $\mathfrak{D}$-component of ξ on M is geodesic.

First, let us consider Case I, that is, $\xi \in \mathfrak{D}$. To consider this case, we introduce a theorem due to Lee and Suh [9] as follows:

Theorem C. *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

By Theorem C, for Case I we know that *M is locally congruent to a real hypersurface of Type* (B) *under our assumptions.* Now let us check the converse problem, that is, whether or not the model space M_B of Type (B) given in Theorem A satisfies the commuting condition

$$(*) \qquad \varphi\varphi_1 AX = A\varphi\varphi_1 X$$

for any tangent vector field $X \in \mathfrak{D}^{\perp}$. To check this, let us assume that M_B satisfies the condition $(*)$.

From the structure of the tangent space T_xM_B at any point x on M_B we see that the distribution $\mathfrak{D}^{\perp}$ is equal to the eigenspace T_β where $T_\beta = \mathrm{Span}\{\xi_\mu;\ \mu = 1,2,3\}$ (see Proposition B in Section 2). Thus putting $X = \xi_2$ in $(*)$, the left-hand side becomes

$$0 = \varphi\varphi_1 A\xi_2 - A\varphi\varphi_1\xi_2 = \beta\varphi\xi_3 - A\varphi\xi_3 = \beta\varphi\xi_3,$$

because $\varphi_1\xi_2 = \xi_3$ and $T_\gamma = \mathrm{Span}\{\varphi_\mu\xi;\ \mu = 1,2,3\}$ where $\gamma = 0$. It means that $\beta = 0$ because $\varphi\xi_3$ is a unit vector field. But from Proposition B we see that $\beta = 2\cot(2r)$ where $r \in (0, \pi/4)$, so that the function β does not vanish. This gives a contradiction. Therefore, we assert that *there exists no Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with the commuting shape operator $\varphi\varphi_1 AX = A\varphi\varphi_1 X$ if the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$.*

Next we consider Case II, that is, the case $\xi \in \mathfrak{D}^{\perp}$. Accordingly, we may put $\xi = \xi_1$. From this, differentiating this equation along any direction Y, we have $q_\mu(Y) = 2g(AY, \xi_\mu)$ for $\mu = 2, 3$ and moreover,

$$AY = \alpha\eta(Y)\xi + 2g(AY,\xi_2)\xi_2 + 2g(AY,\xi_3)\xi_3 - \varphi\varphi_1 AY$$

for any tangent vector field Y on M. From this, together with our assumption $(*)$, it follows that

$$(4.1) \qquad AY = \alpha\eta(Y)\xi + 2g(AY,\xi_2)\xi_2 + 2g(AY,\xi_3)\xi_3 - A\varphi\varphi_1 Y$$

for any tangent vector field $Y \in \mathfrak{D}^{\perp}$. Putting $Y = \xi_2$ in (4.1) gives

$$A\xi_2 = g(A\xi_2,\xi_2)\xi_2 + g(A\xi_2,\xi_3)\xi_3,$$

because $A\varphi\varphi_1\xi_2 = A\varphi\xi_3 = A\varphi_3\xi_1 = A\xi_2$. Similarly, substituting $Y = \xi_3$ in (4.1), we have

$$A\xi_3 = g(A\xi_3,\xi_2)\xi_2 + g(A\xi_3,\xi_3)\xi_3.$$

Summing up these statements, we can assert that *under our assumption* $(*)$*, the distribution $\mathfrak{D}^{\perp}$ is invariant under the shape operator A of M in $G_2(\mathbb{C}^{m+2})$.* Then by Theorem A we see that *M is locally congruent to a real hypersurface of Type* (A).

Now it remains only to show whether a real hypersurface M_A of Type (A) satisfies the condition $(*)$ or not. Since the distribution $\mathfrak{D}^\perp$ is composed of two eigenspaces T_α and T_β where $T_\alpha = \mathrm{Span}\{\xi\}$ and $T_\beta = \mathrm{Span}\{\xi_2, \xi_3\}$, we consider the following two cases:

Case A-1. $X \in T_\alpha$, that is, $X = \xi$.

This case is trivial by the assumptions $A\xi = \alpha\xi$ and $\xi = \xi_1$.

Case A-2. $X \in T_\beta$, that is, $X = \xi_2$ or $X = \xi_3$.

Since $\varphi\varphi_1\xi_\mu = \xi_\mu$ for $\mu = 2, 3$, the condition $(*)$ holds.

From these two cases we see that *the shape operator A for a real hypersurface M_A of Type* (A) *satisfies the commuting condition* $(*)$ *for any tangent vector field $X \in \mathfrak{D}^\perp$.*

Summing up these discussions, gives a complete proof of our main theorem in the introduction. $\square$

References

[1] *D. V. Alekseevskij*: Compact quaternion spaces. Funkts. Anal. Prilozh. *2* (1968), 11–20. (In Russian.) zbl MR

[2] *J. Berndt*: Riemannian geometry of complex two-plane Grassmannians. Rend. Semin. Mat., Torino *55* (1997), 19–83. zbl MR

[3] *J. Berndt, Y. J. Suh*: Real hypersurfaces in complex two-plane Grassmannians. Monatsh. Math. *127* (1999), 1–14. zbl MR

[4] *J. Berndt, Y. J. Suh*: Real hypersurfaces with isometric Reeb flow in complex two-plane Grassmannians. Monatsh. Math. *137* (2002), 87–98. zbl MR

[5] *I. Jeong, H. J. Lee, Y. J. Suh*: Anti-commuting real hypersurfaces in complex two-plane Grassmannians. Bull. Aust. Math. Soc. *78* (2008), 199–210. zbl MR

[6] *S. Kobayashi, K. Nomizu*: Foundations of Differential Geometry I. Interscience Publishers, a division of John Wiley and Sons, New York, 1963. zbl MR

[7] *S. Kobayashi, K. Nomizu*: Foundations of Differential Geometry Vol. II. Interscience Tracts in Pure and Applied Mathematics No. 15, Vol. II, Interscience Publishers, a division of John Wiley and Sons, New York, 1969. zbl MR

[8] *H. Lee, S. Kim, Y. J. Suh*: Real hypersurfaces in complex two-plane Grassmannians with certain commuting condition. Czech. Math. J. *62* (2012), 849–861. zbl MR

[9] *H. Lee, Y. J. Suh*: Real hypersurfaces of type B in complex two-plane Grassmannians related to the Reeb vector. Bull. Korean Math. Soc. *47* (2010), 551–561. zbl MR

[10] *J. D. Pérez, I. Jeong, Y. J. Suh*: Real hypersurfaces in complex two-plane Grassmannians with commuting normal Jacobi operator. Acta Math. Hung. *117* (2007), 201–217. zbl MR

[11] *J. D. Pérez, Y. J. Suh*: The Ricci tensor of real hypersurfaces in complex two-plane Grassmannians. J. Korean Math. Soc. *44* (2007), 211–235. zbl MR

[12] *J. D. Pérez, Y. J. Suh, Y. Watanabe*: Generalized Einstein real hypersurfaces in complex two-plane Grassmannians. J. Geom. Phys. *60* (2010), 1806–1818. zbl MR

[13] *Y. J. Suh*: Real hypersurfaces in complex two-plane Grassmannians with commuting shape operator. Bull. Aust. Math. Soc. *68* (2003), 379–393. zbl MR

[14] *Y. J. Suh*: Real hypersurfaces in complex two-plane Grassmannians with harmonic curvature. J. Math. Pures Appl. *100* (2013), 16–33. MR

[15] *Y. J. Suh*: Real hypersurfaces in complex two-plane Grassmannians with parallel Ricci tensor. Proc. Roy. Soc. Edinburgh Sect. A *142* (2012), 1309–1324. MR
[16] *Y. J. Suh*: Real hypersurfaces in complex two-plane Grassmannians with ξ-invariant Ricci tensor. J. Geom. Phys. *61* (2011), 808–814. zbl MR
[17] *Y. J. Suh*: Real hypersurfaces in complex two-plane Grassmannians with Reeb parallel Ricci tensor. J. Geom. Phys. *64* (2013), 1–11. zbl MR
[18] *Y. J. Suh*: Real hypersurfaces of Type B in complex two-plane Grassmannians. Monatsh. Math. *147* (2006), 337–355. zbl MR

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