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Arc index via the alternating tangle decomposition

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ABSTRACT

We introduce the alternating tangle decomposition of a diagram of a link L and improve the upper bound of arc index $\alpha(L)$ by using information of the alternating tangle decomposition. Also we get the exact arc index of a class of links by combining the upper bound with Morton and Beltrami's lower bound of the arc index.

Keywords: Arc index; arc-presentation; wheel diagram; alternating tangle decomposition.

Mathematics Subject Classification 2010: 57M25, 57M27

1. Introduction

Consider the open-book decomposition of S^3 which has open disks as pages and an unknotted circle as a binding. It can be easily shown that every link L can be embedded in an open-book with finitely many pages so that it meets each page in a simple arc. Such an embedding is called an *arc-presentation* of L, see Fig. 1. The *arc index* $\alpha(L)$ of a link L is the minimum number of pages in any arc-presentation for L.

While a link diagram is a presentation of a link with a finite number of singular points (crossings) with multiplicity 2, an arc-presentation with n-pages (and hence a wheel diagram) is a presentation of a link with only one singular point (the binding) with multiplicity n, in which every edge incident to the binding is assigned with a real number (the relative height with respect to the binding). Brunn [6] is the first person who used such a presentation of knots, and Cromwell [7] gave the formal definition of the arc index of a link.

Recently there are many researchers who are studying the multi-crossing projections of links, which is a presentation of a link with finite singular points with

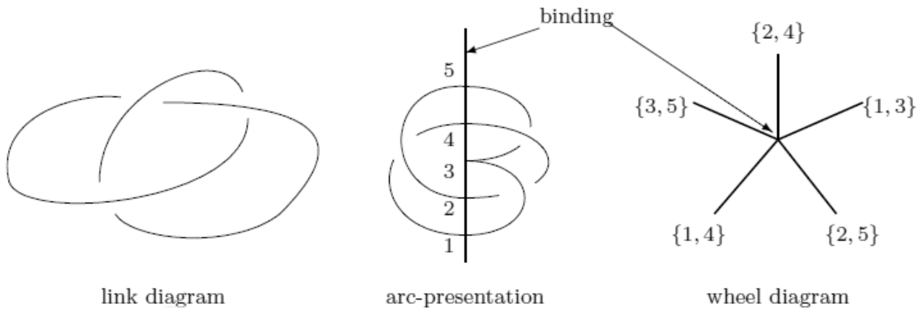

Fig. 1. Various presentations of a link.

multiplicity n [1–3]. Arc index is also closely related with the Thurston–Bennequin number, knot Floer homology and Khovanov homology [9, 10, 12].

In 1996, Cromwell and Nutt [8] found an upper bound on the arc index in terms of the minimal crossing number $c(L)$ and, in 2000, the author and Park [4] showed that for any prime link L, $\alpha(L) \leq c(L) + 2$, and this inequality is strict if and only if L is not alternating. Beltrami [5] improved the upper bound of $\alpha(L)$ for a special class of links, in which all of them are adequate.

In this paper we will introduce the *alternating tangle decomposition* of a link diagram D and improve the upper bound of $\alpha(L)$ by using the alternating tangle decomposition.

Theorem 1.1. *Let D be a connected reduced link diagram with the alternating tangle decomposition $(G_D; T_1, \ldots, T_n)$. Suppose that T_i is strongly reduced and fat for each i. Then*

$$\alpha(D) \leq c(D) + 2n - \nu,$$

where ν denotes the number of non-alternating edges in D.

By combining the above theorem and the lower bound of the arc index of a link obtained by Morton and Beltrami [11], one can get the exact arc index of a class of links.

2. Alternating Tangle Decomposition

Let D be a diagram of a link. Then one can see that D consists of finite number of alternating tangles $T_1, \ldots, T_n$ which are connected by non-alternating edges. By contracting each alternating tangle to a vertex, we get a planar graph G_D, called a *connecting graph* of D. Note that G_D is even-valent and bipartite. We will denote D as $(G_D; T_1, \ldots, T_n)$ and call it *the alternating tangle decomposition* of D, see Fig. 2. Notice that every diagram can be decomposed into an alternating

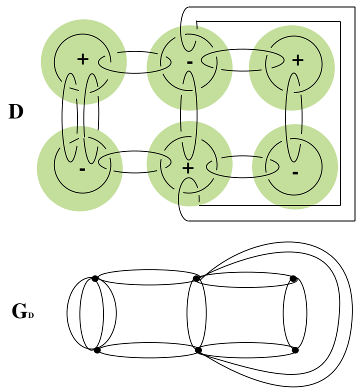

Fig. 2. Alternating tangle decomposition.

tangle decomposition. Conversely, let Γ be a planar even-valent bipartite graph with vertices $v_1, \ldots, v_n$ and let $T_1, \ldots, T_n$ be n alternating tangles. If valency of v_i is k_i and if T_i is a k_i-tangle, then one can construct a link diagram D whose alternating tangle decomposition is $(G_D; T_1, \ldots, T_n)$. Notice that a link diagram presented by an alternating tangle decomposition is not unique.

An alternating tangle T is said to be *strongly alternating* if both of its natural closures are reduced and alternating. A tangle T is said to be *fat* if it does not have any isthmus (cut edge). See Fig. 3.

One can see that if each T_i is strongly alternating, the diagram $(G_D; T_1, \ldots, T_n)$ is adequate, and hence it is a minimal diagram. An *n-semi-alternating* diagram of a link is the diagram which is decomposed into two strongly alternating n-tangles.

In 2002, Beltrami [5] calculated the arc index of semi-alternating links, which is a special case of one of our main results.

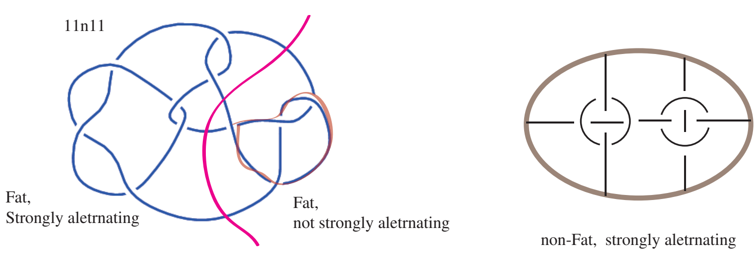

Fig. 3. Fat tangle, strongly alternating tangle.

Proposition 2.1. (1) *If L is a semi-alternating* ($= 2$*-semi-alternating*) *link*, *then* $\alpha(L) = c(L)$.

(2) *If L is an n-semi-alternating link and if the constructing tangles are fat*, *then* $\alpha(L) = c(L) - 2n + 4$.

3. Bound for the Arc Index

Note that a diagram is a presentation of a link with a finite number of singular points (crossings) with multiplicity 2 and an arc-presentation with n-pages (and hence a wheel diagram) is a presentation of a link with only one singular point (the binding) with multiplicity n, in which every edge incident to the binding is assigned with a real number (= the relative height with respect to the binding). It is clear that an arc-presentation is presented by a wheel diagram precisely and vise versa.

In [4], an algorithm was presented for constructing an arc-presentation from a link diagram of a link by fixing a singular point (= a crossing) and assigning relative heights on the edges incident to the crossing, and after then contracting neighboring crossings to the fixed crossing with an assignment of relative heights successively. Note that the contraction of a neighboring crossing to the fixed crossing corresponds to contractions of edges between the two crossings.

To explain the algorithm efficiently, we introduce two basic contraction moves, the *edge contraction* and the *triangle contraction*.

A *knot and spoke diagram* [4] is a planar graph G with specific vertex c_0, called *the binding vertex*, satisfying the following conditions:

(i) all vertices except c_0 are either univalent or 4-valent;
(ii) every univalent vertex is adjacent to c_0 by an edge, called a *spoke*, labeled by two different numbers;
(iii) every 4-valent vertex has under–over information as a link diagram;
(iv) every edge incident to c_0 is labeled by a number if it is not a spoke;
(v) every labeled number is used exactly twice.

By considering the labeled numbers as a relative heights, one can easily see that a knot and spoke diagram can be realized as a link L in $\mathbb{R}^3$. Also if there are only spokes, it is a wheel diagram, and hence an arc-presentation of L. From now on, we will introduce two basic contraction deformations of a knot and spoke diagram to get a wheel diagram.

Definition 3.1 (Edge contraction). Let D be a knot and spoke diagram with the specific vertex c_0. Choose an edge e which is incident to c_0. Note that the end point of e meeting c_0 is assigned the relative height, say a, and the other end point of e meets three other edges, say e_1, e_2, e_3, so that e_1e_3 and ee_2 form parts of the link. By contracting the edge e to c_0, we get a new diagram D/e, in which e_1, e_2 and e_3 are incident to c_0. For being D/e a knot and spoke diagram, we assign relative

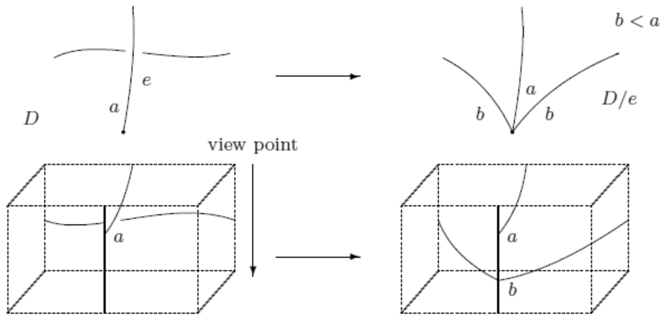

Fig. 4. Edge contraction.

heights at e_1, e_2 and e_3 as follows:

(i) assign the relative height a at e_2;
(ii) assign the relative height b at e_1 and e_3 that is less than (respectively, greater than) any height already used if e_1e_3 is undercross (respectively, overcross) ee_2.

See Fig. 4 for details.

Definition 3.2 (Triangle contraction). Let D be a knot and spoke diagram with the specific vertex c_0. Suppose that there are two edges e and f which is incident to c_0 and an edge h so that e, f and h forms a triangle. Suppose that both end points of h are undercross or both overcross. Note that the end points of e and f meeting c_0 are assigned the relative heights, say a and b, respectively. Let e_1, e_2, e_3 and e_4 denote the edges of D that e_1he_4, ee_2 and fe_3 form a part of the link. By contracting the triangle $T = efh$ to c_0, we get a new diagram D/T, in which e_1, e_2, e_3 and e_4 are incident to c_0. For being D/T a knot and spoke diagram, we assign relative heights at e_1, e_2, e_3 and e_4 as follows:

(i) assign the relative height a at e_2;
(ii) assign the relative height b at e_3;
(iii) assign the relative height c at e_1 and e_4 that is less than (respectively, greater than) any height already used if both end points of h are undercross (respectively, overcross).

See Fig. 5 for details.

In the edge contraction or the triangle contraction, if the other end of the edge e_1 is incident to c_0, in other word, if e and e_1 form a *bigon*, then e_1 is changed to a loop in the resulting diagram. We need to change the edge e_1 into a spoke to get the knot and spoke diagram D/e or D/T, as seen in Fig. 6. Notice that $R(D/e) = R(D)-1, S(D/e) = S(D)+1$ and $R(D/T) = R(D)-2, S(D/T) = S(D)+1$, where $R(D)$ and $S(D)$ denote the number of regions in $D \subset S^2$ and the number of spokes in D, respectively.

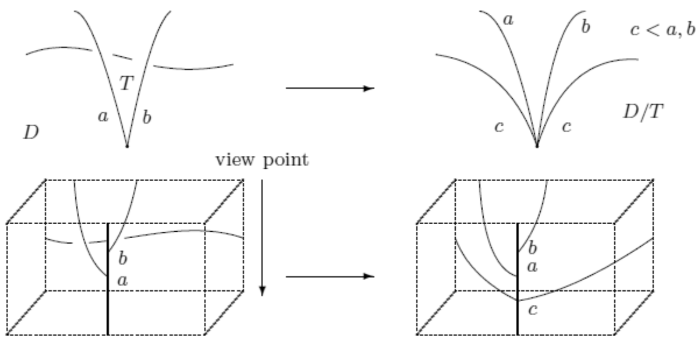

Fig. 5. Triangle contraction.

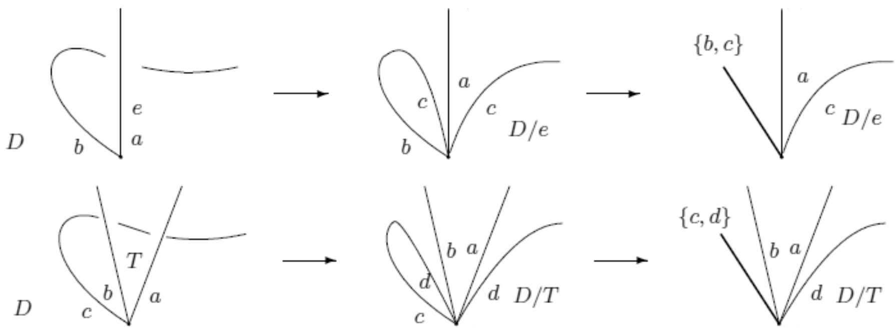

Fig. 6.

Lemma 3.3. *Let D be a knot and spoke diagram with $R(D)$ regions in $D \subset S^2$ and $S(D)$ spokes. Then we have the following conditions*:

(1) $R(D/e) + S(D/e) = R(D) + S(D)$.
(2) $R(D/T) + S(D/T) = R(D) + S(D) - 1$.

Definition 3.4. Let D be a knot and spoke diagram with the binding vertex v. An edge $e \in E(D)$ is said to be *contractible* if the number of components of $D \backslash \{v\}$ equals to the number of components of $(D/e) \backslash \{v\}$.

In [4], Park and the author proved the following lemma which does the key role in the proof of the main theorem.

Lemma 3.5. *Let D be a knot and spoke diagram with the binding vertex v. If $v(D) \geq 2$, then there exist at least two contractible edges which are incident to v. Furthermore, if e is not contractible, then the diagram D can be depicted as Fig.* 7 *and each of D_1 and D_2 contains at least one contractible edge.*

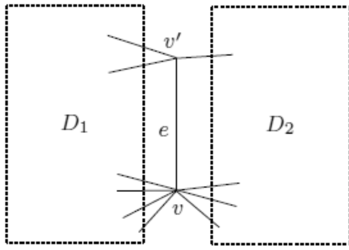

Fig. 7.

Lemma 3.6. *Let D be a knot and spoke diagram with the binding vertex v. Let E be a connected subgraph of D as topological graphs. If there is an edge e, incident to v, whose other end is not in E, then there is a bf contractible edge, incident to v, whose other end is not in E.*

Proof. If e is contractible, we have done. Now suppose that e is not contractible. Then D is divided into two subdiagrams D_1 and D_2 as seen in Fig. 7. Since E is connected and does not contain the other end v' of e, E is contained in one of D_1 and D_2, say D_1. By Lemma 3.5, there exists a contractible edge $e' = vv''$ such that v'' is in D_2. Clearly, v'' is not in D_1 and hence not in E. □

The following is one of the main results of the paper.

Theorem 3.7. *Let D be a connected reduced link diagram with the alternating tangle decomposition $(G_D; T_1, \ldots, T_n)$. Suppose that T_i is fat for each i. Then*

$$\alpha(D) \leq c(D) + 2n - \nu,$$

where ν denotes the number of non-alternating edges in D.

Proof. We will construct an arc-presentation of D with $c(D) + 2n - \nu$ arcs. Recall that the knot and spoke diagram with spokes only is an arc-presentation, and that any link L with diagram D admits an arc-presentation with $c(D) + 2$ arcs. Also notice that each triangle contraction reduces the number of arcs by 1, by Lemma 3.3. Hence we need to find possible triangle contractions as much as possible.

Let γ_i denote the *boundary* of T_i which is obtained from T by removing the outside arcs, see Fig. 8. Suppose that T_i is strongly alternating for each i so that D is an adequate diagram. Since each T_i is strongly alternating and fat, the boundary γ_i of T_i is a simple closed curve. Without loss of generality, we may assume that our binding vertex v is on γ_1, and that every edge in T_1 which is incident to v meets γ, by Lemma 3.6 when E is taken γ as a subgraph.

If T_1 is incident to another alternating tangle, say T_2, by non-alternating edges $f_1, f_2, \ldots, f_m$, see Fig. 9.

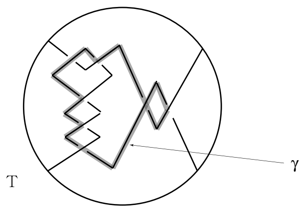

Fig. 8.

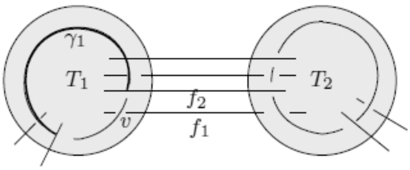

Fig. 9.

If $m = 1$, then T_1 and T_2 are connected by some path which is different with f_1. For, if there are no connection between T_1 and T_2 except f_1, then f_1 is a cut edge of the original diagram D, which is impossible because D is a link diagram. Since f_1 is contractible, one can get a new diagram of the shape in Fig. 11 by contracting f_1.

If $m \geq 2$, then it is clear that f_1 is contractible. By contracting f_1, we get the diagram in Fig. 10.

By applying Lemma 3.6 again to the connected subgraph $E = \gamma_2$, one can contract each edge of T_2 whose one end is v and the other end is not in γ_2, so that all edges in the resulting tangle which are incident to v, meet γ_2. Note that the edge f_2 and the vertex v form a triangle T, on which we will apply the triangle contraction. Since, after the triangle contraction, the resulting diagram is still of the form in Fig. 10, one can repeat the same process to get the diagram in which two tangles T_1 and T_2 are amalgamated into a new tangle $T_1 * T_2$ in Fig. 11.

Notice that we applied the triangle contraction $m - 1$ times and that the last triangle contraction concerned with f_m results in a cut vertex so that it does not decrease the number of arcs of our arc-presentation. Indeed, there are $m - 2$ applicable triangle contractions.

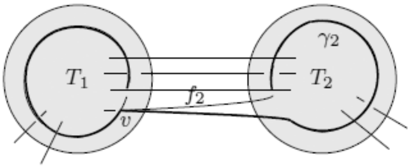

Fig. 10.

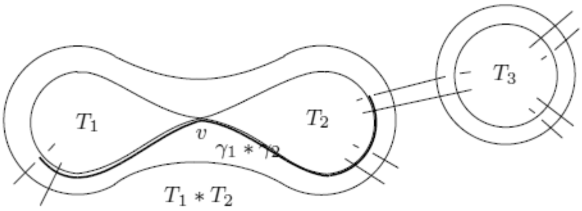

Fig. 11.

If there is another tangle, say T_3, which is connected to $T_1 * T_2$ by non-alternating edges, one can apply the above process to the non-alternating edges $g_1, \ldots, g_l$ between $T_1 * T_2$ and T_3 to get the diagram in which three tangles T_1, T_2 and T_3 are amalgamated into a tangle $T_1 * T_2 * T_3$ in Fig. 12. Note that the boundary of $T_1 * T_2 * T_3$ forms a bouquet of three circles. Notice that there are $l - 2$ applicable triangle contractions.

One can apply this process inductively to get $T_1 * T_2 * T_n$. Here, consider the simple graph Γ_D obtained from G_D by changing multiple edges into a single edge. If Γ_D is a tree, then, the resulting diagram $T_1 * T_2 * T_n$ is of the form in Fig. 13.

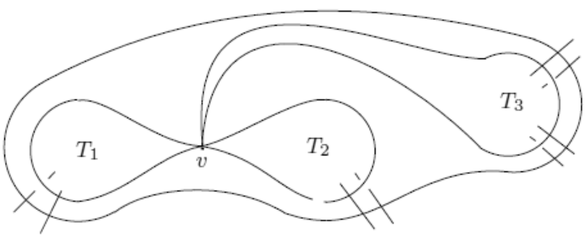

Fig. 12.

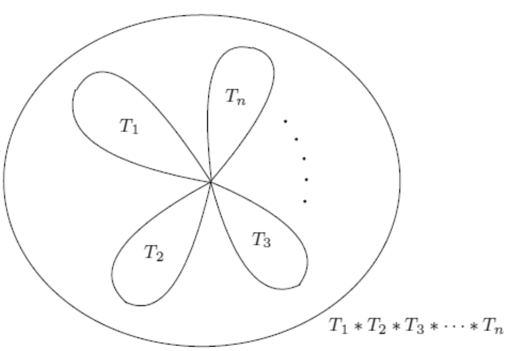

Fig. 13.

If Γ_D is not a tree, then there exists a cycle in Γ_D. Consider an innermost cycle with boundary $e_1e_2\cdots e_k$ which is read by cyclical order in any orientation. Without loss of generality, we may assume that two tangles T_i and T_{i+1} are connected by $n(e_k)$ non-alternating edges in D which correspond to e_i in Γ_D. Note that $T_1 * T_2 * \cdots * T_k$ is of the shape at the left diagram in Fig. 14, on which we can apply triangle contractions $n(e_k)$ times to get the right diagram in Fig. 14.

Notice that throughout the above process, we found $\nu - 2(n-1)$ applicable triangle contractions so that we can get an arc-presentation with $c(D) + 2 - \{\nu - 2(n-1)\} = c(D) - \nu + 2n$ arcs.

Finally, suppose that T_i is not strongly alternating for some i. Then the tangle T_i can be depicted as the left in Fig. 15. By changing all cut crossings of T_i as the right in Fig. 15 according to the crossing, we can get a new alternating tangle T_i', which is still alternating and fat. Notice that $c(T_i') = c(T_i) + k$ and $R(T_i') = R(T_i) + k$ where k is the number of cut crossings of T_i. Since T_i' is strongly alternating and the new diagram $D' = (G_D; T_1', \ldots, T_n')$ has n tangles and ν non-alternating edges. Hence by the previous case, it admits an arc-presentation with $c(D') + 2n - \nu$ arcs.

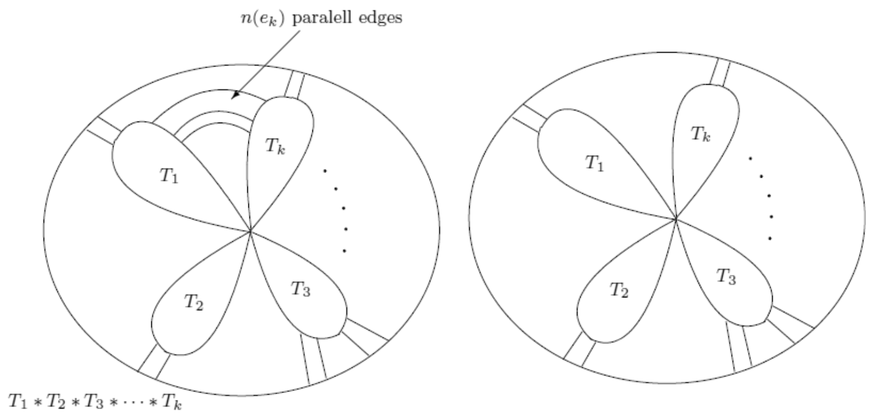

Fig. 14.

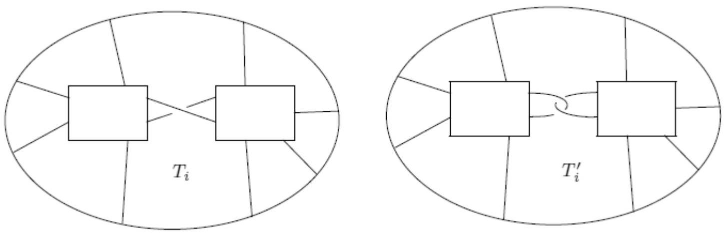

Fig. 15.

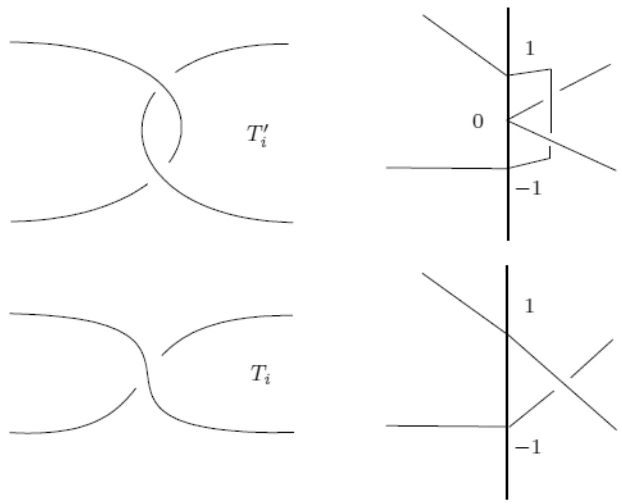

Fig. 16.

By starting the our process at one of the newborn crossings of D', without loss of generality, we may assume that the arc-presentation of T_i' looks like Fig. 16. By slightly changing the arc-presentation of T_i' as in Fig. 16, one can get an arc-presentation of T_i, whose number of arcs is $(-1)+$ that of T_i'. Hence, if k is the number of cut crossings of all T_i's, then we have

$$\alpha(D) \leq \alpha(D') - k \leq c(D') + 2n - \nu - k = c(D) + 2n - \nu. \qquad \square$$

From the proof of the above theorem we can construct an arc-presentation for $(G_D; T_1, \ldots, T_n)$ even though T_i is not fat for some i.

Corollary 3.8. *Let D be a connected reduced link diagram with the alternating tangle decomposition $(G_D; T_1, \ldots, T_n)$. Suppose that the boundary circle of T_i has m_i double points for each i. Then*

$$\alpha(D) \leq c(D) + 2n - \nu + 2m,$$

where ν denotes the number of non-alternating edges in D and $m = m_1 + \cdots + m_n$.

In 1998, Morton and Beltrami [11] gave the following lower bound for the arc index.

Proposition 3.9. *Let $\alpha(L)$ denote the arc index of L. Then*

$$\alpha(L) \geq \mathrm{breadth}_a\ F_L(a, z) + 2,$$

where $F_L(a, z)$ is the Kauffman polynomial of L.

In 1988, Thistlethwaite [13] gave a lower bound for $\mathrm{breadth}_a\ F_L(a,z)+2$. Let G denote the graph derived from the checkerboard shading of D by placing a vertex in each shaded region and connecting them through the crossings in the usual way. The edges of G are labeled $+$ or $-$ according to the sense of the crossings. Let G_+ denote the subgraph of G consisting of all vertices of G and the positive edges of G and let $\overline{G_+}$ denote the quotient graph obtained from G_+ by identifying those pairs of vertices which are ends of a path consisting of negative edges of G. The graphs G_- and $\overline{G_-}$ are defined likewise.

Proposition 3.10. *For a connected adequate link diagram D of L with the associated graph G,*

$$\mathrm{breadth}_a F_L(a,z)+2 \geq \mathrm{rank}(G_+)+\mathrm{rank}(G_-)+V(\overline{G_+})+V(\overline{G_-}).$$

Since $(G_D;T_1,\ldots,T_n)$ is adequate when each T_i is strongly reduced, we can obtain the following lower bound for the arc index.

Theorem 3.11. *Let D be a connected reduced link diagram with the alternating tangle decomposition $(G_D;T_1,\ldots,T_n)$. Suppose that T_i is strongly reduced for each i, and the incidents between all two vertices of G_D are even. Then*

$$\alpha(D) \geq c(D)+2n-\nu,$$

where ν denotes the number of non-alternating edges in D.

Proof. Note that D is adequate because T_i is strongly reduced for each i. Let Γ_D denote the graph obtained from G_D by identifying two consecutive multiple edges. Then from the construction of G_+ and G_-, one can see that $\mathrm{rank}(G_+)+\mathrm{rank}(G_-)=\mathrm{rank}(G_D)-\mathrm{rank}(\Gamma_D)$. Since $\overline{G_+}$ (respectively, $\overline{G_+}$) is the quotient graph obtained from G_+ (respectively, G_-) by identifying those pairs of vertices which are ends of a path consisting of negative (respectively, positive) edges of G, $V(\overline{G_+})+V(\overline{G_-})=V(G_D)-E(\Gamma_D)+V(\Gamma_D)$. Since G_D and Γ_D are planar, $\mathrm{rank}(G_D)+V(G_D)=E(G_D)+1, \mathrm{rank}(\Gamma_D)+V(\Gamma_D)=E(\Gamma_D)+1$. Since $V(\Gamma_D)=n, E(\Gamma_D)=\frac{\nu}{2}$, we get $\mathrm{rank}(G_+)+\mathrm{rank}(G_-)+V(\overline{G_+})+V(\overline{G_-})=c(D)+2n-\nu$. □

Corollary 3.12. *Let D be a connected reduced link diagram with the alternating tangle decomposition $(G_D;T_1,\ldots,T_n)$. Suppose that T_i is strongly reduced and fat for each i, and suppose that the number of edges between all two vertices of G_D are even. Then*

$$\alpha(D) = c(D)+2n-\nu,$$

where ν denotes the number of non-alternating edges in D.

Example 3.13. For the diagram in Fig. 17, we know that $n=6$, $c(D)=6+6+8+8+4+4=36$ and $\nu=E(G_D)=18$. Since all alternating tangles are

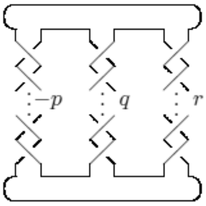

Fig. 17. $P(-p, q, r)$.

strongly reduced and fat and since for any two tangles T_i and T_j, the number of non-alternating edges between them is even, by the above corollary, we have $\alpha(D) = c(D) + 2n - \nu = 36 + 2 \times 6 - 18 = 30$.

Remark 3.14. The condition being strongly alternating and fat is essential. In 2012, Jin gave a talk about the arc index of pretzel links $P(-p, q, r), p, q, r \geq 2$. He showed that if $p, r \geq 3$, then $\alpha(P(-p, 2, r)) = c(P(-p, 2, r))$, and if $q \geq 3$, then $\alpha(P(-p, 2, r)) < c(P(-p, 2, r))$ in all the cases that he treated. Notice that the pretzel link $P(-p, q, r)$ consists of two alternating tangles, one of which is neither strongly alternating nor fat.

Acknowledgments

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