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Pseudo anti-commuting and Ricci soliton real hypersurfaces in complex two-plane Grassmannians

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ABSTRACT

In this paper, first we introduce a new notion of pseudo anti-commuting for real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and prove a complete classification theorem, which gives a shrinking Ricci soliton with potential Reeb flow on Hopf real hypersurfaces and a tube over a totally real totally geodesic $\mathbb{Q}P^n$, $m = 2n$ in $G_2(\mathbb{C}^{m+2})$.

0. Introduction

In the geometry of real hypersurfaces in complex space forms $M_m(c)$ or in quaternionic space forms $Q_m(c)$ Okumura [1], Kimura [2], Montiel and Romero [3] (resp. Martinez and Pérez [4]) considered real hypersurfaces in $M_n(c)$ (resp. in $Q_m(c)$) with commuting shape operator, that is, $A\phi = \phi A$, or commuting Ricci tensor, $S\phi = \phi S$, where S and ϕ (resp. A and ϕ_i) denote the Ricci tensor and the structure tensor of real hypersurfaces in $M_m(c)$ (resp. in $Q_m(c)$).

In a quaternionic projective space $\mathbb{Q}P^m$ Pérez and Suh [5] have classified real hypersurfaces in QP^m with commuting Ricci tensor $S\phi_i = \phi_i S$, $i = 1, 2, 3$, where S (resp. ϕ_i) denotes the Ricci tensor (resp. the structure tensor) of M in $\mathbb{Q}P^m$, is locally congruent to A_1, A_2-type, that is, a tube over $\mathbb{Q}P^k$ with radius $0 < r < \frac{\pi}{2}$, $k \in \{0, \dots, m-1\}$. The almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_i = -J_i N$, $i = 1, 2, 3$, where J_i, $i = 1, 2, 3$, denote a quaternionic Kähler structure of $\mathbb{Q}P^m$ and N a unit normal field of M in $\mathbb{Q}P^m$. Moreover, Pérez and Suh [6] have considered the notion of $\nabla_{\xi_i} R = 0$, $i = 1, 2, 3$, where R denotes the curvature tensor of a real hypersurface M in $\mathbb{Q}P^m$, and proved that M is locally congruent to a tube of radius $\frac{\pi}{4}$ over $\mathbb{Q}P^k$.

In paper [7] the author considered a real hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor, $S\phi = \phi S$, where S and ϕ denote the Ricci tensor and the structure tensor of M in $G_2(\mathbb{C}^{m+2})$, respectively. The curvature tensor $R(X, Y)Z$ of M in $G_2(\mathbb{C}^{m+2})$ can be derived from the curvature tensor $\bar{R}(X, Y)Z$ of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ for any vector fields X, Y and Z on M. Then by contraction and using the geometric structure

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$JJ_i = J_iJ$, $i = 1, 2, 3$ between the Kähler structure J and the quaternionic Kähler structure J_i, $i = 1, 2, 3$, we can derive the Ricci tensor S given by (see Section 3)

$$g(SX, Y) = \sum_{i=1}^{4m-1} g(R(e_i, X)Y, e_i),$$

where $\{e_1, \dots, e_{4m-1}\}$ denotes a basis of the tangent space T_xM of M, $x \in M$, in $G_2(\mathbb{C}^{m+2})$.

When the Ricci tensor S and the structure tensor ϕ commutes like $S\phi = -\phi S$, the Ricci tensor is said to be anti-commuting. Motivated by such a notion of anti-commuting, we consider a new notion so called *pseudo-anti commuting* Ricci tensor if the Ricci tensor satisfies the formula

$$S\phi + \phi S = 2k\phi, \quad k = \text{const}.$$

It is known that Einstein, pseudo-Einstein real hypersurfaces in the sense of Kon [8], Cecil and Ryan [9], real hypersurfaces of type (B), which is a tube over a totally real totally geodesic real projective space $\mathbb{R}H^n$, $m = 2n$, in $M_m(c)$ satisfy the formula (see Yano and Kon [10]). Moreover, it can be easily checked that Einstein hypersurfaces and some special kind of pseudo Einstein hypersurfaces in $G_2(\mathbb{C}^{m+2})$, and hypersurfaces of type (B), which is a tube over a totally real totally geodesic quaternionic projective space $\mathbb{Q}H^n$, $m = 2n$ in $G_2(\mathbb{C}^{m+2})$ satisfy this formula (see Berndt and Suh [11], Pérez, Suh and Watanabe [12]).

On the other hand, the ambient space $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J (see Berndt and Suh [11], [13]). So, in $G_2(\mathbb{C}^{m+2})$ we have two natural geometrical conditions for real hypersurfaces that $[\xi] = \text{Span}\,\{\xi\}$ or $\mathfrak{D}^\perp = \text{Span}\,\{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. By using such kinds of geometric conditions and the result in Alekseevskii [14], Berndt and Suh [11] have proved the following.

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*
(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

When the structure vector field ξ of M in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator A, M is said to be a *Hopf hypersurface*. In such a case the integral curves of the structure vector field ξ are geodesics (see Berndt and Suh [13]). The flow generated by the integral curves of the structure vector field ξ for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be a *geodesic Reeb flow*.

On the other hand, we say that the Reeb vector field is Killing, that is, $\mathcal{L}_\xi g = 0$ for the Lie derivative along the direction of the structure vector field ξ, which gives a characterization of real hypersurfaces of type (A) in Theorem A.

When the Ricci tensor S of M in $G_2(\mathbb{C}^{m+2})$ satisfies the formula $S\phi + \phi S = 2k\phi$, $k = \text{const}$, we say that M has a *pseudo anti-commuting Ricci tensor*.

In the proof of Theorem A we have proved that the one-dimensional distribution $[\xi]$ belongs to either the 3-dimensional distribution $\mathfrak{D}^\perp$ or to the orthogonal complement $\mathfrak{D}$ such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$. The case (A) in Theorem A is just the case that the one dimensional distribution $[\xi]$ belongs to the distribution $\mathfrak{D}^\perp$. Of course they satisfy that the Reeb vector ξ is Killing, that is, the structure tensor ϕ commutes with the shape operator A.

Recently, we have known that a solution of the Ricci flow equation $\frac{\partial}{\partial t} g(t) = -2\text{Ric}(g(t))$, is given by

$$\frac{1}{2}(\mathfrak{L}_\xi g)(X, Y) + \text{Ric}(X, Y) = kg(X, Y),$$

where k is a constant and $\mathfrak{L}_\xi$ denotes the Lie derivative along the direction of the Reeb vector field ξ. Then the solution is said to be a *Ricci soliton*, and surprisingly, it satisfies the pseudo-anti commuting condition $S\phi + \phi S = 2k\phi$.

In this paper we consider such a converse problem and want to give a complete classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying $S\phi + \phi S = 2k\phi$. In order to do this, first we assert the following theorem.

Theorem 1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with pseudo anti-commuting Ricci tensor, $m \geq 3$. Then we have one of the following*

(i) $k = 4m + 2 + \frac{\alpha}{2}(h - \alpha)$ *for $\xi \in \mathfrak{D}^\perp$, where $\alpha = g(A\xi, \xi)$ and h denotes the mean curvature of M.*
(ii) *M is locally congruent to a tube of radius r over a totally geodesic and totally real quaternionic projective space $\mathbb{Q}P^m$ in $G_2(\mathbb{C}^{m+2})$ for $\xi \in \mathfrak{D}$.*

When the constant k is equal to $4m + 2 + \frac{\alpha}{2}(h - \alpha)$, we will show that a nice geometric and cosmological structure could be given for hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying certain condition. In order to do this, let us recall an n-dimensional Riemannian manifold (M, g) is said to be a *Ricci soliton* if there exists a smooth vector field $V \in T_xM$, $x \in M$ that satisfies for any $X, Y \in TM$

$$\frac{1}{2}(\mathfrak{L}_V g)(X, Y) + \text{Ric}(X, Y) = kg(X, Y),$$

where $\mathfrak{L}_V g$ denotes the Lie derivative of g with respect to the vector field V and k a constant (see Chow et al. [15]). We will denote the *Ricci soliton* by (M, g, V, k) and call the vector field V as the potential vector field of the Ricci soliton. A Ricci soliton (M, g, V, k) is said to be a stable, expanding or shrinking according to $k = 0$, $k < 0$ or $k > 0$.

When the potential vector field V of the Ricci soliton (M, g, V, k) is a Killing vector field, M becomes an Einstein manifold. It is known that the Ricci tensor S of an Einstein hypersurface M in $G_2(\mathbb{C}^{m+2})$ is given by $S = ag$ for a constant a, that is, $\mathrm{Ric}(X, Y) = ag(X, Y)$ for any X and Y on M and a Riemannian metric g defined on M. Naturally the Ricci tensor S commutes with the structure tensor ϕ, that is, $S\phi = \phi S$. So by virtue of a theorem due to the author [7] it becomes a hypersurfaces of type (A) in $G_2(\mathbb{C}^{m+2})$. But by Proposition B in Section 5 it can be easily checked that any tubes of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ cannot be Einstein (see [12]). This means that among real hypersurfaces of type (A) there do not exist any Ricci solitons in $G_2(\mathbb{C}^{m+2})$ with the Killing potential vector field.

But, besides of this one, we can also assert that there do not exist any Ricci soliton on real hypersurfaces of type B mentioned (ii) in Theorem 1. Then as an application of Theorem 1 in the direction of Math. Physics, we give another theorem as follows:

Theorem 2. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with potential Reeb field ξ and Ricci soliton constant k. Then $k = 4(m+1) > 0$ and the Ricci soliton (M, g, ξ, k) becomes a shrinking Ricci soliton.*

By Theorem 2 and using the result given in Chow and etc. (see p. 7 in [15]), we know that any *shrinking Ricci soliton* on a closed n-manifold has positive curvature. Then as another geometric result from such a topological point of a view, by Theorem 2 we assert the following.

Corollary. *Let M be a closed Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with potential Reeb field ξ and Ricci soliton constant k. Then the Ricci soliton (M, g, ξ, k) has a positive scalar curvature.*

In Section 2 we recall Riemannian geometry of complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and in Section 3 we will show some fundamental properties of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. The formula for the Ricci tensor S and its covariant derivative ∇S will be shown explicitly in this section. In Sections 4 and 5 we will give a complete proof of our Theorem 1 according to the geodesic Reeb flow satisfying $\xi \in \mathfrak{D}$ or the geodesic Reeb flow satisfying $\xi \in \mathfrak{D}^\perp$. Finally, in Section 6 we introduce the notion of Ricci soliton given by Chow et al. [15] and make its applications to real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ and prove our Theorem 2. Moreover, related to the pseudo-anti commuting, we will give some remarks about proper pseudo-Einstein, Lie ξ invariant and harmonic curvature, and finally non existence of Ricci soliton on ruled real hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [11,13,16,7,17,17]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan–Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\mathrm{tr}(JJ_1) = 0$. This fact will be used in next sections.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken module three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

Let $p \in G_2(\mathbb{C}^{m+2})$ and W a subspace of $T_pG_2(\mathbb{C}^{m+2})$. We say that W is a quaternionic subspace of $T_pG_2(\mathbb{C}^{m+2})$ if $JW \subset W$ for all $J \in \mathfrak{J}_p$. And we say that W is a totally complex subspace of $T_pG_2(\mathbb{C}^{m+2})$ if there exists a one-dimensional subspace $\mathfrak{V}$ of $\mathfrak{J}_p$ such that $JW \subset W$ for all $J \in \mathfrak{V}$ and $JW \perp W$ for all $J \in \mathfrak{V}^\perp \subset \mathfrak{J}_p$. Here, the orthogonal complement of $\mathfrak{V}$ in $\mathfrak{J}_p$ is taken with respect to the bundle metric and orientation on $\mathfrak{J}$ for which any local oriented orthonormal frame field of $\mathfrak{J}$ is a canonical local basis of $\mathfrak{J}$. A quaternionic (resp. totally complex) submanifold of $G_2(\mathbb{C}^{m+2})$ is a submanifold all of whose tangent spaces are quaternionic (resp. totally complex) subspaces of the corresponding tangent spaces of $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\bar{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned} \tag{1.2}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$.

2. Some fundamental formulas

In this section we derive some fundamental formulas which will be used in the proof of our main theorem. Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, that is, a submanifold in $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal field of M and A the shape operator of M with respect to N.

The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M. Using the above expression (1.2) for the curvature tensor $\bar{R}$, the Gauss and the Codazzi equations are respectively given by

$$\begin{aligned}R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ &\quad + \sum_{\nu=1}^{3}\{g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y)\phi_\nu Z\} \\ &\quad + \sum_{\nu=1}^{3}\{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X - g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\} - \sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\} \\ &\quad - \sum_{\nu=1}^{3}\{\eta(X)g(\phi_\nu\phi Y,Z) - \eta(Y)g(\phi_\nu\phi X,Z)\}\xi_\nu + g(AY,Z)AX - g(AX,Z)AY\end{aligned}$$

and

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi + \sum_{\nu=1}^{3}\big\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X,Y)\xi_\nu\big\} \\ &\quad + \sum_{\nu=1}^{3}\big\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\big\} + \sum_{\nu=1}^{3}\big\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\big\}\xi_\nu,\end{aligned}$$

where R denotes the curvature tensor of M in $G_2(\mathbb{C}^{m+2})$.

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations (see [12,16,7,18]):

$$\begin{aligned}&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \qquad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\ &\phi\xi_\nu = \phi_\nu\xi, \qquad \eta_\nu(\phi X) = \eta(\phi_\nu X), \\ &\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\ &\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}.\end{aligned} \tag{2.1}$$

Now let us put

$$JX = \phi X + \eta(X)N, \qquad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (1.1) and (2.1) we have that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \qquad \nabla_X\xi = \phi AX, \tag{2.2}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.3}$$

$$(\nabla_X\phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX,Y)\xi_\nu. \tag{2.4}$$

Summing up these formulas, we find the following

$$\begin{aligned}\nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu) \\ &= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu) \\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX - g(AX,\xi)\xi_\nu + \eta(\xi_\nu)AX.\end{aligned} \tag{2.5}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1, 2, 3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{2.6}$$

3. Proof of main theorem

In this section let us consider a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor, that is, $S\phi = \phi S$.

Now let us contract Y and Z in the equation of Gauss in Section 2. Then the Ricci tensor S of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is given by

$$\begin{aligned} SX &= \sum_{i=1}^{4m-1} R(X, e_i)e_i \\ &= (4m+10)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu + \sum_{\nu=1}^{3}\{(\mathrm{Tr}\phi_\nu\phi)\phi_\nu\phi X - (\phi_\nu\phi)^2X\} \\ &\quad - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(X)\phi_\nu\phi\xi_\nu\} - \sum_{\nu=1}^{3}\{(\mathrm{Tr}\,\phi_\nu\phi)\eta(X) - \eta(\phi_\nu\phi X)\}\xi_\nu + hAX - A^2X, \end{aligned} \tag{3.1}$$

where h denotes the trace of the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. From the formula $JJ_\nu = J_\nu J$, $\mathrm{Tr}\,JJ_\nu = 0$, $\nu = 1, 2, 3$ we calculate the following for any basis $\{e_1, \ldots, e_{4m-1}, N\}$ of the tangent space of $G_2(\mathbb{C}^{m+2})$

$$\begin{aligned} 0 &= \mathrm{Tr}\,JJ_\nu \\ &= \sum_{k=1}^{4m-1} g(JJ_\nu e_k, e_k) + g(JJ_\nu N, N) \\ &= \mathrm{Tr}\,\phi\phi_\nu - \eta_\nu(\xi) - g(J_\nu N, JN) \\ &= \mathrm{Tr}\,\phi\phi_\nu - 2\eta_\nu(\xi) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} (\phi_\nu\phi)^2X &= \phi_\nu\phi(\phi\phi_\nu X - \eta_\nu(X)\xi + \eta(X)\xi_\nu) \\ &= \phi_\nu(-\phi_\nu X + \eta(\phi_\nu X)\xi) + \eta(X)\phi_\nu{}^2\xi \\ &= X - \eta_\nu(X)\xi_\nu + \eta(\phi_\nu X)\phi_\nu\xi + \eta(X)\{-\xi + \eta_\nu(\xi)\xi\}. \end{aligned} \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1), we have

$$\begin{aligned} SX &= (4m+10)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\} + hAX - A^2X \\ &= (4m+7)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\} + hAX - A^2X. \end{aligned} \tag{3.4}$$

Now let us take a covariant derivative of $S\phi + \phi S = 2k\phi$, $k = \text{const}$. Then it gives that

$$(\nabla_Y S)\phi X + S(\nabla_Y\phi)X + (\nabla_Y\phi)SX + \phi(\nabla_Y S)X = 2k(\nabla_Y\phi)X. \tag{3.5}$$

Then the first term of (3.5) becomes

$$\begin{aligned} (\nabla_Y S)\phi X &= -3g(\phi AY, \phi X)\xi - 3\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(\phi X) - q_{\nu+1}(Y)\eta_{\nu+2}(\phi X) + g(\phi_\nu AY, \phi X)\}\xi_\nu \\ &\quad - 3\sum_{\nu=1}^{3}\eta_\nu(\phi X)\{q_{\nu+2}(Y)\xi_{\nu+1} - q_{\nu+1}(Y)\xi_{\nu+2} + \phi_\nu A\phi X\} \\ &\quad + \sum_{\nu=1}^{3}\big[Y(\eta_\nu(\xi))\phi_\nu\phi^2X + \eta_\nu(\xi)\{-q_{\nu+1}(Y)\phi_{\nu+2}\phi^2X \\ &\quad + q_{\nu+2}(Y)\phi_{\nu+1}\phi^2X + \eta_\nu(\phi^2X)AY - g(AY, \phi^2X)\xi_\nu\} \\ &\quad - \eta_\nu(\xi)g(AY, \phi X)\phi_\nu\xi - g(\phi AY, \phi_\nu\phi X)\phi_\nu\xi \\ &\quad + \{q_{\nu+1}(Y)\eta(\phi_{\nu+2}\phi X) - q_{\nu+2}(Y)\eta(\phi_{\nu+1}\phi X) - \eta_\nu(\phi X)\eta(AY) + \eta(\xi_\nu)g(AY, \phi X)\}\phi_\nu\xi \\ &\quad - \eta(\phi_\nu\phi X)\{q_{\nu+2}(Y)\phi_{\nu+1}\xi - q_{\nu+1}(Y)\phi_{\nu+2}\xi + \phi_\nu\phi AY - \eta(AY)\xi_\nu + \eta(\xi_\nu)AY\} \\ &\quad - g(\phi AY, \phi X)\eta_\nu(\xi)\xi_\nu\big] + (Yh)A\phi X + h(\nabla_Y A)\phi X - (\nabla_Y A^2)\phi X. \end{aligned}$$

The second term of (3.5) becomes

$$\begin{aligned} S(\nabla_Y\phi)X &= \eta(X)\big[(4m+7)AY - 3\eta(AY)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(AY)\xi_\nu + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi AY - \eta(\phi_\nu AY)\phi_\nu\xi - \eta(AY)\eta_\nu(\xi)\xi_\nu\} \\ &\quad + hA^2Y - A^3Y\big] - g(AY, X)\big[(4m+7)\xi - 3\xi - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu + hA\xi - A^2\xi\big]. \end{aligned}$$

The third term of (3.5) gives

$$(\nabla_Y\phi)SX = \eta(SX)AY - g(AY, SX)\xi,$$

and the fourth term of (3.5) is given by

$$\begin{aligned}\phi(\nabla_Y S)X =& -3\eta(X)\phi^2AY - 3\sum_{\nu=1}^{3}\big\{q_{\nu+2}(Y)\eta_{\nu+1}(X) - q_{\nu+1}(Y)\eta_{\nu+2}(X) + g(\phi_\nu AY, \phi X)\big\}\phi\xi_\nu \\ &- 3\sum_{\nu=1}^{3}\eta_\nu(X)\big\{q_{\nu+2}(Y)\phi\xi_{\nu+1} - q_{\nu+1}(Y)\phi\xi_{\nu+2} + \phi\phi_\nu AY\big\} \\ &+ \sum_{\nu=1}^{3}\Big[Y\big(\eta_\nu(\xi)\big)\phi\phi_\nu\phi X + \eta_\nu(\xi)\big\{-q_{\nu+1}(Y)\phi\phi_{\nu+2}\phi X \\ &+ q_{\nu+2}(Y)\phi\phi_{\nu+1}\phi X + \eta_\nu(\phi X)\phi AY - g(AY, \phi X)\phi\xi_\nu\big\} \\ &+ \eta_\nu(\xi)\big\{\eta(X)\phi\phi_\nu AY - g(AY, X)\phi\phi_\nu\xi\big\} - g(\phi AY, \phi_\nu X)\phi\phi_\nu\xi \\ &+ \big\{q_{\nu+1}(Y)\eta(\phi_{\nu+2}X) - q_{\nu+2}(Y)\eta(\phi_{\nu+1}X) - \eta_\nu(X)\eta(AY) + \eta(\xi_\nu)g(AY, X)\big\}\phi\phi_\nu\xi \\ &- \eta(\phi_\nu X)\big\{q_{\nu+2}(Y)\phi\phi_{\nu+1}\xi - q_{\nu+1}(Y)\phi\phi_{\nu+2}\xi + \phi\phi_\nu\phi AY - \eta(AY)\phi\xi_\nu + \eta(\xi_\nu)\phi AY\big\} \\ &- g(\phi AY, X)\eta_\nu(\xi)\phi\xi_\nu - \eta(X)Y(\eta_\nu(\xi))\phi\xi_\nu - \eta(X)\eta_\nu(\xi)\phi\nabla_Y\xi_\nu\Big] \\ &+ (Yh)\phi AX + h\phi(\nabla_Y A)X - \phi(\nabla_Y A^2)X.\end{aligned}$$

Putting $X = \xi$ into (3.5) and using the structure vector ξ is principal, that is, $A\xi = \alpha\xi$, then we have

$$\begin{aligned}S(\nabla_Y\phi)\xi + (\nabla_Y\phi)S\xi + \phi(\nabla_Y S)\xi =& \Big[(4m+7)AY - 3\eta(AY)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(AY)\xi_\nu \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi AY - \eta(\phi_\nu\phi AY)\phi_\nu\xi - \alpha\eta(Y)\eta_\nu(\xi)\xi_\nu\} \\ &+ hA^2Y - A^3Y\Big] - \alpha\eta(Y)\Big[4(m+1)\xi - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu + (\alpha h - \alpha^2)\xi\Big] \\ &+ \Big[\{4(m+1) + h\alpha - \alpha^2\} - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)^2\Big]AY - 3\eta(X)\phi^2AY \\ &- \Big\{\{4(m+1)\alpha + h\alpha^2 - \alpha^3\}\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(AY)\Big\}\xi \\ &- 3\sum_{\nu=1}^{3}\{q_{\nu+2}(Y)\eta_{\nu+1}(\xi) - q_{\nu+1}(Y)\eta_{\nu+2}(\xi) + \eta_\nu(\phi AY)\}\phi\xi_\nu \\ &- 3\sum_{\nu=1}^{3}\eta_\nu(\xi)\{q_{\nu+2}(Y)\phi\xi_{\nu+1} - q_{\nu+1}(Y)\phi\xi_{\nu+2} + \phi\phi_\nu AY\} \\ &+ \sum_{\nu=1}^{3}\Big[\eta_\nu(\xi)\{\phi\phi_\nu AY - \alpha\eta(Y)\phi^2\xi_\nu\} - g(\phi AY, \phi\xi_\nu)\phi^2\xi_\nu \\ &- Y(\eta_\nu(\xi))\phi\xi_\nu - \eta_\nu(\xi)\phi\nabla_Y\xi_\nu\Big] + h\phi(\nabla_Y A)\xi - \phi(\nabla_Y A^2)\xi.\end{aligned}$$

From this, putting $Y = \xi$ into the above formula, we have the following

$$0 = \sum_{\nu=1}^{3}\{q_{\nu+2}(\xi)\eta_{\nu+1}(\xi) - q_{\nu+1}(\xi)\eta_{\nu+2}(\xi)\}\phi\xi_\nu + \sum_{\nu=1}^{3}\eta_\nu(\xi)\{q_{\nu+2}(\xi)\phi\xi_{\nu+1} - q_{\nu+1}(\xi)\phi\xi_{\nu+2} + \alpha\phi^2\xi_\nu\}.$$

Now in order to show that ξ belongs to either the distribution $\mathfrak{D}$ or to the distribution $\mathfrak{D}^\perp$, let us assume that $\xi = X_1 + X_2$ for some $X_1 \in \mathfrak{D}$ and $X_2 \in \mathfrak{D}^\perp$. Then it follows that

$$\begin{aligned}0 =& \sum_{\nu=1}^{3}\Big\{q_{\nu+2}(\xi)\eta_{\nu+1}(\xi) - q_{\nu+1}(\xi)\eta_{\nu+2}(\xi)\Big\}(\phi_\nu X_1 + \phi_\nu X_2) \\ &+ \sum_{\nu=1}^{3}\eta_\nu(\xi)\Big\{q_{\nu+2}(\xi)(\phi_{\nu+1}X_1 + \phi_{\nu+1}X_2) - q_{\nu+1}(\xi)(\phi_{\nu+2}X_1 + \phi_{\nu+2}X_2) - \alpha\xi_\nu + \alpha\eta(\xi_\nu)(X_1 + X_2)\Big\}.\end{aligned} \tag{3.6}$$

Then by comparing $\mathfrak{D}$ and $\mathfrak{D}^{\perp}$ component of (3.6), we have respectively

$$0 = \sum_{\nu=1}^{3}\{q_{\nu+2}(\xi)\eta_{\nu+1}(\xi) - q_{\nu+1}(\xi)\eta_{\nu+2}(\xi)\}\phi_\nu X_1 + \alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)^2 X_1 + \sum_{\nu=1}^{3}\eta_\nu(\xi)\{q_{\nu+2}(\xi)\phi_{\nu+1}X_1 - q_{\nu+1}(\xi)\phi_{\nu+2}X_1\}, \tag{3.7}$$

$$0 = \sum_{\nu=1}^{3}\{q_{\nu+2}(\xi)\eta_{\nu+1}(\xi) - q_{\nu+1}(\xi)\eta_{\nu+2}(\xi)\}\phi_\nu X_2 + \sum_{\nu=1}^{3}\eta_\nu(\xi)\{q_{\nu+2}(\xi)\phi_{\nu+1}X_2 - q_{\nu+1}(\xi)\phi_{\nu+2}X_2 - \alpha\xi_\nu + \alpha\eta(\xi_\nu)X_2\}. \tag{3.8}$$

Taking an inner product (3.7) with X_1, we have

$$\alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)^2 = 0. \tag{3.9}$$

Then $\alpha = 0$ or $\eta_\nu(\xi) = 0$ for $\nu = 1, 2, 3$. So for a non-vanishing geodesic Reeb flow we have $\eta_\nu(\xi) = 0$, $\nu = 1, 2, 3$. This means that $\xi \in \mathfrak{D}$, which makes a contradiction for our assumption $\xi = X_1 + X_2$. Including this one, we are able to assert the following.

Lemma 3.1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with pseudo anti-commuting Ricci tensor. Then the Reeb vector ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^{\perp}$.*

Proof. When the geodesic Reeb flow is non-vanishing, that is $\alpha \neq 0$, (3.9) gives $\xi \in \mathfrak{D}$. When the geodesic Reeb flow is vanishing, we differentiate $A\xi = 0$. Then by Berndt and Suh [13] we know that

$$\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y) = 0.$$

From this, by replacing Y by ϕY, it follows that

$$\sum_{\nu=1}^{3}\eta_\nu^2(\xi)\eta(Y) = 0.$$

So if there are some $Y \in \mathfrak{D}$ such that $\eta(Y) \neq 0$, then $\eta_\nu(\xi) = 0$ for $\nu = 1, 2, 3$. This means that $\xi \in \mathfrak{D}$. If $\eta(Y) = 0$ for any $Y \in \mathfrak{D}$, then we know $\xi \in \mathfrak{D}^{\perp}$. □

4. Real hypersurfaces with geodesic Reeb flow satisfying $\xi \in \mathfrak{D}$

Let us consider a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ with pseudo-commuting Ricci tensor, that is, $S\phi + \phi S = 2k\phi$, $k =$ const. Then in this section, by Lemma 3.1, we consider pseudo-commuting hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\xi \in \mathfrak{D}$. Then by a theorem due to Lee and Suh [19], M is locally congruent to a tube of radius r over a totally real and totally geodesic $\mathbb{Q}P^m$ in $G_2(\mathbb{C}^{m+2})$.

Concerned with such kind of tube we are able to recall a proposition given by Berndt and Suh [11] as follows:

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \qquad \beta = 2\cot(2r), \qquad \gamma = 0, \qquad \lambda = \cot(r), \qquad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \qquad m(\beta) = 3 = m(\gamma), \qquad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi, \qquad T_\beta = \mathfrak{J}J\xi, \qquad T_\gamma = \mathfrak{J}\xi,\ T_\lambda,\ T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^{\perp}, \qquad \mathfrak{J}T_\lambda = T_\lambda, \qquad \mathfrak{J}T_\mu = T_\mu, \qquad JT_\lambda = T_\mu.$$

Now it remains only to check whether the Ricci tensor for real hypersurfaces of type (B) in Theorem A is pseudo anti-commuting or not. In order to do this, first let us calculate the following

$$SX = (4m+7)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\} + hAX - A^2X.$$

From this, by using the formula in Section 2, we have

$$S\phi X = (4m+7)\phi X - 3\sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu X - \eta_\nu(X)\phi_\nu\xi\} + hAX - A^2X \tag{4.1}$$

and

$$\phi SX = (4m+7)\phi X - 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi\xi_\nu - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu X - \eta(\phi_\nu X)\xi_\nu\} + h\phi AX - \phi A^2X. \tag{4.2}$$

Then $S\phi + \phi S = 2k\phi$ becomes

$$\begin{aligned}2k\phi X &= 2(4m+7)\phi X - 2\sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu - 2\sum_{\nu=1}^{3}\eta_\nu(X)\phi\xi_\nu\\ &\quad - 2\sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu X + h(A\phi+\phi A)X - (A^2\phi+\phi A^2)X\}.\end{aligned} \tag{4.3}$$

Now by proposition A let us check the formula (4.3) as follows:
Case I. $X = \xi \in \mathfrak{D}$

$$0 = -2\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi\xi_\nu - 2\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu\xi = -4\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi\xi_\nu.$$

The right side also vanishes. So we have this case.
Case II. $X = \xi_1 \in \mathfrak{D}^\perp$

Proposition A gives $A\phi\xi_1 = 0$. Then it satisfies

$$\begin{aligned}2k\phi\xi_1 &= 2(4m+7)\phi\xi_1 - 2\{\eta_2(\phi\xi_1)\xi_2 + \eta_3(\phi\xi_1)\xi_3\} - 2\phi\xi_1 + h\phi A\xi_1 - \phi A^2\xi_1\\ &= \left\{2(4m+7) - 2 + \beta h - \beta^2\right\}\phi\xi_1\end{aligned}$$

for $2k = 2(4m+7) - 2 + \beta h - \beta^2$. This also holds for ξ_2 and ξ_3.
Case III. $X = \phi\xi_1 \in T_\gamma$, $\gamma = 0$.

Then $A\phi\xi_1 = 0$ implies that (4.3) holds

$$\begin{aligned}-2k\xi_1 &= 2(4m+7)(-\xi_1+\xi) + 2\xi_1 + hA\phi^2\xi_1 - A^2\phi^2\xi_1\\ &= \{-2(4m+7) + 2 - \beta h + \beta^2\}\xi_1\end{aligned}$$

for $2k = 2(4m+7) - 2 + \beta h - \beta^2$. This also holds for $\phi\xi_2$ and $\phi\xi_3$.
Case IV. $X \in T_\lambda$, $\lambda = \cot r$.

Then $AX = \lambda X$, $A\phi X = \mu\phi X$, $A^2\phi X = \mu\phi X$ and $\phi A^2X = \lambda^2\phi X$. Using these formulas, we have

$$\begin{aligned}2k\phi X &= 2(4m+7)\phi X + h(A\phi+\phi A)X - (A^2\phi+\phi A^2)X\\ &= \{2(4m+7) + h\beta - (\beta^2+2)\}\phi X.\end{aligned}$$

This case also becomes $2k = 2(4m+7) - 2 + \beta h - \beta^2$.
Case V. $X \in T_\mu$, $\mu = -\tan r$.

Then $AX = \mu X$, $A\phi X = \lambda\phi X$ and $A^2\phi X = \lambda^2\phi X$ give for (4.3) as follows:

$$\begin{aligned}2k\phi X &= 2(4m+7)\phi X + h(A\phi+\phi A)X - (A^2\phi+\phi A^2)X\\ &= \left\{2(4m+7) + h(\mu+\lambda) - (\mu^2+\lambda 2)\right\}\phi X.\end{aligned}$$

This case also becomes $2k = 2(4m+7) - 2 + \beta h - \beta^2$.

So summing up all cases mentioned above, real hypersurfaces of type (B) satisfy pseudo anti-commuting condition $S\phi + \phi S = 2k\phi$ for $2k = 2(4m+7) - 2 + \beta h - \beta^2$, where $\beta = 2\cot 2r$ and $h = \mathrm{Tr}A$ denotes the mean curvature of type (B).

5. Pseudo anti-commuting real hypersurfaces with $\xi \in \mathfrak{D}^\perp$

Now let us consider a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ with pseudo anti-commuting Ricci tensor and $\xi \in \mathfrak{D}^\perp$. Now differentiating $S\phi + \phi S = 2k\phi$ gives

$$(\nabla_Y S)\phi X + S(\nabla_Y \phi)X + (\nabla_Y \phi)SX + \phi(\nabla_Y S)X = 2k(\nabla_Y \phi)X.$$

In this section by Lemma 3.1 we only discuss the geodesic Reeb flow ξ belongs to the distribution $\mathfrak{D}^\perp$. Since we have assumed that $\xi \in \mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$, there exists an Hermitian structure $J_1 \in \mathfrak{J}$ such that $JN = J_1 N$, that is, $\xi = \xi_1$. Then it follows that

$$\phi\xi_2 = \phi_2\xi = \phi_2\xi_1 = -\xi_3, \qquad \phi\xi_3 = \phi_3\xi_1 = -\xi_2. \tag{5.1}$$

From this, together with the expression of (3.4) and $\xi \in \mathfrak{D}^\perp$, we have

$$\begin{aligned}
&(4m+1)g(AX,Y)\xi - 3\Big[\{q_3(Y)\eta_3(X) + q_2(Y)\eta_2(X)\}\xi_1 - q_1(Y)\eta_2(X)\xi_2 - q_1(Y)\eta_3(X)\xi_3\Big] \\
&\quad + 2\eta(X)\eta_2(AY)\xi_2 + 2\eta(X)\eta_3(AY)\xi_3 + \sum_{\nu=1}^{3}\eta_\nu(X)\phi_\nu\phi AY \\
&\quad + (Yh)A\phi X + h(\nabla_Y A)\phi X - (\nabla_Y A^2)\phi X + \eta(X)\{hA^2Y - A^3Y\} \\
&\quad - \{g(AY,SX) + \eta_3(X)\eta_3(AY) + \eta_2(X)\eta_2(AY)\}\xi + 4\Big[g(\phi_2 AY, X)\xi_3 - g(\phi_3 AY, X)\xi_2\Big] \\
&\quad - 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi\phi_\nu AY + 4\sum_{\nu=1}^{3} g(\phi AY, \phi_\nu X)\xi_\nu + \eta_3(X)\phi_2 AY - \eta_2(X)\phi_3 AY \\
&\quad + (Yh)\phi AX + h\phi(\nabla_Y A)X - \phi(\nabla_Y A^2)X \\
&= 2k\{\eta(Y)AX - g(AY,X)\xi\}.
\end{aligned} \tag{5.2}$$

Now putting $X = \xi$ in (5.2), we have

$$\begin{aligned}
&(4m+1)g(A\xi,Y)\xi + 2\eta_2(AY)\xi_2 + 2\eta_3(AY)\xi_3 + \phi_1\phi AY + hA^2Y - A^3Y \\
&- g(AY,S\xi)\xi + 4\Big\{g(\phi_2 AY,\xi)\xi_3 - g(\phi_3 AY,\xi)\xi_2\Big\} - 3\phi\phi_1 AY + 4g(\phi AY,\phi_2\xi)\xi_2 + 4g(\phi AY,\phi_3\xi)\xi_3 \\
&+ h\phi(\nabla_Y A)\xi - \phi(\nabla_Y A^2)\xi = 0.
\end{aligned}$$

From this, if we use the following formulas

$$\begin{aligned}
S\xi &= 4(m+1)\xi - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu + hA\xi - A^2\xi \\
&= (4m + h\alpha - \alpha^2)\xi
\end{aligned}$$

and

$$g(AY, S\xi) = \alpha(4m + h\alpha - \alpha^2)\eta(Y),$$

then it follows that

$$\phi_1\phi AY + hA^2Y - A^3Y = -10\eta_2(AY)\xi_2 - 10\eta_3(AY)\xi_3 + h\alpha AY - \alpha^2 AY + h\phi A\phi AY - \phi A^2\phi AY + 3\phi\phi_1 AY. \tag{5.3}$$

On the other hand, by the equation of Codazzi in [11] (see p. 6), we have

$$\begin{aligned}
A\phi AY &= \phi Y + \sum_{\nu=1}^{3}\{\eta_\nu(Y)\phi\xi_\nu + \eta_\nu(\phi Y)\xi_\nu + \eta_\nu(\xi)\phi_\nu Y - 2\eta(Y)\eta_\nu(\xi)\phi\xi_\nu - 2\eta_\nu(\xi)\eta_\nu(\phi Y)\xi\} + \alpha(A\phi + \phi A)Y \\
&= \phi Y + \phi_1 Y + \eta_2(Y)\phi\xi_2 + \eta_3(Y)\phi\xi_3 + \eta_2(\phi Y)\xi_2 + \eta_3(\phi Y)\xi_3 + \alpha(A\phi + \phi A)Y.
\end{aligned} \tag{5.4}$$

So for any $Y \in \mathfrak{D}$ (5.4) gives that $A\phi AY = \phi Y + \phi_1 Y + \alpha(A\phi + \phi A)Y$. This implies

$$\phi A^2\phi AY = \phi A\phi Y + \phi A\phi_1 Y + \alpha\phi A(A\phi + \phi A)Y.$$

From this, together with (5.3), it follows that

$$\begin{aligned}
\phi_1\phi AY + hA^2Y - A^3Y = &-10\eta_2(AY)\xi_2 - 10\eta_3(AY)\xi_3 + \alpha hAY - \alpha^2 AY + \alpha h\phi(A\phi + \phi A)Y \\
&+ h(-Y + \phi\phi_1 Y) - \phi A\phi Y - \phi A\phi_1 Y - \alpha\phi A(A\phi + \phi A)Y + 3\phi\phi_1 AY.
\end{aligned} \tag{5.5}$$

On the other hand, we calculate the following

$$S\phi Y = (4m+7)\phi Y - 3\eta_2(\phi Y)\xi_2 - 3\eta_3(\phi Y)\xi_3 + \phi_1\phi^2 Y - \eta(\phi_2\phi Y)\phi_2\xi - \eta(\phi_3\phi Y)\phi_3\xi + hA\phi Y - A^2\phi Y,$$

$$\phi SY = (4m+7)\phi Y - 3\sum_{\nu=1}^{3}\eta_\nu(Y)\phi\xi_\nu + \phi\phi_1\phi Y - \eta(\phi_2 Y)\phi_2\xi - \eta(\phi_3 Y)\phi_3\xi + h\phi AY - \phi A^2 Y.$$

So for any $Y \in \mathfrak{D}$ the condition $S\phi + \phi S = 2k\phi$ implies that

$$2(4m+7)\phi Y - \phi_1 Y + hA\phi Y - A^2\phi Y + \phi\phi_1\phi Y + h\phi AY - \phi A^2 Y = 2k\phi Y.$$

Then by replacing Y by ϕY for $Y \in \mathfrak{D}$ we have

$$A^3Y - hA^2Y - \{2(4m+7) - 2k\}AY = A\phi_1\phi Y + A\phi\phi_1 Y - hA\phi A\phi Y + A\phi A^2\phi Y. \tag{5.6}$$

Now by using (5.4) for $Y \in \mathfrak{D}$, the terms in the right side becomes respectively

$$A\phi A\phi Y = -Y + \phi_1\phi Y + \alpha(A\phi + \phi A)\phi Y$$

and

$$A\phi A^2\phi Y = \phi A\phi Y + \phi_1 A\phi Y + \eta_2(A\phi Y)\phi\xi_2 + \eta_3(A\phi Y)\phi\xi_3 + \eta_2(\phi A\phi Y)\xi_2 + \eta_3(\phi A\phi Y)\xi_3 + \alpha(A\phi + \phi A)\phi Y.$$

From these, together with (5.5) and (5.6), we have

$$\begin{aligned}
&\phi_1\phi AY - 2\{4m+7-k\}AY - A\phi_1\phi Y - A\phi\phi_1 Y + hA\phi A\phi Y - A\phi A^2\phi Y \\
&\quad = -10\eta_2(AY)\xi_2 - 10\eta_3(AY)\xi_3 + \alpha hAY - \alpha^2 AY + \alpha h\phi(A\phi + \phi A)Y \\
&\qquad + h(-Y + \phi\phi_1 Y) - \phi A\phi Y - \phi A\phi_1 Y - \alpha\phi A(A\phi + \phi A)Y + 3\phi\phi_1 AY.
\end{aligned}$$

Substituting the above formulas into this, we have

$$\begin{aligned}
&\phi_1\phi AY - \{2(4m+7-k) - \alpha(\alpha - h)\}AY - A\phi_1\phi Y - A\phi\phi_1 Y \\
&\qquad + h\{-Y + \phi_1\phi Y + \alpha(A\phi + \phi A)\phi Y\} - \phi A\phi Y - \phi_1 A\phi Y - \alpha(A\phi + \phi A)A\phi Y \\
&\qquad - \eta_2(A\phi Y)\phi\xi_2 - \eta_3(A\phi Y)\phi\xi_3 - \eta_2(\phi A\phi Y)\xi_2 - \eta_3(\phi A\phi Y)\xi_3 \\
&\quad = -10\eta_2(AY)\xi_2 - 10\eta_3(AY)\xi_3 + \alpha h\phi(A\phi + \phi A)Y + h(-Y + \phi\phi_1 Y) \\
&\qquad - \phi A\phi Y - \phi A\phi_1 Y - \alpha\phi A(A\phi + \phi A)Y + 3\phi\phi_1 AY.
\end{aligned}$$

From this, if we use the following formulas obtained from (5.4)

$$\alpha A\phi A\phi Y = -\alpha Y + \alpha\phi_1\phi Y + \alpha^2(A\phi + \phi A)\phi Y$$

and

$$\alpha\phi A\phi AY = -\alpha Y + \alpha\phi\phi_1 Y + \alpha^2\phi(A\phi + \phi A)Y,$$

then it follows that

$$\begin{aligned}
&\phi_1\phi AY - \{2(4m+7-k) - \alpha(\alpha - h)\}AY - A\phi_1\phi Y - A\phi\phi_1 Y - \phi_1 A\phi Y + \alpha Y - \alpha\phi_1\phi Y - \alpha^2(A\phi + \phi A)\phi Y \\
&\qquad - \eta_2(A\phi Y)\phi\xi_2 - \eta_3(A\phi Y)\phi\xi_3 - \eta_2(\phi A\phi Y)\xi_2 - \eta_3(\phi A\phi Y)\xi_3 \\
&\quad = -10\eta_2(AY)\xi_2 - 10\eta_3(AY)\xi_3 - \phi A\phi_1 Y + \alpha Y - \alpha\phi\phi_1 Y - \alpha^2\phi(A\phi + \phi A)Y + 3\phi\phi_1 AY.
\end{aligned}$$

From this, let us take an inner product with ξ_2, then for any $Y \in \mathfrak{D}$ we have

$$\begin{aligned}
&-2\eta_2(AY) - 2g(A\phi_1\phi Y, \xi_2) + \eta_3(A\phi Y) + \eta_3(A\phi_1 Y) \\
&\qquad - \{2(4m+7-k) - \alpha(\alpha - h)\}\eta_2(AY) - \eta_3(A\phi Y) - \eta_2(\phi A\phi Y) \\
&\quad = -10\eta_2(AY).
\end{aligned} \tag{5.7}$$

Then in this section we know that the distribution $\mathfrak{D}$ can be decomposed into two distributions $\mathfrak{D}_1$ and $\mathfrak{D}_2$ defined in such a way that

$$\mathfrak{D}_1 = \{Y \in \mathfrak{D} | \phi Y = \phi_1 Y\}$$

and

$$\mathfrak{D}_2 = \{Y \in \mathfrak{D} | \phi Y = -\phi_1 Y\}.$$

So first let us consider the distribution $\mathfrak{D}_1$.

Then by a direct calculation in (5.7) for any $Y \in \mathfrak{D}_1$, we have

$$\{2(4m+2-k)-\alpha(\alpha-h)\}\eta_2(A\phi Y)=0. \tag{5.8}$$

Now let use the similar method as in taking ξ_2 in above formula. So if we take an inner product ξ_3 to the above formula, then it follows that

$$\begin{aligned}&-2\eta_3(AY)-2g(A\phi_1\phi Y,\xi_3)-\eta_2(A\phi Y)-\eta_2(A\phi_1 Y)\\&\quad-\{2(4m+7-k)-\alpha(\alpha-h)\}\eta_3(AY)+\eta_2(A\phi Y)-\eta_3(\phi A\phi Y)\\&=-10\eta_3(AY).\end{aligned} \tag{5.9}$$

Then by a straightforward calculation in (5.9) for any $Y \in \mathfrak{D}_1$, we have

$$\{2(4m+2-k)-\alpha(\alpha-h)\}\eta_3(AY)=0. \tag{5.10}$$

From this, we assert the following.

Lemma 5.1. *Let M be a Hopf pseudo anti-commuting real hypersurface in $G_2(\mathbb{C}^{m+2})$ with the Reeb vector field ξ belonging to the distribution $\mathfrak{D}^\perp$. Then $k=4m+2+\frac{\alpha}{2}(h-\alpha)$ or M is locally congruent to a tube of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, where h denotes the mean curvature of M and $\alpha=g(A\xi,\xi)$.*

Proof. Now we consider for the case $k \neq 4m+2+\frac{\alpha}{2}(h-\alpha)$. Then (5.8) and (5.10) give $\eta_\nu(AY)=0$ for any $Y \in \mathfrak{D}_1$. Then in order to show that $g(A\mathfrak{D},\mathfrak{D}^\perp)=0$ it is sufficient to show that $\eta_\nu(AY)=0$ for any $Y \in \mathfrak{D}_2$.

So for any $Y \in \mathfrak{D}_2$ (5.7) and (5.9) give the following respectively

$$\Big\{6-2(4m+7-k)+\alpha(\alpha-h)\Big\}\eta_2(AY)=2\eta_3(A\phi Y) \tag{5.11}$$

and

$$\Big\{6-2(4m+7-k)+\alpha(\alpha-h)\Big\}\eta_3(AY)=-2\eta_2(A\phi Y). \tag{5.12}$$

Now let us put $b=6-2(4m+7-k)+\alpha(\alpha-h)$. Then we consider the following two cases.

Case I. $b \neq 0$

Then (5.11) and (5.12) give for any $Y \in \mathfrak{D}_2$

$$\eta_2(AY)=\frac{2}{b}\eta_3(A\phi Y),\qquad \eta_3(AY)=-\frac{2}{b}\eta_2(A\phi Y).$$

Since the distribution $\mathfrak{D}_2$ is invariant by the structure tensor ϕ, if we replace the vector Y by ϕY, it becomes

$$\eta_2(A\phi Y)=-\frac{2}{b}\eta_3(AY),\qquad \eta_3(A\phi Y)=\frac{2}{b}\eta_2(AY).$$

Then it gives $\eta_2(AY)=\frac{4}{b^2}\eta_2(AY)$. This implies $\eta_2(AY)=\eta_3(AY)=0$ for $Y \in \mathfrak{D}_2$ when $b \neq 2$. When $b=2$, (5.8) and (5.10) give naturally $\eta_2(AY)=0$ and $\eta_3(AY)=0$ for any $Y \in \mathfrak{D}_1$ respectively. Moreover, from (5.11) and (5.12) it follows that

$$\eta_2(AY)=\eta_3(A\phi Y),\qquad \eta_3(AY)=-\eta_2(A\phi Y)$$

for any $Y \in \mathfrak{D}_2$. Then by using the same method given in Suh [7, pp. 1803–1804] we can prove that $\eta_2(AY)=0$ and $\eta_3(AY)=0$ for any $Y \in \mathfrak{D}_2$.

Case II. $b=0$.

(5.11) and (5.12) give $\eta_2(A\phi Y)=0$ and $\eta_3(A\phi Y)=0$ for any $Y \in \mathfrak{D}_2$ respectively. The invariance of the distribution $\mathfrak{D}_2$ under the structure tensor ϕ gives also $\eta_2(AY)=\eta_3(AY)=0$ for any $Y \in \mathfrak{D}_2$.

Summing up Cases I and II, we conclude that the distribution $\mathfrak{D}$ is invariant by the shape operator, that is, $g(A\mathfrak{D},\mathfrak{D}^\perp)=0$. Then by Theorem A due to Berndt and Suh [11], M is locally isometric to a tube of radius r over the totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. This gives our assertion. □

Now we want to prove that real hypersurfaces of type (A), that is, a tube of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, does not admit any pseudo anti-commuting structures.

Related to this kind of hypersurfaces in Theorem A we introduce another proposition due to Berndt and Suh [11] as follows:

Proposition B. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D}\subset\mathfrak{D}$, $A\xi=\alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN=J_1N$. Then M has three (if $r=\pi/2$) or four (otherwise) distinct constant principal curvatures*

$$\alpha=\sqrt{8}\cot(\sqrt{8}r),\qquad \beta=\sqrt{2}\cot(\sqrt{2}r),\qquad \lambda=-\sqrt{2}\tan(\sqrt{2}r),\qquad \mu=0$$

with some $r \in (0, \pi/\sqrt{8})$. *The corresponding multiplicities are*

$$m(\alpha) = 1, \qquad m(\beta) = 2, \qquad m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces we have

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN, \\ T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N, \\ T_\lambda &= \{X | X \perp \mathbb{H}\xi, JX = J_1X\}, \\ T_\mu &= \{X | X \perp \mathbb{H}\xi, JX = -J_1X\}. \end{aligned}$$

In the paper [13] due to Berndt and Suh we have given a characterization of real hypersurfaces of type A in Theorem A when the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor ϕ, which is equivalent to the condition that *the Reeb flow on M is isometric*, that is $\mathcal{L}_\xi g = 0$, where $\mathcal{L}$ (resp. g) denotes the Lie derivative (resp. the induced Riemannian metric) of M in the direction of the Reeb vector field ξ. Namely, Berndt and Suh [11] proved the following.

Theorem B. *Let M be a connected orientable real hypersurface in* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. *Then the Reeb flow on M is isometric if and only if M is an open part of a tube around some totally geodesic* $G_2(\mathbb{C}^{m+1})$ *in* $G_2(\mathbb{C}^{m+2})$.

Now let us check that real hypersurfaces of type (A) mentioned in Proposition B and Theorem B whether they satisfy pseudo anti-commuting, that is, $S\phi + \phi S = 2k\phi$. Then by Theorem B for the commuting shape operator, that is, $A\phi = \phi A$, the commuting Ricci tensor $S\phi = \phi S$ implies that $S\phi = \phi S = k\phi$, which is given by

$$\begin{aligned} S\phi X &= (4m+7)\phi X - 3\sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu X - \eta_\nu(X)\phi_\nu\xi\} + hA\phi X - A^2\phi X \\ &= k\phi X. \end{aligned} \tag{5.13}$$

Now by using Proposition B, we check case by case whether two sides in (5.13) are equal to each other as follows:

Case I. $X = \xi = \xi_1$

In this case it can be easily checked that two sides are equal to each other.

Case II. $X = \xi_2, \xi_3$

Then by putting $X = \xi_2$ in (5.13) we have

$$\begin{aligned} k\phi\xi_2 = S\phi\xi_2 &= (4m+7)\phi\xi_2 - 3\sum_{\nu=1}^{3}\eta_\nu(\phi\xi_2)\xi_\nu - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\xi_2 - \eta_\nu(\xi_2)\phi_\nu\xi\} + hA\phi\xi_2 - A^2\phi\xi_2 \\ &= -(4m+7)\xi_3 + 3\xi_3 - 2\xi_3 - h\beta\xi_3 + \beta^2\xi_3 \\ &= -\{(4m+6) + h\beta - \beta^2\}\xi_3 \end{aligned}$$

which gives $k = 4m + 6 + h\beta - \beta^2$.

Case III. $X \in T_\lambda, \lambda = -\sqrt{2}\tan(\sqrt{2}r)$.

Then $AX = \lambda X$, $A\phi X = \lambda\phi X$ and $A^2\phi X = \lambda^2\phi X$ gives

$$\begin{aligned} k\phi X = S\phi X &= (4m+7)\phi X - \phi_1 X + h\lambda\phi X - \lambda^2\phi X \\ &= (4m + 6 + h\lambda - \lambda^2)\phi X. \end{aligned}$$

This case becomes $k = 4m + 6 + h\lambda - \lambda^2$.

Case IV. $X \in T_\mu, \mu = 0$.

Then $A\phi X = 0$ and $A^2\phi X = 0$ gives

$$k\phi X = S\phi X = (4m+7)\phi X - \phi_1 X = (4m+8)\phi X.$$

This means $k = 4m + 8$.

By comparing Cases II and III for the constant k we know $\cot^2(\sqrt{2}r) = m - 1$. On the other hand, from Cases III and IV we have

$$4m + 8 = 4m + 6 + h(-\sqrt{2}\tan\sqrt{2}r) - 2\tan^2\sqrt{2}r,$$

which gives

$$\begin{aligned} 2 &= (3\sqrt{2}\cot\sqrt{2}r - \sqrt{2}(2m-1)\tan\sqrt{2}r)(-\sqrt{2}\tan\sqrt{2}r) - 2\tan^2\sqrt{2}r \\ &= -6 + 2(2m-1)\tan^2\sqrt{2}r - 2\tan^2\sqrt{2}r. \end{aligned}$$

Then it follows that $\tan^2\sqrt{2}r = \frac{2}{m-1}$, which gives a contradiction. So real hypersurfaces of type (A) in $G_2(\mathbb{C}^{m+2})$ in Lemma 5.1 cannot be appeared. Now we prove the following.

Theorem 5.2. *Let M be a Hopf pseudo anti-commuting real hypersurface in* $G_2(\mathbb{C}^{m+2})$ *with Reeb vector field* ξ *belonging to the distribution* $\mathfrak{D}^\perp$. *Then* $k = 4m + 2 + \frac{\alpha}{2}(h - \alpha)$.

6. Ricci soliton on real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

Let us recall that an n-dimensional Riemannian manifold (M, g) is said to be *Ricci soliton* if there exists a smooth vector field $V \in T_xM$, $x \in M$ that satisfies

$$\frac{1}{2}(\mathfrak{L}_V g)(X, Y) + \mathrm{Ric}(X, Y) = kg(X, Y), \quad X, Y \in TM$$

where $\mathfrak{L}_V g$ denotes the Lie derivative of g with respect to the vector field V and k a constant (see Chow et al. [15]). We will denote the *Ricci soliton* by (M, g, V, k) and call the vector field V as the potential vector field of the Ricci soliton. A Ricci soliton (M, g, V, k) is said to be a stable, expanding or shrinking according to $k = 0$, $k < 0$ or $k > 0$. It is known that the Ricci tensor S of an Einstein hypersurface M in $G_2(\mathbb{C}^{m+2})$ is given by $S = ag$ for a constant a, that is, $\mathrm{Ric}(X, Y) = ag(X, Y)$ for any X and Y on M and a Riemannian metric g defined on M.

In this section we consider a Ricci soliton on real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with the potential Reeb vector field ξ. Then the Ricci soliton formula gives the following for any vector fields on M

$$\mathrm{Ric}(X, Y) + \frac{1}{2}(\mathfrak{L}_\xi g)(X, Y) = kg(X, Y), \tag{6.1}$$

where $\mathfrak{L}_\xi$ denotes the Lie derivative along the direction of the Reeb vector field ξ. The formula (6.1) becomes

$$\begin{aligned}(4m + 7 - k)X - 3\eta(X)\xi - 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta(\phi_\nu X)\phi_\nu\xi \\ - \eta(X)\eta_\nu(\xi)\xi_\nu\} + hAX - A^2X + \frac{1}{2}(\phi A - A\phi)X = 0.\end{aligned} \tag{6.2}$$

Moreover, from (6.2) it follows that

$$\begin{aligned}S\phi X &= k\phi X + \frac{1}{2}(A\phi - \phi A)\phi X \\ &= k\phi X - \frac{1}{2}(\phi A\phi X + AX - \alpha\eta(X)\xi)\end{aligned}$$

and

$$\begin{aligned}\phi SX &= k\phi X + \frac{1}{2}(\phi A\phi - \phi^2 A)X \\ &= k\phi X + \frac{1}{2}(\phi A\phi X + AX - \alpha\eta(X)\xi).\end{aligned}$$

Then it follows that the Ricci soliton satisfies $(S\phi + \phi S)X = 2k\phi X$, $k =$ const. By using Lemma 3.1, the Reeb vector ξ belongs either to the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$. Then we assert the following.

Lemma 6.1. *Let M be a Hopf Ricci soliton on real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with the potential Reeb field ξ. Then the soliton constant k is given by*

$$k = 4m + h\alpha - \alpha^2 \quad \textit{for } \xi \in \mathfrak{D}^\perp,$$
$$k = 4(m + 1) + h\alpha - \alpha^2 \quad \textit{for } \xi \in \mathfrak{D}$$

Proof. The Ricci soliton constant k in (6.1) becomes

$$k = \mathrm{Ric}(\xi, \xi) = g(S\xi, \xi) = 4(m + 1) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)^2 + hg(A\xi, \xi) - g(A^2\xi, \xi), \tag{6.3}$$

because $(\mathfrak{L}_\xi g)(\xi, \xi) = 0$. We know that the Hopf Ricci soliton satisfies the pseudo anti-commuting, so Lemma 3.1 gives that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$. Then putting $X = \xi$ in (6.2), we assert the results in our lemma according to the Reeb vector $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$ as follows: For $\xi = \xi_1 \in \mathfrak{D}^\perp$ in (6.3) the Ricci curvature $\mathrm{Ric}(\xi, \xi) = g(S\xi, \xi)$ becomes

$$k = \mathrm{Ric}(\xi, \xi) = g(S\xi, \xi) = 4(m + 1) - 4 + h\alpha - \alpha^2,$$

so $k = 4m + h\alpha - \alpha^2$. For $\xi \in \mathfrak{D}$, (6.3) gives $k = 4(m + 1) + h\alpha - \alpha^2$. This gives our assertion. □

From (6.2), together with (6.3), we have

$$\Big\{4\sum_{\nu=1}^{3}\eta_\nu(\xi)^2-\alpha(h-\alpha)+3\Big\}X-3\eta(X)\xi-3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu$$
$$+\sum_{\nu=1}^{3}\big\{\eta_\nu(\xi)\phi_\nu\phi X-\eta(\phi_\nu X)\phi_\nu\xi-\eta(X)\eta_\nu(\xi)\xi_\nu\big\}+hAX-A^2X+\frac{1}{2}(\phi A-A\phi)X=0. \tag{6.4}$$

Then for $\xi\in\mathfrak{D}$, (6.4) gives the following

$$\big\{3-\alpha(h-\alpha)\big\}X-3\eta(X)\xi-3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu-\sum_{\nu=1}^{3}\eta(\phi_\nu X)\phi_\nu\xi+hAX-A^2X+\frac{1}{2}(\phi A-A\phi)X=0. \tag{6.5}$$

Since the Hopf Ricci soliton satisfies $S\phi+\phi S=2k\phi$, by Theorem 1 we have two cases (i) $k=4m+2+\frac{\alpha}{2}(h-\alpha)$ and (ii) real hypersurfaces of type (B) in Theorem A. Now first we consider the latter case (ii). Then we can use all the properties given in Proposition A in Section 4. So we can apply Proposition A to (6.5). Now, putting $X=\xi_2$ in (6.5), we have

$$-\alpha(h-\alpha)\xi_2+(h\beta-\beta^2)\xi_2=0.$$

From this we know that $h=\alpha+\beta=\alpha+3\beta+4(n-1)(\lambda+\mu)$. This implies $\cot r=\tan r$, which means $r=\frac{\pi}{4}$. It gives us a contradiction. So the Ricci soliton cannot be appeared in real hypersurfaces of type (B) in $G_2(\mathbb{C}^{m+2})$.

Next we consider the first case (i) for $\xi\in\mathfrak{D}^\perp$. Since the Ricci soliton satisfies pseudo anti-commuting, we have $S\phi+\phi S=2k\phi$, so Theorem 5.2 gives $k=4m+2+\frac{\alpha}{2}(h-\alpha)$. From this, compared with the first result $k=4m+h\alpha-\alpha^2$ in Lemma 6.1 for the Hopf Ricci soliton, we know that $4=\alpha(h-\alpha)$. So the soliton constant k becomes $k=4(m+1)>0$. This gives that $(M,\xi,g,4(m+1))$ becomes a shrinking Ricci soliton. Then we give a complete proof of Theorem 2 in the introduction. □

Remark 6.2. In the paper due to Pérez, Suh and Watanabe [12] we have given a classification of pseudo-Einstein real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. But it is proper pseudo-Einstein with $c\neq 0$ so it does not satisfy the pseudo-anti commuting formula, because the quaternionic Kähler structure is included in $G_2(\mathbb{C}^{m+2})$.

Remark 6.3. Related to the properties of the Ricci tensor in $G_2(\mathbb{C}^{m+2})$ we have proved the non-existence of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel Ricci tensor in [17]. Motivated by such a geometric property we give a characterization of type (A) in Theorem A by the invariant Ricci tensor, that is, $\mathcal{L}_\xi S=0$ along the flow in the direction of the Reeb vector field ξ (see [18]). Moreover, in [20] we gave a complete classification of real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with harmonic curvature, that is, $\delta S=0$, where δ denotes the adjoint coderivative defined on M in $G_2(\mathbb{C}^{m+2})$.

Remark 6.4. When we consider a ruled real hypersurface $M_R=\Sigma*\times G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, the expression of the shape operator $A\xi=\beta\xi_2$, $A\xi_2=\beta\xi$ and $AX=0$ for any X orthogonal to $\xi=\xi_1$ and ξ_2 (see [21]). So the ruled real hypersurface M_R is not Hopf. Of course, the trace of the shape operator h vanishes and the principal curvature $\alpha=g(A\xi,\xi)$ also vanishes. From such a view point, by Theorem 2, there do not exist any Ricci soliton on ruled real hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

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