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Similarity indices and the Miyazawa polynomials of virtual links

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ABSTRACT

Y. Miyazawa introduced a two-variable polynomial invariant of virtual knots in 2006 [Magnetic graphs and an invariant for virtual links, *J. Knot Theory Ramifications* **15** (2006) 1319–1334] and then generalized it to give a multi-variable one via decorated virtual magnetic graph diagrams in 2008. A. Ishii gave a simple state model for the two-variable Miyazawa polynomial by using pole diagrams in 2008 [A multi-variable polynomial invariant for virtual knots and links, *J. Knot Theory Ramifications* **17** (2008) 1311–1326]. H. A. Dye and L. H. Kauffman constructed an arrow polynomial of a virtual link in 2009 which is equivalent to the multi-variable Miyazawa polynomial [Virtual crossing number and the arrow polynomial, preprint (2008), arXiv:0810.3858v3, http://front.math.ucdavis.edu.]. We give a bracket model for the multi-variable Miyazawa polynomial via pole diagrams and polar tangles similarly to the Ishii's state model for the two-variable polynomial. By normalizing the bracket polynomial we get the multi-variable Miyazawa polynomial $f_K \in \mathbb{Z}[A, A^{-1}, K_1, K_2, \ldots]$ of a virtual link K. n-similar knots take the same value for any Vassiliev invariant of degree $< n$. We show that $f_{K_1} \equiv f_{K_2} \bmod (A^4 - 1)^n$ if two virtual links K_1 and K_2 are n-similar. Also we give a necessary condition for a virtual link to be periodic by using n-similarity of virtual tangles and the Miyazawa polynomial.

Keywords: Virtual link; multi-variable Miyazawa polynomial; polar link; similarity index.

Mathematics Subject Classification 2010: 57M25, 57M27

1. Introduction

A *link* is an embedding of circles S^1 in the three-dimensional Euclidean space $\mathbb{R}^3$. A link K_1 is said to be *isotopic* or *equivalent* to a link K_2 if there is an ambient isotopy of $\mathbb{R}^3$ which transforms K_1 to K_2. In particular, a link with one component is called a *knot*.

A basic problem in knot theory is to distinguish a pair of links, and mathematicians have been trying to find invariants of links. In 1980s, many quantum

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polynomial invariants of knots were introduced [2, 4, 9, 11–13]. In 1990, Vassiliev introduced finite type invariants of knots, called Vassiliev invariants, by using singularity theory and algebraic topology [24].

In 1996, Kauffman introduced virtual knot theory which is an extension of knot theory, and gave several invariants of virtual knots [14]. In particular, he extended the bracket polynomial of links to virtual links and defined Vassiliev invariants of virtual knots.

Miyazawa found a two-variable polynomial invariant of virtual links [16] and independently generalized it by constructing a multi-variable polynomial invariant of virtual links [17]. He gave a lower bound on the virtual crossing number by using the multi-variable polynomial.

Dye and Kauffman also defined the arrow polynomial of a virtual link which is a generalization of the bracket polynomial [3]. The polynomial gives an invariant of (oriented) virtual knots and links. By changing variables suitably, we may get the arrow polynomial from the Miyazawa polynomial and vice versa.

A *virtual link diagram* is closed curves generically immersed in the two-dimensional Euclidean space. A double point of the curve is either a (classical) crossing or a virtual crossing. A virtual crossing is denoted by a singular point surrounded by a small circle. Two virtual link diagrams are said to be *equivalent* if there is a finite sequence of Reidemeister moves and virtual moves transforming one of the two diagrams to the other diagram. See Figs. 1 and 2. *Virtual links* are defined as the equivalence classes of virtual link diagrams under the equivalence relation. A virtual link with one circle component is called a *virtual knot*. Any two equivalent classical knots are isotopic [7, 14]. In this paper, all virtual links are assumed to be oriented.

It is a well-known open problem whether Vassiliev invariants can distinguish all knots or not. Goussarov [5, 6], Habiro [8] and Stanford [22] independently showed that two knots have the same value for all Vassiliev invariants of degree $< n$ if and only if they are n-similar. Thus Vassiliev invariants distinguish all of the knots if and only if for any two different knots K and K' there is a positive integer n such that K and K' are not n-similar. Similarly to the case of classical knots, any pair of virtual links cannot be distinguished by all Vassiliev invariants of degree $< n$.

Ishii gave a bracket polynomial for the Miyazawa polynomial by using pole diagrams and reconstructed the Miyazawa multi-variable polynomial by using pole

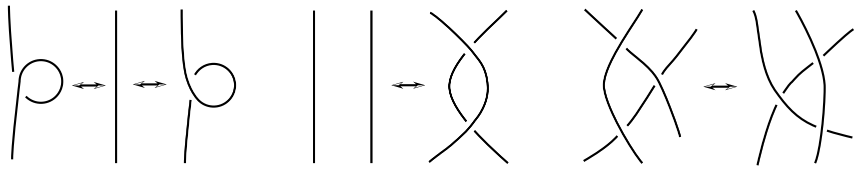

Fig. 1. Reidemeister moves.

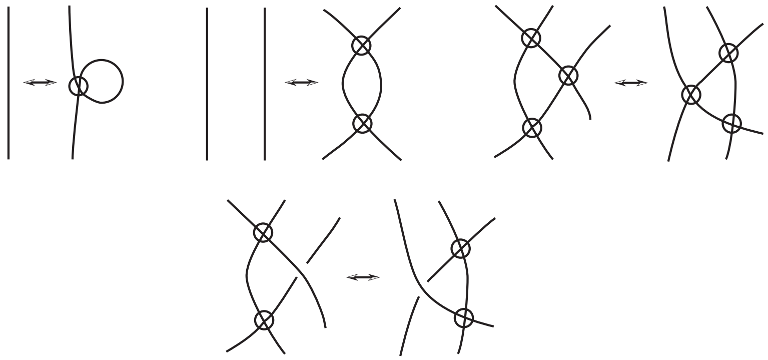

Fig. 2. Virtual moves.

diagrams [10]. A closed curve of a virtual link diagram is called a *strand*. A *pole* on a strand of a virtual link diagram is a unit normal vector with the initial point on the strand. A virtual link diagram allowed to have poles is called a *pole diagram* or a *polar link diagram*. Ishii showed that the virtual crossing number of a virtualized alternating link is determined by its diagram [10].

We can naturally extend Reidemeister moves and virtual moves of virtual links to polar link diagrams. A local move on polar link diagrams as shown in Fig. 3 is called a *polar move*.

Two polar link diagrams are said to be *equivalent* if they are related by a finite sequence of Reidemeister moves, virtual moves and polar moves. A *polar link* is defined to be an equivalent class of polar link diagrams under Reidemeister moves, virtual moves and polar moves.

Miyazawa introduced a decorated virtual magnetic graph (DVMG) diagram and defined a multi-variable polynomial invariant for virtual links via DVMG diagrams [16]. An oriented bivalent graph G in $\mathbb{R}^3$ is said to be *magnetic* if the edges of each component of G are oriented alternately. A diagram of G is called a *magnetic graph diagram*. G is allowed to have components consisting of closed edges with no vertices. A *virtual magnetic graph diagram* (VMG) is a magnetic graph diagram allowed to have virtual crossings. A vertex of a VMG diagram is said to be *oriented* if one of the two edges incident to the vertex is chosen for the vertex. A VMG diagram

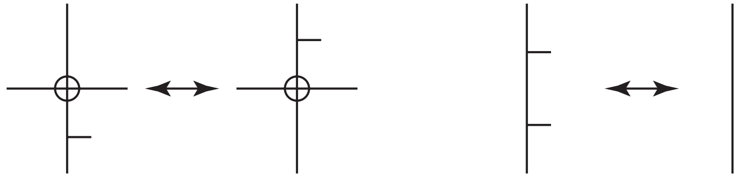

Fig. 3. Polar moves.

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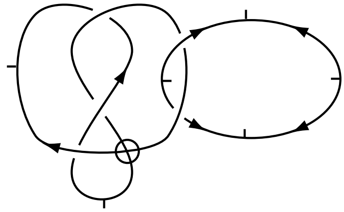

Fig. 4. An oriented pole diagram.

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is said to be *DVMG diagram* if each vertex of the VMG diagram is oriented. In particular, a DVMG diagram without vertices is a virtual link diagram, which may have classical crossings.

Let P be a polar link diagram. Denote by $G(P)$ the 2-valent graph obtained by collapsing each pole of P to a vertex. The preimage of an edge of the graph $G(P)$ in the correspondence is called an *edge* of P. A polar link diagram P whose edges are oriented is said to be *oriented* if the edges of each component of P are oriented alternately. See Fig. 4. From now on, any polar link diagram is assumed to be oriented.

In Sec. 2, we introduce polar tangles and then we define products and closures of polar tangles. Similarly to the Ishii's model for the Miyazawa polynomial, we redefine the multi-variable Miyazawa polynomial for polar links.

In Sec. 3, we introduce a skein module of polar tangles by using the relations of the multi-variable Miyazawa polynomial and then introduce n-similarity for polar tangles and polar links. We show that $f_K \equiv f_L \bmod (A^4 - 1)^n$ for any n-similar virtual links K and L, where f_K is the multi-variable Miyazawa polynomial of a virtual link K. Also we show that if T_1 and T_2 are n-similar virtual (k,k)-tangle diagrams such that T_1^2 is well-defined, then $f_{\overline{T_1^{p^r}}} \equiv f_{\overline{T_2^{p^r}}} \bmod (p, (A^4-1)^{np^r})$ for all primes p and for all positive integers n and r.

2. Polar Tangles and the Multi-variable Miyazawa Polynomial

Let a and b be real numbers with $a < b$. Let $\mathbf{I}$ be the closed interval $[a,b]$ and let k be a positive integer. Fix k points in the upper plane $\mathbf{I}^2 \times \{b\}$ of the cube $\mathbf{I}^3$ and the corresponding k points in the lower plane $\mathbf{I}^2 \times \{a\}$. A (k,k)-*tangle* is obtained by embedding oriented curves and oriented circles in $\mathbf{I}^3$ so that the endpoints of the curves are the fixed $2k$ points. A (k,k)-*tangle diagram* is a projection of a tangle onto a plane with the information of over-strands and under-strands at double points.

A *singular* (k,k)-tangle diagram is a (k,k)-tangle diagram which may have some transverse self-intersections called *double points*.

Given two singular (k,k)-tangle diagrams S and T, the *tangle product* ST is defined to be the tangle obtained by gluing the lower line segment of the rectangle containing S to the upper line segment of the rectangle containing T. The *closure*

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$\overline{T}$ of a tangle diagram T is the knot or link obtained by attaching k parallel strands connecting the k points on the upper line segment of the rectangle and their corresponding k points on the lower line segment of the rectangle in the exterior of the rectangle containing T.

A *virtual tangle diagram* is a tangle diagram allowed to have virtual crossings, denoted as a 4-valent vertex with small circle around it as previously mentioned. Two virtual (k, k)-tangle diagrams are said to be *equivalent* if there is a finite sequence of Reidemeister moves and virtual moves transforming one to the other one. *Virtual* (k, k)*-tangles* are defined to be the equivalence classes of virtual (k, k)-tangle diagrams under the equivalence relation. In particular, a virtual link is a virtual $(0, 0)$-tangle.

A (k, k)-tangle is said to be *oriented* if all of the curves and circles are oriented. We define the *product* and *closure* for oriented tangles if the orientations of curves are matched in the meeting points.

A *pole* on a strand of a virtual tangle diagram is a unit normal vector with the initial point on the strand. A virtual tangle diagram allowed to have poles is called a *polar* (k, k)*-tangle diagram* or *polar tangle diagram* briefly. A *singular* polar (k, k)-tangle diagram is a polar (k, k)-tangle diagram which may have some double points. (*Singular*) *polar tangles* are defined to be the equivalence classes of (singular) polar tangle diagrams under Reidemeister moves, virtual moves and polar moves. An oriented polar tangle is defined similarly to that of oriented polar links. See Fig. 5 for oriented polar $(3, 3)$-tangles. From now on, we assume that any polar tangles are oriented.

Similarly to the case of tangle diagrams we define the product and closure for polar tangle diagrams when the orientations are matched at the meeting points.

Two polar (k, k)-tangles S and T are said to *have the same boundary orientation* if the local orientation of the curve at each endpoint of S is the same as that of the curve at the corresponding endpoint of T. See Fig. 6.

For a polar (k, k)-tangle T, assume that $T^2 = TT$ is defined. T^n will denote the n-times self-product of T for each $n \in \mathbb{N}$, and T^0 will denote the trivial tangle having the same boundary orientation with T. Note that, for a polar (k, k)-tangle T, if the closure $\overline{T}$ is well-defined then T^n and $\overline{T^n}$ are also well-defined.

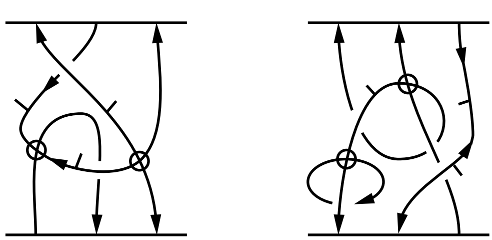

Fig. 5. Oriented polar $(3, 3)$-tangles.

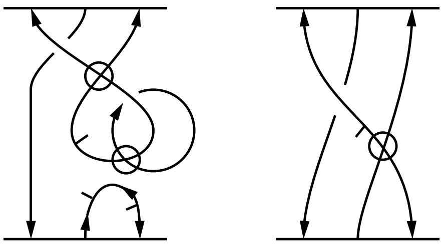

Fig. 6. $(3,3)$-Tangles having the same boundary orientation.

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A polar (k,k)-tangle T is said to be 1-*trivial* if there exists a diagram D of T, which can be changed to the trivial diagram by changing some crossings of D by using Reidemeister moves and virtual moves.

Dye and Kauffman defined the arrow polynomial based on an oriented state expansion of an oriented virtual link diagram. We extend the multi-variable Miyazawa polynomial for polar link diagrams as follows.

Definition 2.1. We define the bracket polynomial of a polar link diagram by using the following relations.

(1) $\langle L_+\rangle = A\langle L_0\rangle + A^{-1}\langle L_\infty\rangle$ and $\langle L_-\rangle = A^{-1}\langle L_0\rangle + A\langle L_\infty\rangle$, where L_+, L_-, L_0, and L_∞ are polar link diagrams as shown in Fig. 7 and A is an indeterminate.
(2) $\langle C_1\rangle = \langle C_2\rangle$ and $\langle C_3\rangle = \langle C_4\rangle$, where C_1, C_2, C_3 and C_4 are diagrams as shown in Fig. 8.
(3) $\langle O\rangle = 1$ and $\langle L\cup O_m\rangle = (-A^2-A^{-2})K_m\langle L\rangle$, where K_m's are indeterminates, O is the trivial knot diagram without crossings and O_m is the polar link diagram with $2m$ poles as shown in Fig. 8.

If a function v from the set of all polar link diagrams to a set takes the same value for any pair of equivalent polar link diagrams, then it is called an *invariant* of polar links. We define the *sign* of a crossing of a polar link diagram as shown in Fig. 9. The *writhe* $w(K)$ of a polar link diagram K is defined to be the sum of signs of all crossings of K.

Miyazawa and Dye and Kauffman showed independently that the polynomial $\langle K\rangle$ of a virtual link is invariant under the Reidemeister moves II and III and the

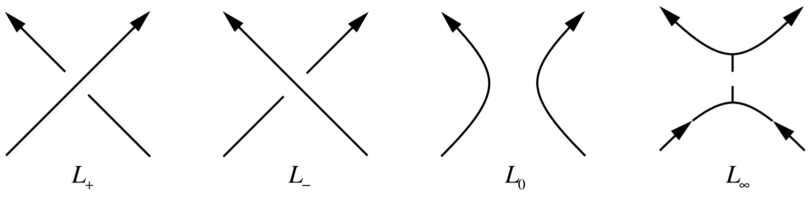

Fig. 7. Splicing a crossing.

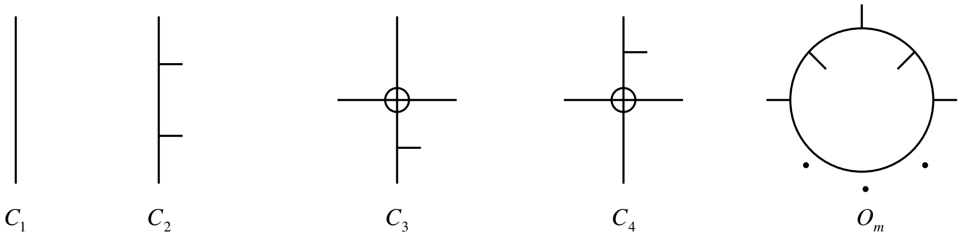

Fig. 8. Poles of polar diagrams.

Fig. 9. The sign of a crossing.

virtual moves in different approaches. They obtained an invariant f_K by normalizing $\langle K\rangle$ by the formula

$$f_K = (-A^3)^{-w(K)}\langle K\rangle.$$

We extend the polynomials $\langle K\rangle$ and f_K for polar link diagrams and polar links, respectively. These two polynomials take values in the polynomial ring $\mathbb{Z}[A, A^{-1}, K_1, K_2, \ldots]$.

3. Similarity Indices of Virtual Knots and Tangles

Two virtual link diagrams are *homotopic* if they are related by a finite sequence of Reidemeister moves, virtual moves and crossing changes of self-crossings. A crossing of a virtual link diagram is called a *self-crossing* if the over-strand and under-strand involved with the crossing belong to the same component. If two virtual links have diagrams which are homotopic then they are said to be *homotopic*. Since crossing change is an unknotting operation for classical knots, any pair of classical knots is homotopic. A *flat virtual link* is the homotopy class of a virtual link. We will see that similarity of virtual links is a generalization of homotopy of virtual links.

In 1990, Ohyama [18] introduced the n-triviality of a knot for each natural number n, gave several necessary conditions for a knot K to be n-trivial by using the coefficients of the Conway polynomial. Ohyama and Ogushi [21] showed that for each natural number n, there are infinitely many n-trivial knots. Taniyama [23] extended the notion of n-triviality of a knot to the n-similarity of links naturally. Ohyama introduced the n-similarity for tangles in [20]. Goussarov [5, 6] introduced

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the n-similarity of knots and showed that it is an equivalence relation on knots and that the set of knots modulo n-similarity forms a group under the connected sum.

Definition 3.1. A *singular virtual link diagram* is a virtual link diagram with finitely many double points and a *singular virtual link* is the equivalence class of a virtual link diagram under the relation generated by Reidemeister moves, virtual moves and moves involved with singular points as illustrated in Fig. 10. Similarly, we define singular polar (k, k)-tangle diagrams and singular polar (k, k)-tangles.

Given a singular virtual knot, link or polar link diagram D, let $A_1, \ldots, A_m$ be disjoint nonempty sets of crossings of D. For $\epsilon_i = \pm$ $(1 \leq i \leq m)$, let $D(A_1^{\epsilon_1}, \ldots, A_m^{\epsilon_m})$ denote the singular knot, link or polar link diagram obtained from D by changing all the crossings in A_i only if $\epsilon_i = -$.

Definition 3.2. A singular virtual link K_1 is said to be *n-similar* to a singular virtual link K_2 for a positive integer n, if there exist a singular virtual link diagram D of K_1 and disjoint nonempty sets $A_1, \ldots, A_n$ of crossings of D such that $D(A_1^{\epsilon_1}, \ldots, A_n^{\epsilon_n})$ is a diagram of K_2 when $\epsilon_i = -$ for some $i \in \{1, \ldots, n\}$.

The *similarity index* of virtual links K and L is the largest such n. In particular if K is the trivial virtual link, then K is said to be *n-trivial*. The *triviality index* $O(K)$ of K is the largest nonnegative integer n such that K is n-trivial.

Ohyama gave knots which are n-trivial as shown in Fig. 11 [18, 21].

Similarly as in the case of singular virtual links, we can define the n-similarity and the similarity index for singular polar (k, k)-tangle diagrams when we replace "link" by "polar tangle" in the above definition. See [18–20, 23] for n-similarity

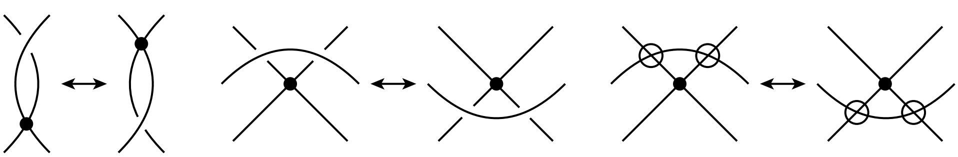

Fig. 10. Moves for singular virtual link diagrams.

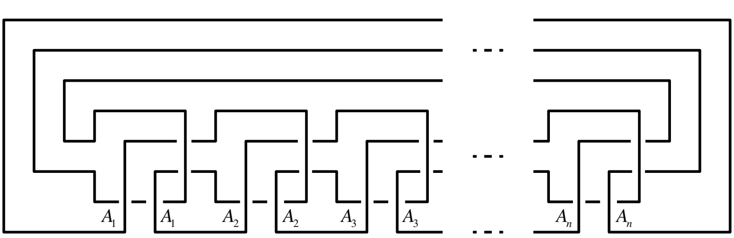

Fig. 11. An n-trivial knot.

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of knots, links and tangles. If a polar link or polar (k,k)-tangle is n-similar to the trivial link or tangle, it is said to be *n-trivial*. The *triviality index* $O(K)$ of a polar link or polar tangle K is also defined to be the similarity index of K and the trivial one. Now we see that two virtual links are homotopic if and only if they are 1-similar.

Let L_+, L_-, L_0 and L_∞ be polar link diagrams as shown in Fig. 7. Since $f_L = (-A^3)^{-w(L)}$ for any polar link diagram L,

$$\langle L_+\rangle = A\langle L_0\rangle + A^{-1}\langle L_\infty\rangle \quad \text{and} \quad \langle L_-\rangle = A^{-1}\langle L_0\rangle + A\langle L_\infty\rangle.$$

Thus we get the following.

Lemma 3.3. *For the quadruple $(L_+, L_-, L_0, L_\infty)$ of polar link diagrams as in Fig. 7, we have equalities*

$$f_{L_+} = -A^{-2}f_{L_0} - A^{-4}f_{L_\infty} \quad \textit{and} \quad f_{L_-} = -A^{2}f_{L_0} - A^{4}f_{L_\infty}.$$

Dye and Kauffman noted that the arrow polynomial gives a flat virtual link invariant if we take $A = 1$ [3]. Let g, h and i be polynomials in the ring $\mathbb{Z}[A, A^{-1}, K_1, K_2, \ldots]$. Let I be the ideal of $\mathbb{Z}[A, A^{-1}, K_1, K_2, \ldots]$ generated by the element i. If $g - h$ is an element of I, we denote it by $g \equiv h \bmod i$. We get a necessary condition for a given pair of virtual links to be homotopic by using the multi-variable Miyazawa polynomial as in the following

Lemma 3.4. *If L_1 and L_2 are homotopic polar links then*

$$f_{L_1} \equiv f_{L_2} \mod A^4 - 1.$$

Proof. It is enough to show that the f-polynomial is invariant modulo $A^4 - 1$ for a crossing change. Let L_+, L_-, L_0 and L_∞ be polar link diagrams as shown in Fig. 7. We compute the f-polynomial for the two links L_+, L_- which differ by a crossing change. Since $f_{L_+} = -A^{-2}f_{L_0} - A^{-4}f_{L_\infty}$ and $f_{L_-} = -A^{2}f_{L_0} - A^{4}f_{L_\infty}$ by Lemma 3.3, we have

$$\begin{aligned} f_{L_+} - f_{L_-} &= (A^2 - A^{-2})f_{L_0} + (A^4 - A^{-4})f_{L_\infty} \\ &\equiv 0 \mod (A^4 - 1). \end{aligned}$$

Therefore, we have $f_{L_+} \equiv f_{L_-} \bmod A^4 - 1$. □

Since every virtual link is a polar link, we have the following.

Theorem 3.5. *If L_1 and L_2 are homotopic virtual links then*

$$f_{L_1} \equiv f_{L_2} \mod A^4 - 1.$$

Example 3.6. Let K_1 and K_2 be the two virtual links as shown in Fig. 12. Since $f_{K_1} = 1 - A^4 + (-A^{-4} - 1)K_1^2 \equiv -2K_1^2 \bmod (A^4 - 1)$ and $f_{K_2} = -A^2 - A^{-2} \equiv -2A^2 \bmod (A^4 - 1)$, we see that K_1 and K_2 are not homotopic.

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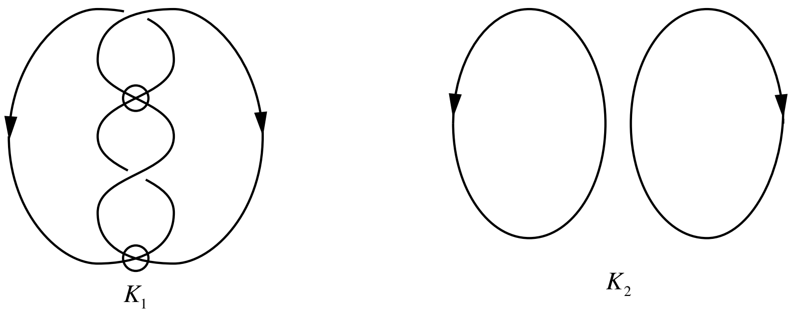

Fig. 12. Non-homotopic links.

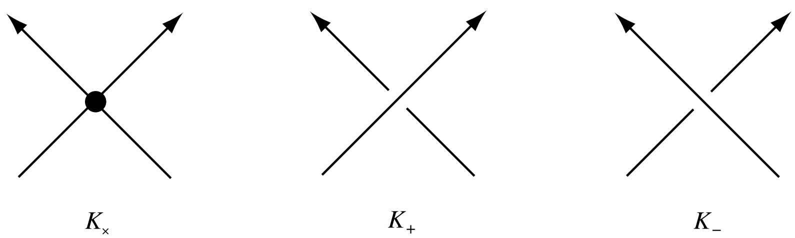

Fig. 13. $K_{(\times)}$, K_+ and K_-.

A polar link invariant v taking values in a module can be extended to a singular polar link invariant by using the Vassiliev skein relation: $v(K_\times) = v(K_+) - v(K_-)$, where $K_\times, K_+$ and K_- are singular polar link diagrams which are identical except the indicated local parts in Fig. 13.

Definition 3.7. A polar link invariant v taking values in a module is called a *Vassiliev invariant of degree* n if n is the smallest nonnegative integer such that v vanishes on singular polar links with more than n double points. A polar link invariant v is called a *Vassiliev invariant* if v is a Vassiliev invariant of degree n for some nonnegative integer n.

For each singular polar (k, k)-tangle T, let $\mathcal{T}_T$ denote the set of all singular polar (k, k)-tangles which have the same boundary orientation with T. Also $\mathbb{Z}\mathcal{T}_T$ denotes the free $\mathbb{Z}$-module generated by $\mathcal{T}_T$. Let $\mathcal{V}$ be the submodule of $\mathbb{Z}\mathcal{T}_T$ generated by the relation $L_\times = L_+ - L_-$, where $L_\times, L_+$ and L_- are singular polar (k, k)-tangles as shown in Fig. 14. For a singular polar (k, k)-tangle $T' \in \mathcal{T}_T$, we denote the equivalence class of T' in $\mathbb{Z}\mathcal{T}_T/\mathcal{V}$ by $[T']$.

Let $\mathbb{H}$ be the ring $\mathbb{Z}[A, A^{-1}, K_1, K_2, \ldots]$ of polynomials with indeterminates $A, K_1, K_2, \ldots$. For a given set S, we denote by $\mathbb{H}S$ the free $\mathbb{H}$-module generated by the set S.

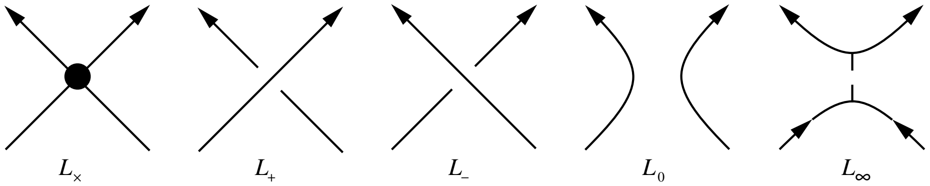

Fig. 14. Crossings and splicings.

For the module $\mathbb{H}\mathcal{T}_T$, define $\mathcal{R}$ to be the submodule of $\mathbb{H}\mathcal{P}$ generated by the relations:

$$L_\times = L_+ - L_-, \quad L_+ = -A^{-2}L_0 - A^{-4}L_\infty \quad \text{and} \quad L_- = -A^2L_0 - A^4L_\infty,$$

where $L_\times, L_+, L_-, L_0$ and L_∞ are the tangles which are identical except the indicated local parts as shown in Fig. 14. Let $\mathbb{H}\mathcal{T}_T/\mathcal{R}$ denote the quotient module of $\mathbb{H}\mathcal{T}_T$ by $\mathcal{R}$. For $T' \in \mathcal{T}_T, [T']$ denotes the equivalence class in the quotient module $\mathbb{H}\mathcal{T}_T/\mathcal{R}$. Recall that we have defined $[T']$ in the module $\mathbb{Z}\mathcal{T}_T/\mathcal{V}$. It would be clear from context that whether $[T']$ belongs to $\mathbb{Z}\mathcal{T}_T/\mathcal{V}$ or $\mathbb{H}\mathcal{T}_T/\mathcal{R}$.

Note that if T^2 is well-defined, the product operation on $\mathcal{T}_T$ induces the product operation on $\mathbb{Z}\mathcal{T}_T$.

Notations. Let T be a polar (k,k)-tangle diagram allowed to have singular crossings.

(1) Let $\{x_1, \ldots, x_l\}$ be a set of crossings of T. Let $T(\{x_1^-, \ldots, x_{j-1}^-, x_j^\times, x_{j+1}^+, \ldots, x_l^+\})$ be the singular polar (k,k)-tangle diagram obtained from T by changing the crossings $x_1, \ldots, x_{j-1}$ and collapsing the crossing x_j to a double point.

(2) Let $S = \sum_{i=1}^m a_iT_i \in \mathbb{Z}\mathcal{T}_T$ be a $\mathbb{Z}$-linear combination of polar (k,k)-tangle diagrams $T_1, \ldots, T_m$ allowed to have singular points. Assume that each T_i has crossings labeled by $x_1, \ldots, x_l$. We define $S(\{x_1^-, \ldots, x_{j-1}^-, x_j^\times, x_{j+1}^+, \ldots, x_l^+\})$ by the equation

$$\begin{aligned} &S(\{x_1^-, \ldots, x_{j-1}^-, x_j^\times, x_{j+1}^+, \ldots, x_l^+\}) \\ &\quad = \sum_{i=1}^m a_iT_i(\{x_1^-, \ldots, x_{j-1}^-, x_j^\times, x_{j+1}^+, \ldots, x_l^+\}). \end{aligned}$$

(3) Let $\{A_1, A_2, \ldots, A_n\}$ be a collection of disjoint nonempty sets of crossings of T. Let $A_j = \{x_{j1}, \ldots, x_{j\alpha_j}\}$ for $j = 1, \ldots, n$ and ϵ_{jk} be the sign of the crossing x_{jk} for $k = 1, \ldots, \alpha_j$. We define $T(A_1^\times, \ldots, A_n^\times) \in \mathbb{Z}\mathcal{T}_T$ inductively by the following rules:

$$T(A_1^\times) = \sum_{j=1}^{\alpha_1} \epsilon_{1j} T(\{x_{11}^-, \ldots, x_{1j-1}^-, x_{1j}^\times, x_{1j+1}^+, \ldots, x_{1\alpha_1}^+\}) \quad \text{and}$$

for $k = 2, \ldots, n$,

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$$T(A_1^\times, \ldots, A_k^\times) = \sum_{j=1}^{\alpha_k} \epsilon_{kj} T(A_1^\times, \ldots, A_{k-1}^\times)$$
$$\times (\{x_{k1}^-, \ldots, x_{kj-1}^-, x_{kj}^\times, x_{kj+1}^+, \ldots, x_{k\alpha_k}^+\}).$$

Lemma 3.8. *Let T_1 and T_2 be two n-similar polar (k, k)-tangle diagrams. Assume that a collection $\{A_1, A_2, \ldots, A_n\}$ of disjoint nonempty sets of crossings of T_1 gives the n-similarity. Then we have the following equality in $\mathbb{Z}\mathcal{T}_T/\mathcal{V}$*

$$[T_1] - [T_2] = [T_1(A_1^\times, \ldots, A_n^\times)].$$

Proof. We use mathematical argument on n. Let T_1 and T_2 be 1-similar via a set $A_1 = \{x_{11}, \ldots, x_{1\alpha_1}\}$ of crossings of T_1. For each $j = 1, \ldots, \alpha_1$, we have formula in $\mathbb{Z}\mathcal{T}_T/\mathcal{V}$ as following

$$\begin{aligned}
&\epsilon_{1j}[T_1(\{x_{11}^-, \ldots, x_{1j-1}^-, x_{1j}^\times, x_{1j+1}^+, \ldots, x_{1\alpha_1}^+\})] \\
&\quad = [T_1(\{x_{11}^-, \ldots, x_{1j-1}^-, x_{1j}^+, x_{1j+1}^+, \ldots, x_{1\alpha_1}^+\})] \\
&\qquad - [T_1(\{x_{11}^-, \ldots, x_{1j-1}^-, x_{1j}^-, x_{1j+1}^+, \ldots, x_{1\alpha_1}^+\})].
\end{aligned}$$

Now we sum the equalities for $j = 1, \ldots, \alpha_1$ and get the equalities

$$\begin{aligned}
[T_1] - [T_2] &= [T_1(\{x_{11}^+, \ldots, x_{1\alpha_1}^+\})] - [T_1(\{x_{11}^-, \ldots, x_{1\alpha_1}^-\})] \\
&= \sum_{j=1}^{\alpha_1} \epsilon_{1j}[T_1(\{x_{11}^-, \ldots, x_{1j-1}^-, x_{1j}^\times, x_{1j+1}^+, \ldots, x_{1\alpha_1}^+\})] \\
&= T_1(A_1^\times).
\end{aligned}$$

Assume that the statement holds for $(n-1)$-similar polar (k, k)-tangle diagrams. Let T_1 and T_2 be n-similar polar (k, k)-tangle diagrams and the family $\{A_1, A_2, \ldots, A_n\}$ of disjoint nonempty sets of crossings of T_1 gives the n-similarity, where $A_j = \{x_{j1}, \ldots, x_{j\alpha_j}\}$ for $j = 1, \ldots, n$. Since for $j = 1, \ldots, \alpha_n$,

$$\begin{aligned}
&\epsilon_{nj}[T_1(A_1^\times, \ldots, A_{n-1}^\times)(\{x_{n1}^-, \ldots, x_{nj-1}^-, x_{nj}^\times, x_{nj+1}^+, \ldots, x_{n\alpha_n}^+\})] \\
&\quad = [T_1(A_1^\times, \ldots, A_{n-1}^\times)(\{x_{n1}^-, \ldots, x_{nj-1}^-, x_{nj}^+, x_{nj+1}^+, \ldots, x_{n\alpha_n}^+\})] \\
&\qquad - [T_1(A_1^\times, \ldots, A_{n-1}^\times)(\{x_{n1}^-, \ldots, x_{nj-1}^-, x_{nj}^-, x_{nj+1}^+, \ldots, x_{n\alpha_n}^+\})],
\end{aligned}$$

we get the following equalities in $\mathbb{Z}\mathcal{T}_T/\mathcal{V}$

$$\begin{aligned}
&[T_1(A_1^\times, \ldots, A_n^\times)] \\
&\quad = \sum_{j=1}^{\alpha_n} \epsilon_{nj}[T_1(A_1^\times, \ldots, A_{n-1}^\times) \\
&\qquad \times (\{x_{n1}^-, \ldots, x_{nj-1}^-, x_{nj}^\times, x_{nj+1}^+, \ldots, x_{n\alpha_n}^+\})]
\end{aligned}$$

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$$\begin{aligned}
&= [T_1(A_1^\times, \dots, A_{n-1}^\times)(\{x_{n1}^+, \dots, x_{n\alpha_n}^+\})] \\
&\quad - [T_1(A_1^\times, \dots, A_{n-1}^\times)(\{x_{n1}^-, \dots, x_{n\alpha_n}^-\})] \\
&= ([T_1] - [T_2]) - ([T_2] - [T_2]) \\
&= [T_1] - [T_2].
\end{aligned}$$

Note that $[T_1(A_1^\times, \dots, A_{n-1}^\times)(\{x_{n1}^+, \dots, x_{n\alpha_n}^+\})] = [T_1] - [T_2]$, by inductive hypothesis, because $\{A_1, \dots, A_{n-1}\}$ gives an $(n-1)$-similarity of T_1 and T_2 for $T_1(\{x_{n1}^+, \dots, x_{n\alpha_n}^+\})$, and that $[T_1(A_1^\times, \dots, A_{n-1}^\times)(\{x_{n1}^-, \dots, x_{n\alpha_n}^-\})] = [T_2] - [T_2]$, by inductive hypothesis, because $\{A_1, \dots, A_{n-1}\}$ gives an $(n-1)$-similarity of T_2 and T_2 for $T_1(\{x_{n1}^-, \dots, x_{n\alpha_n}^-\})$. ☐

Let $\mathcal{P}$ be the set of singular polar link diagrams modulo the equivalence relation of polar link diagrams and let $\mathbb{H}\mathcal{P}$ be the free $\mathbb{H}$-module generated by $\mathcal{P}$. We extend the invariant f on $\mathbb{H}\mathcal{P}$ linearly. Let $\mathcal{R}$ be the submodule of $\mathbb{H}\mathcal{P}$ generated by the set of relations:

$$L_\times = L_+ - L_-, \quad L_+ = -A^{-2}L_0 - A^{-4}L_\infty \quad \text{and}$$
$$L_- = -A^2 L_0 - A^4 L_\infty,$$

where $L_\times, L_+$ and L_- are singular polar links which are identical except for the shown local part in Fig. 14. By Lemma 3.3, we can induce a map $\overline{f}$ from f on the quotient module $\mathbb{H}\mathcal{P}/\mathcal{R}$ by the assigning rule $\overline{f}_{[L]} = f_L$ for each $L \in \mathcal{P}$. For a polar link diagram P, we denote the equivalence class of P in $\mathbb{H}\mathcal{P}/\mathcal{R}$ by $[P]$.

A polar (k,k)-tangle diagram L is said to be *n-singular* if it has n double points. For two elements a and b in commutative ring R, a is said to be a *multiple* of b, denoted by $b|a$, if there exists an element $c \in R$ satisfying $a = bc$.

Lemma 3.9. *If L is an n-singular polar (k,k)-tangle diagram, then there exist $a_i \in \mathbb{H}$ and $(n-1)$-singular polar (k,k)-tangle diagrams L_i $(i = 1, \dots, l)$ such that $(A^4 - 1) \mid a_i$ and $[L] = \sum_{i=1}^{1} a_i[L_i]$ in the module $\mathbb{H}\mathcal{T}_T/\mathcal{R}$.*

Proof. Let $L = (L_\times), L_+, L_-, L_0$ and L_∞ be polar (k,k)-tangle diagrams which differ locally as shown in Fig. 14.

Then in the module $\mathbb{H}\mathcal{T}_T/\mathcal{R}$, by Lemma 3.3 we have

$$\begin{aligned}
[L_\times] &= [L_+] - [L_-] \\
&= (-A^{-2}P_{L_0} - A^{-4}P_{L_\infty}) - (-A^2 P_{L_0} - A^4 P_{L_\infty}) \\
&= (A^2 - A^{-2})[L_0] + (A^4 - A^{-4})[L_\infty] \\
&= (A^4 - 1)A^{-2}[L_0] + (A^4 - 1)(1 + A^{-4})[L_\infty].
\end{aligned}$$

If $L_\times$ is an n-singular polar link diagram then we see that L_0 and L_∞ are $(n-1)$-singular polar link diagrams. ☐

In particular, an n-singular polar link is an n–singular polar $(0,0)$-tangle and we can apply Lemma 3.9 for singular polar links.

Lemma 3.10. *Let K and L be two n-similar polar link diagrams. Then we have a congruence relation*

$$f_K \equiv f_L \mod (A^4-1)^n.$$

Proof. If K and L are n-similar polar link diagrams then by Lemma 3.8, $[K]-[L]$ is given as a linear combination of equivalence classes of n-singular polar link diagrams. By applying Lemma 3.9 repeatedly we see that there exist $b_i \in \mathbb{Z}[A, A^{-1}, K_1, \ldots]$ and polar link diagrams L_i $(i = 1, \ldots, l)$ such that

$$(A^4-1)^n \mid b_i \quad \text{and} \quad [K]-[L] = \sum_{i=1}^{1} b_i[L_i]$$

in the module $\mathbb{H}\mathcal{P}/\mathcal{R}$. By Lemma 3.3, the f-polynomial vanishes for any elements of $\mathcal{R}$. Therefore, we see that

$$f_K - f_L = \overline{f}_{[K]} - \overline{f}_{[L]} = \overline{f}_{[K]-[L]} = \sum_{i=1}^{l} b_i \overline{f}_{[L_i]} = \sum_{i=1}^{l} b_i f_{L_i} \equiv 0 \mod (A^4-1)^n. \qquad \square$$

Since every virtual link diagram is a polar link diagram, we have the following.

Theorem 3.11. *Let K and L be two n-similar virtual link diagrams. Then we have a congruence relation*

$$f_K \equiv f_L \mod (A^4-1)^n.$$

Definition 3.12. A polar link L is said to be *n-periodic* if there exists a diagram of the polar link which is unchanged under the rotation of $\frac{2\pi}{n}$.

If L is n-periodic then there exists a polar tangle T such that $L = \overline{T^n}$.

We give a necessary condition for two polar (k,k)-tangles T_1 and T_2 to be n-similar by using periodic polar links. Assume that two polar tangles T_1 and T_2 are n-similar then they have the same boundary orientation. Therefore, the closure of T_1 is well-defined if and only if the closure of T_2 is well-defined. Moreover, the closure of T_1 is well-defined if and only if T_1^2 is well-defined. We extend the closing operation for the set of singular polar tangles to the module $\mathbb{Z}\mathcal{T}_{T_1}$ linearly. Let α and β be elements in the ring $\mathbb{Z}[A, A^{-1}, K_1, K_2, \ldots]$. For two polynomials $f_1, f_2 \in \mathbb{Z}[A, A^{-1}, K_1, K_2, \ldots]$ denote $f_1 \equiv f_2 \bmod (\alpha, \beta)$ if $f_1 - f_2$ belongs to the ideal of the ring $\mathbb{Z}[A, A^{-1}, K_1, K_2, \ldots]$ generated by $\alpha - \beta$.

Lemma 3.13. *Let p be a prime and n and r be positive integers. Assume that T_1 and T_2 are n-similar polar (k,k)-tangle diagrams such that T_1^2 is well-defined. Then*

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we get the following relation

$$f_{\overline{T_1^{p^r}}} \equiv f_{\overline{T_2^{p^r}}} \mod (p, (A^4-1)^{np^r}).$$

Proof. In the module $\mathbb{Z}\mathcal{T}_T/\mathcal{V}$, by Lemma 3.8 we have

$$[T_1] - [T_2] = [T_1(A_1^\times, \ldots, A_n^\times)].$$

Then we get a congruence relation in the quotient module $\mathbb{Z}\mathcal{T}_T/\mathcal{V}$:

$$\begin{aligned}[T_1^{p^r}] - [T_2^{p^r}] &\equiv [(T_1 - T_2)^{p^r}] \mod p\\ &= [(T_1(A_1^\times, \ldots, A_n^\times))^{p^r}].\end{aligned}$$

Since $\overline{T_1^{p^r}} - \overline{T_2^{p^r}} \equiv \overline{(T_1(A_1^\times, \ldots, A_n^\times))^{p^r}} \mod p$ in $\mathbb{H}\mathcal{P}/\mathcal{R}$ and since the quotient map $\overline{f} : \mathbb{H}\mathcal{P}/\mathcal{R} \to \mathcal{P}$ is induced from the linear map $f : \mathbb{H}\mathcal{P} \to \mathcal{P}$ which vanishes on $\mathcal{R}$, we get a congruence relation

$$\begin{aligned}f_{\overline{T_1^{p^r}}} - f_{\overline{T_2^{p^r}}} &= \overline{f}_{\overline{[T_1^{p^r}]}} - \overline{f}_{\overline{[T_2^{p^r}]}}\\ &\equiv \overline{f}_{\overline{[(T_1(A_1^\times,\ldots,A_n^\times))^{p^r}]}} \mod p\\ &= f_{\overline{(T_1(A_1^\times,\ldots,A_n^\times))^{p^r}}}.\end{aligned}$$

Since $\overline{(T_1(A_1^\times, \ldots, A_n^\times))^{p^r}}$ is a linear combination of np^r-singular links,

$$f_{\overline{(T_1(A_1^\times,\ldots,A_n^\times))^{p^r}}} \equiv 0 \mod (A^4-1)^{np^r} \quad \text{by Lemma 3.9.}$$

Hence we get the formula $f_{\overline{T_1^{p^r}}} \equiv f_{\overline{T_2^{p^r}}} \mod (p, (A^4-1)^{np^r})$. □

Since every virtual tangle diagram is a polar tangle diagram, we have the following.

Theorem 3.14. *Let p be a prime and n and r be positive integers. Assume that T_1 and T_2 are n-similar virtual (k,k)-tangle diagrams such that T_1^2 is well-defined. Then we get the following relation*

$$f_{\overline{T_1^{p^r}}} \equiv f_{\overline{T_2^{p^r}}} \mod (p, (A^4-1)^{np^r}).$$

For a polar tangle diagram T, the mirror image of T, denoted by T^*, is the tangle diagram obtained by changing all crossings of T. For a polar link L, from the definition of the f-polynomial, we get

$$f_{L^*}(A, K_1, K_2, \ldots) = f_L(A^{-1}, K_1, K_2, \ldots).$$

Example 3.15. Let T_1 and T_2 be the tangle diagrams as shown in Fig. 15. Then T_2 is the mirror image of T_1 and T_1 and T_2 are 1-similar. Let L be the 2-periodic virtual link $\overline{T^2}$. Suppose that T_1 and T_2 are 2-similar. Then by Theorem 3.13, we have

$$f_L(A, K_1, K_2, \ldots) \equiv f_L(A^{-1}, K_1, K_2, \ldots) \mod (2, (A^4-1)^4).$$

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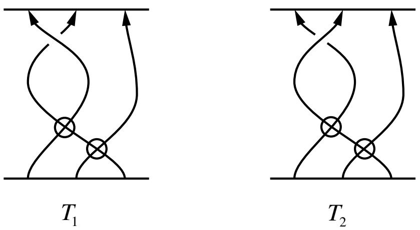

Fig. 15. 1-similar tangles.

But $f_L(A, K_1, K_2, \ldots)$ and $f_{L^*}(A, K_1, K_2, \ldots)$ are not congruent mod $(2, (A^4-1)^4)$ because

$$\begin{aligned} f_L &= (-A^{10} - A^6)K_2 - 2(A^8 + A^4)K_1 + A^4 \\ &\equiv (A^{10} + A^6)K_2 + A^4 \mod (2, (A^4-1)^4) \end{aligned}$$

and

$$\begin{aligned} f_{L^*} &= (-A^{-10} - A^{-6})K_2 - 2(A^{-8} + A^{-4})K_1 + A^{-4} \\ &\equiv (A^{10} + A^6)K_2 + A^{12} \mod (2, (A^4-1)^4). \end{aligned}$$

Therefore, we see that the similarity index of T_1 and T_2 is 1.

If T and T^* are n-similar and $L = \overline{T^p}$, then by applying Lemma 3.13, we get the following.

Corollary 3.16. *Let p be a prime and let n and r be positive integers. Assume that a polar link $L = \overline{T^{p^r}}$ is p^r-periodic and T and T^* are n-similar polar tangles. Then*

$$f_L(A, K_1, K_2, \ldots) \equiv f_L(A^{-1}, K_1, K_2, \ldots) \mod (p, (A^4-1)^{p^r}).$$

For any virtual tangle T, T and T^* are 1-similar. Therefore, we get a necessary condition for a virtual link to be periodic by using the multi-variable Miyazawa polynomial as in the following.

Corollary 3.17. *Let p be a prime and n and r be positive integers. If a virtual link L is p^r-periodic, then*

$$f_L(A, K_1, K_2, \ldots) \equiv f_L(A^{-1}, K_1, K_2, \ldots) \mod (p, (A^4-1)^{p^r}).$$

In 2009, Kim, Lee and Seo gave several necessary conditions for a virtual link to be periodic by using the two-variable Miyazawa polynomial [15]. Corollary 3.17 is a generalization of one of their theorems for periodic virtual links.

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