# HOPF HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH GENERALIZED TANAKA-WEBSTER PARALLEL NORMAL JACOBI OPERATOR

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Abstract. We study the classifying problem of immersed submanifolds in Hermitian symmetric spaces. Typically in this paper, we deal with real hypersurfaces in a complex two-plane Grassmannian  $G_2(\mathbb{C}^{m+2})$  which has a remarkable geometric structure as a Hermitian symmetric space of rank 2. In relation to the generalized Tanaka-Webster connection, we consider a new concept of the parallel normal Jacobi operator for real hypersurfaces in  $G_2(\mathbb{C}^{m+2})$  and prove non-existence of real hypersurfaces in  $G_2(\mathbb{C}^{m+2})$  with generalized Tanaka-Webster parallel normal Jacobi operator.

*Keywords*: real hypersurface; complex two-plane Grassmannian; Hopf hypersurface; generalized Tanaka-Webster connection; normal Jacobi operator; generalized Tanaka-Webster parallel normal Jacobi operator

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## INTRODUCTION

In complex projective spaces or in quaternionic projective spaces, many differential geometers studied real hypersurfaces with parallel curvature tensor ([7]). From a new perspective, it is investigated to classify real hypersurfaces in complex two-plane Grassmannians with parallel normal Jacobi operator, that is,  $\nabla \overline{R}_N = 0$  ([8], [10] and [6]).

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As a prevailing notion, in a Riemannian manifold  $(\overline{M}, \overline{g})$ , a vector field X along a geodesic  $\gamma$  of  $\overline{M}$  is called a *Jacobi field* if it satisfies the second order Jacobi equation

$$\overline{\nabla}_{\dot{\gamma}}^2 X + \overline{R}(X, \dot{\gamma})\dot{\gamma} = 0,$$

where  $\dot{\gamma}$  is the vector tangent to  $\gamma$ . For any tangent vector field X at  $x \in \overline{M}$ , the Jacobi operator  $\overline{R}_X$  is defined by

$$(\overline{R}_X Y)(x) = (\overline{R}(Y, X)X)(x),$$

for any vector field  $Y \in T_x \overline{M}$ .

On the other hand, let us put a unit normal vector field N to a hypersurface M into the curvature tensor  $\overline{R}$  of the ambient space  $\overline{M}$ . In [8], for any tangent vector field X on M, the normal Jacobi operator  $\overline{R}_N$  is defined by

$$\overline{R}_N(X) = \overline{R}(X, N)N$$

The ambient space, a complex two-plane Grassmannian  $G_2(\mathbb{C}^{m+2})$  consists of all complex two-dimensional linear subspaces in  $\mathbb{C}^{m+2}$ . This Riemannian symmetric space is the unique compact irreducible Riemannian manifold equipped with both a Kähler structure J and a quaternionic Kähler structure  $\mathfrak{J}$  not containing J. Then, naturally, we could consider two geometric conditions for hypersurfaces Min  $G_2(\mathbb{C}^{m+2})$ : that both the one-dimensional distribution  $[\xi] = \text{Span}\{\xi\}$  and the three-dimensional distribution  $\mathfrak{D}^{\perp} = \text{Span}\{\xi_1, \xi_2, \xi_3\}$  are invariant under the shape operator A of M ([3]), where the Reeb vector field  $\xi$  is defined by  $\xi = -JN$ , Ndenotes a local unit normal vector field of M in  $G_2(\mathbb{C}^{m+2})$  and the almost contact 3-structure vector fields  $\xi_{\nu}$  are defined by  $\xi_{\nu} = -J_{\nu}N$ ,  $\nu = 1, 2, 3$ , where  $\{J_1, J_2, J_3\}$ denotes a local basis of  $\mathfrak{J}$ . The distribution  $\mathfrak{D}$  denotes the orthogonal complement of  $\mathfrak{D}^{\perp}$  in  $T_xM$ ,  $x \in M$  which becomes the maximal quaternionic subbundle of  $T_xM$ ,  $x \in M$ . If X is a tangent vector on M, we may put

$$JX = \varphi X + \eta(X)N, \quad J_{\nu}X = \varphi_{\nu}X + \eta_{\nu}(X)N$$

where  $\varphi X$  (resp.  $\varphi_{\nu} X$ ) is the tangential part of JX (resp.  $J_{\nu} X$ ) and  $\eta(X) = g(X, \xi)$ (resp.  $\eta_{\nu}(X) = g(X, \xi_{\nu})$ ) is the coefficient of normal part of JX (resp.  $J_{\nu} X$ ). In this case, we call  $\varphi$  the structure tensor field of M.

By using the result in Alekseevskij [1], Berndt and Suh [3] proved the following:

**Theorem A.** Let M be a connected orientable real hypersurface in  $G_2(\mathbb{C}^{m+2})$ ,  $m \ge 3$ . Then both  $[\xi]$  and  $\mathfrak{D}^{\perp}$  are invariant under the shape operator of M if and only if

- (A) M is an open part of a tube around a totally geodesic  $G_2(\mathbb{C}^{m+1})$  in  $G_2(\mathbb{C}^{m+2})$ , or
- (B) *m* is even, say m = 2n, and *M* is an open part of a tube around a totally geodesic  $\mathbb{H}P^n$  in  $G_2(\mathbb{C}^{m+2})$ .

By using the normal Jacobi operator, Jeong, Kim and Suh considered the notion of *parallel normal Jacobi operator*, that is,  $\nabla_X \overline{R}_N = 0$  along any vector field X on M in  $G_2(\mathbb{C}^{m+2})$ . Then they gave a non-existence theorem as follows [8]:

**Theorem B.** There exist no Hopf hypersurfaces in  $G_2(\mathbb{C}^{m+2})$ ,  $m \ge 3$ , with parallel normal Jacobi operator.

Recall that the Reeb vector field  $\xi$  is said to be *Hopf* if it is invariant under the shape operator A. The one dimensional foliation of M by the integral manifolds of the Reeb vector field  $\xi$  is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in  $G_2(\mathbb{C}^{m+2})$  if and only if the Hopf foliation of M is totally geodesic. By the formulas in [8], Section 3, it can be easily checked that M is Hopf if and only if the Reeb vector field  $\xi$  is Hopf.

Moreover, Jeong and Suh considered the general notion of the  $\mathfrak{F}$ -parallel normal Jacobi operator defined in such a way that  $\nabla_{\mathfrak{F}} \overline{R}_N = 0$ ,  $\mathfrak{F} = [\xi] \cup \mathfrak{D}^{\perp}$ , which is weaker than the notion of the parallel normal Jacobi operator mentioned above. They gave a non-existence theorem as follows [10]:

**Theorem C.** There exist no connected Hopf real hypersurfaces in  $G_2(\mathbb{C}^{m+2})$ ,  $m \ge 3$ , with  $\mathfrak{F}$ -parallel normal Jacobi operator,  $\mathfrak{F} = [\xi] \cup \mathfrak{D}^{\perp}$ .

Related to the Levi-Civita connection  $\nabla$ , the generalized Tanaka-Webster connection (from now on, GTW connection) for contact metric manifolds was introduced by Tanno ([13]) as a generalization of the connection defined by Tanaka in [12] and, independently, by Webster in [14]. The Tanaka-Webster connection is defined as a canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold. A real hypersurface M in a Kähler manifold has an (integrable) CR-structure associated with the almost contact structure ( $\varphi, \xi, \eta, g$ ) induced on M by the Kähler structure, but, in general, this CR-structure is not guaranteed to be pseudo-Hermitian. Cho defined GTW connection for a real hypersurface of a Kähler manifold (see [4], [5]) by

$$\widehat{\nabla}_X^{(k)} Y = \nabla_X Y + g(\varphi A X, Y) \xi - \eta(Y) \varphi A X - k \eta(X) \varphi Y,$$

with a constant  $k \in \mathbb{R} \setminus \{0\}$  (see [5], [9]).

Using this GTW connection  $\widehat{\nabla}^{(k)}$ , we consider the new notion of generalized Tanaka-Webster parallel normal Jacobi operator (in short, GTW parallel normal Jacobi operator), that is,  $\widehat{\nabla}_X^{(k)} \overline{R}_N = 0$  for any vector field  $X \in T_x M$ . In Section 1 we will prove the following Main Theorem.

**Main Theorem.** There exist no Hopf hypersurface in a complex two-plane Grassmannian  $G_2(\mathbb{C}^{m+2})$ ,  $m \ge 3$ , with GTW parallel normal Jacobi operator.

In Section 2 we define a new notion called the *GTW Reeb-parallel* defined by  $(\widehat{\nabla}_{\xi}^{(k)}\overline{R}_N)Y = 0$  for any tangent vector field Y on M. It is weaker than the GTW parallel normal Jacobi operator. As an interesting result, for  $\xi \in \mathfrak{D}^{\perp}$ , any Hopf hypersurface M in  $G_2(\mathbb{C}^{m+2})$  admits a natural GTW Reeb-parallel normal Jacobi operator.

In this paper, we refer to [1], [2], [3], [8], and [11] for Riemannian geometric structures of  $G_2(\mathbb{C}^{m+2})$  and its geometric quantities.

## 1. Proof of Main Theorem

Let us denote by  $\overline{R}(X,Y)Z$  the curvature tensor in  $G_2(\mathbb{C}^{m+2})$ . Then the normal Jacobi operator  $\overline{R}_N$  of M in  $G_2(\mathbb{C}^{m+2})$  can be defined by  $\overline{R}_N X = \overline{R}(X,N)N$  for any vector field  $X \in T_x M = \mathfrak{D} \oplus \mathfrak{D}^{\perp}$ , where the distribution  $\mathfrak{D}$  denotes the orthogonal complement of  $\mathfrak{D}^{\perp}$  in  $T_x M, x \in M$  (see [8]).

In [8] and [10], the derivative of the normal Jacobi operator is written as

$$(1.1) \qquad (\nabla_X \overline{R}_N)Y = 3g(\varphi AX, Y)\xi + 3\eta(Y)\varphi AX + 3\sum_{\nu=1}^3 \{g(\varphi_\nu AX, Y)\xi_\nu + \eta_\nu(Y)\varphi_\nu AX\} - \sum_{\nu=1}^3 [2\eta_\nu(\varphi AX)(\varphi_\nu\varphi Y - \eta(Y)\xi_\nu) - g(\varphi_\nu AX, \varphi Y)\varphi_\nu \xi - \eta(Y)\eta_\nu(AX)\varphi_\nu \xi - \eta_\nu(\varphi Y)(\varphi_\nu\varphi AX - g(AX, \xi)\xi_\nu)]$$

for any tangent vector fields X and Y on M.

In [5], the author defined the GTW connection  $\widehat{\nabla}^{(k)}$  for M as follows:

(1.2) 
$$\widehat{\nabla}_X^{(k)} Y = \nabla_X Y + g(\varphi A X, Y)\xi - \eta(Y)\varphi A X - k\eta(X)\varphi Y$$

for a non-zero real number k. By using (1.2), we have

$$\begin{aligned} (\widehat{\nabla}_X^{(k)}\overline{R}_N)Y &= \widehat{\nabla}_X^{(k)}(\overline{R}_NY) - \overline{R}_N(\widehat{\nabla}_X^{(k)}Y) \\ &= \nabla_X(\overline{R}_NY) + g(\varphi AX, \overline{R}_NY)\xi - \eta(\overline{R}_NY)\varphi AX - k\eta(X)\varphi\overline{R}_NY \\ &- \overline{R}_N(\nabla_XY + g(\varphi AX, Y)\xi - \eta(Y)\varphi AX - k\eta(X)\varphi Y). \end{aligned}$$

From this, together with the fact that M is Hopf, we obtain

$$(1.3) \qquad (\widehat{\nabla}_{X}^{(k)}\overline{R}_{N})Y = \sum_{\nu=1}^{3} \{3g(\varphi_{\nu}AX,Y)\xi_{\nu} + 3\eta_{\nu}(Y)\varphi_{\nu}AX \\ -2\eta_{\nu}(\varphi AX)\varphi_{\nu}\varphi Y + 5\eta_{\nu}(\varphi AX)\eta(Y)\xi_{\nu} \\ +g(\varphi_{\nu}AX,\varphi Y)\varphi_{\nu}\xi + \eta_{\nu}(\varphi Y)\varphi_{\nu}\varphi AX \\ -\alpha\eta(X)\eta_{\nu}(\varphi Y)\xi_{\nu} + 3\eta_{\nu}(\varphi AX)\eta_{\nu}(Y)\xi \\ -\eta_{\nu}(\xi)g(\varphi AX,\varphi_{\nu}\varphi Y)\xi + \eta_{\nu}(\xi)\eta_{\nu}(\varphi AX)\eta(Y)\xi \\ -\alpha\eta_{\nu}(\xi)\eta(X)\eta_{\nu}(\varphi Y)\xi + \eta_{\nu}(AX)\eta_{\nu}(\varphi Y)\xi \\ +4\eta_{\nu}(\xi)\eta_{\nu}(Y)\varphi AX - 4k\eta(X)\eta_{\nu}(Y)\varphi_{\nu}\xi \\ +k\eta_{\nu}(\xi)\eta(X)\varphi\varphi_{\nu}\varphi Y - k\eta_{\nu}(\xi)\eta(X)\eta(Y)\varphi_{\nu}\xi \\ -k\eta_{\nu}(\xi)g(\varphi AX,Y)\xi_{\nu} + \eta_{\nu}(\xi)\eta(Y)\varphi_{\nu}AX \\ +k\eta_{\nu}(\xi)\eta(X)\varphi_{\nu}Y\} \}$$

for any tangent vector fields X and Y on M.

Let us assume that the normal Jacobi operator  $\overline{R}_N$  on a Hopf hypersurface M in a complex two-plane Grassmann manifold  $G_2(\mathbb{C}^{m+2})$  is *GTW parallel*, that is,

(\*) 
$$(\widehat{\nabla}_X^{(k)} \overline{R}_N) Y = 0$$

for any tangent vector fields X and Y on M.

Here, it is the main goal to show that the Reeb vector field  $\xi$  belongs to either the distribution  $\mathfrak{D}$  or its orthogonal complement  $\mathfrak{D}^{\perp}$  such that  $TM = \mathfrak{D} \oplus \mathfrak{D}^{\perp}$  in  $G_2(\mathbb{C}^{m+2})$  when the normal Jacobi operator is GTW parallel.

From now on, we may write the Reeb vector field  $\boldsymbol{\xi}$  as

(\*\*) 
$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$$

for some unit vector fields  $X_0 \in \mathfrak{D}$  and  $\xi_1 \in \mathfrak{D}^{\perp}$ .

By putting  $X = \xi$  in (1.3) and using the condition (\*), we have

(1.4) 
$$0 = (\widehat{\nabla}_{\xi}^{(k)} \overline{R}_N) Y = \sum_{\nu=1}^3 \{-4\alpha \eta_{\nu}(\varphi Y)\xi_{\nu} + 4\alpha \eta_{\nu}(Y)\varphi_{\nu}\xi - 4k\eta_{\nu}(Y)\varphi_{\nu}\xi + k\eta_{\nu}(\xi)\varphi\varphi_{\nu}\varphi Y - k\eta_{\nu}(\xi)\eta(Y)\varphi_{\nu}\xi - k\eta_{\nu}(\xi)\eta_{\nu}(\varphi Y)\xi + 4k\eta_{\nu}(\varphi Y)\xi_{\nu} + k\eta_{\nu}(\xi)\varphi_{\nu}Y\}$$

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for any tangent vector field Y on M. Taking the inner product with  $\xi$  in (1.4), this becomes

$$4(\alpha - k)\eta(X_0)\eta(\xi_1)g(Y,\varphi_1X_0) = 0$$

for any tangent vector field Y on M, since  $\varphi \xi_1 = \eta(X_0) \varphi_1 X_0$ . Replacing Y by  $\varphi_1 X_0$ in the above equation, we obtain

$$(\alpha - k)\eta(X_0)\eta(\xi_1) = 0.$$

Thus there are 3 cases:

Case 1:  $\eta(X_0) = 0$ , which means that  $\xi$  belongs to the distribution  $\mathfrak{D}^{\perp}$ .

Case 2:  $\eta(\xi_1) = 0$ , which means that  $\xi$  belongs to the distribution  $\mathfrak{D}$ .

Finally, in the case of  $\eta(X_0)\eta(\xi_1) \neq 0$ , the only possible situation is the following one:

Case 3:  $\alpha = k$ . In this case,  $\alpha$  becomes a non-zero constant real number. From [3], Section 4, we get

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_{\nu}(\xi)\eta_{\nu}(\varphi Y)$$

for any Y tangent to M. This gives

$$0 = \eta(\xi_1)\varphi\xi_1 = \eta(\xi_1)\varphi_1\xi = \eta(\xi_1)\eta(X_0)\varphi_1X_0$$

Because of the assumptions in Case 3, this yields  $\varphi_1 X_0 = 0$ . Therefore  $-X_0 + \eta(X_0)\xi_1 = 0$ . That is,  $X_0 = \eta(X_0)\xi_1$ , which is impossible. Thus we have just proved that the Reeb vector field  $\xi$  belongs either to the distribution  $\mathfrak{D}$  or the distribution  $\mathfrak{D}^{\perp}$ .

First of all, we consider the case  $\xi \in \mathfrak{D}^{\perp}$ . Without loss of generality, we may put  $\xi = \xi_1$ .

**Lemma 1.1.** Let M be a Hopf hypersurface of  $G_2(\mathbb{C}^{m+2})$  with GTW parallel normal Jacobi operator. If the Reeb vector field  $\xi$  belongs to the distribution  $\mathfrak{D}^{\perp}$ , then  $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$ .

Proof. Since  $\xi$  belongs to the distribution  $\mathfrak{D}^{\perp}$ , using (1.3) and the assumption (\*), we have

(1.5) 
$$0 = \sum_{\nu=1}^{3} \{ 3g(\varphi_{\nu}AX, Y)\xi_{\nu} + 3\eta_{\nu}(Y)\varphi_{\nu}AX - 2\eta_{\nu}(\varphi AX)\varphi_{\nu}\varphi Y + 5\eta_{\nu}(\varphi AX)\eta(Y)\xi_{\nu} + g(\varphi_{\nu}AX,\varphi Y)\varphi_{\nu}\xi + \eta_{\nu}(\varphi Y)\varphi_{\nu}\varphi AX \} \}$$

$$-\alpha\eta(X)\eta_{\nu}(\varphi Y)\xi_{\nu} + 3\eta_{\nu}(\varphi AX)\eta_{\nu}(Y)\xi + \eta_{\nu}(AX)\eta_{\nu}(\varphi Y)\xi$$
$$-4k\eta(X)\eta_{\nu}(Y)\varphi_{\nu}\xi + 4k\eta(X)\eta_{\nu}(\varphi Y)\xi_{\nu}\}$$
$$-g(\varphi AX,\varphi\varphi_{1}Y)\xi - 4\eta_{1}(Y)\varphi AX + k\eta(X)\varphi\varphi_{1}\varphi Y$$
$$-4g(\varphi AX,Y)\xi_{1} + \eta(Y)\varphi_{1}AX + k\eta(X)\varphi_{1}Y$$

for any tangent vector fields X and Y on M.

Restricting Y to the distribution  $\mathfrak{D}$ , (1.5) can be read as

(1.6) 
$$0 = 3g(\varphi_1 A X, Y)\xi_1 + 3g(\varphi_2 A X, Y)\xi_2 + 3g(\varphi_3 A X, Y)\xi_3$$
$$- 2\eta_2(\varphi A X)\varphi_2\varphi Y - 2\eta_3(\varphi A X)\varphi_3\varphi Y - g(\varphi_2 A X, \varphi Y)\xi_3$$
$$+ g(\varphi_3 A X, \varphi Y)\xi_2 - g(A X, \varphi_1 Y)\xi - 4g(\varphi A X, Y)\xi_1$$

for any tangent vector field X on M.

Taking the inner product with  $\xi_2$ , we get

$$3g(\varphi_2 AX, Y) + g(\varphi_3 AX, \varphi Y) = 0$$

for any tangent vector fields X on M and  $Y \in \mathfrak{D}$ , that is,

$$-3A\varphi_2Y - A\varphi_3\varphi Y = 0.$$

Replacing Y by  $\varphi Y\in\mathfrak{D}$  in the above equation, we obtain

(1.7) 
$$A\varphi_3 Y = 3A\varphi_2\varphi Y.$$

Taking the inner product with  $\xi_3$  in (1.6), we get

$$3g(\varphi_3 AX, Y) - g(\varphi_2 AX, \varphi Y) = 0$$

for any tangent vector fields X on M and  $Y \in \mathfrak{D}$ . In other words,

(1.8) 
$$3A\varphi_3 Y = A\varphi_2 \varphi Y.$$

Combining (1.7) and (1.8), we get

$$A\varphi_3 Y = 9A\varphi_3 Y$$

for any tangent vector field  $Y \in \mathfrak{D}$ .

Replacing Y by  $\varphi_3 Y$  in the above equation, we have

$$AY = 0.$$

Hence,  $g(AY, \xi_{\nu}) = 0$  for  $\nu = 1, 2, 3$  and any  $Y \in \mathfrak{D}$ , that is,  $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$ .

In the case of  $\xi \in \mathfrak{D}$ , from [11] we know that M must be locally congruent to a real hypersurface of type (B) under our assumptions. So, we see that M is locally congruent to a model space either of type (A) or type (B) in Theorem A under the assumption of our Main Theorem.

Hence it remains to check whether the normal Jacobi operator  $\overline{R}_N$  of real hypersurfaces of type (A) or type (B) satisfies the condition (\*) for any tangent vector field Y on M or not.

Now, consider  $\xi \in \mathfrak{D}^{\perp}$ . According to the following proposition from [3], a real hypersurface M of type (A) has four distinct constant principal curvatures as follows:

**Proposition A.** Let M be a connected real hypersurface of  $G_2(\mathbb{C}^{m+2})$ . Suppose that  $A\mathfrak{D} \subset \mathfrak{D}$ ,  $A\xi = \alpha\xi$ , and  $\xi$  is tangent to  $\mathfrak{D}^{\perp}$ . Let  $J_1 \in \mathfrak{J}$  be the almost Hermitian structure such that  $JN = J_1N$ . Then M has three (if  $r = \pi/2\sqrt{8}$ ) or four (otherwise) distinct constant principal curvatures

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

with some  $r \in (0, \pi/\sqrt{8})$ . The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces are

$$T_{\alpha} = \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1 = \operatorname{Span}\{\xi\} = \operatorname{Span}\{\xi_1\},$$
  

$$T_{\beta} = \mathbb{C}^{\perp}\xi = \mathbb{C}^{\perp}N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3 = \operatorname{Span}\{\xi_2, \ \xi_3\},$$
  

$$T_{\lambda} = \{X; \ X \perp \mathbb{H}\xi, \ JX = J_1X\},$$
  

$$T_{\mu} = \{X; \ X \perp \mathbb{H}\xi, \ JX = -J_1X\},$$

where  $\mathbb{R}\xi$ ,  $\mathbb{C}\xi$  and  $\mathbb{H}\xi$  denote, respectively, the real, complex and quaternionic span of the structure vector field  $\xi$  and  $\mathbb{C}^{\perp}\xi$  denotes the orthogonal complement of  $\mathbb{C}\xi$ in  $\mathbb{H}\xi$ .

Using this, we consider a unit eigenvector  $X \in T_{\lambda}$ ,  $Y = \xi_2$  and assuming  $\xi = \xi_1 \in \mathfrak{D}^{\perp}$ , we obtain from (1.3)

$$3\lambda\varphi_2 X - \lambda\varphi_3\varphi X = 0.$$

Since X belongs to  $T_{\lambda}$ ,  $\varphi X$  is a tangent vector field on  $T_{\lambda}$ , that is,  $\varphi X = \varphi_1 X$ .

Thus we have  $2\lambda\varphi_2 X = 0$ . Taking the inner product with  $\varphi_2 X$ , we get  $\lambda = 0$ . This gives a contradiction. So we know that no real hypersurface of type (A) in  $G_2(\mathbb{C}^{m+2})$  admits a GTW parallel normal Jacobi operator in the case of  $\xi$  belonging to the distribution  $\mathfrak{D}^{\perp}$ . We make the following remark.

**Remark 1.2.** If the Reeb vector field  $\xi$  belongs to the distribution  $\mathfrak{D}^{\perp}$ , then there exists no hypersurface of type (A) in  $G_2(\mathbb{C}^{m+2})$  with GTW parallel normal Jacobi operator.

Now we check the case  $\xi \in \mathfrak{D}$  supposing that M has a GTW parallel normal Jacobi operator. In order to do this we introduce a proposition due to Berndt and Suh [3]:

**Proposition B.** Let M be a connected real hypersurface of  $G_2(\mathbb{C}^{m+2})$ . Suppose that  $A\mathfrak{D} \subset \mathfrak{D}$ ,  $A\xi = \alpha\xi$ , and  $\xi$  is tangent to  $\mathfrak{D}$ . Then the quaternionic dimension m of  $G_2(\mathbb{C}^{m+2})$  is even, say m = 2n, and M has five distinct constant principal curvatures

 $\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$ 

with some  $r \in (0, \pi/4)$ . The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_{\alpha} = \mathbb{R}\xi = \operatorname{Span}\{\xi\},$$
  

$$T_{\beta} = \mathfrak{J}J\xi = \operatorname{Span}\{\xi_{\nu}; \ \nu = 1, 2, 3\},$$
  

$$T_{\gamma} = \mathfrak{J}\xi = \operatorname{Span}\{\varphi_{\nu}\xi; \ \nu = 1, 2, 3\},$$
  

$$T_{\lambda}, \quad T_{\mu},$$

where

$$T_{\lambda} \oplus T_{\mu} = (\mathbb{HC}\xi)^{\perp}, \quad \mathfrak{J}T_{\lambda} = T_{\lambda}, \quad \mathfrak{J}T_{\mu} = T_{\mu}, \quad JT_{\lambda} = T_{\mu}$$

The distribution  $(\mathbb{HC}\xi)^{\perp}$  is the orthogonal complement of  $\mathbb{HC}\xi$ , where

$$\mathbb{HC}\xi = \mathbb{R}\xi \oplus \mathbb{R}J\xi \oplus \mathfrak{J}\xi \oplus \mathfrak{J}J\xi.$$

If we consider a unit eigenvector  $X \in T_{\lambda}$ ,  $Y = \xi_2$  in (1.3), it becomes

$$\sum_{\nu=1}^{3} \{3\lambda \eta_{\nu}(\xi_2)\varphi_{\nu}X + \lambda g(\varphi_{\nu}X,\varphi\xi_2)\varphi_{\nu}\xi\} = 0.$$

So we have

$$3\lambda\varphi_2 X = 0.$$

Taking the inner product with  $\varphi_2 X$ , we get  $\lambda = 0$ . This gives a contradiction. So this case cannot occur. Also we make the following remark.

**Remark 1.3.** If the Reeb vector field  $\xi$  belongs to the distribution  $\mathfrak{D}$ , then there exists no hypersurface of type (B) in  $G_2(\mathbb{C}^{m+2})$  with GTW parallel normal Jacobi operator.

Hence summing up Lemma 1.1 and Remarks 1.2, 1.3, we complete the proof of Main Theorem.  $\hfill \Box$ 

#### 2. GTW REEB-PARALLEL NORMAL JACOBI OPERATOR

In this section, we consider a new notion which differs from the GTW parallel normal Jacobi operator.

Let us assume that the normal Jacobi operator  $\overline{R}_N$  on Hopf hypersurfaces M in complex two-plane Grassmann manifolds  $G_2(\mathbb{C}^{m+2})$  is *GTW Reeb-parallel* defined by

(2.1) 
$$(\widehat{\nabla}_{\xi}^{(k)}\overline{R}_{N})Y = 0$$

for any tangent vector field Y on M. From this notion, together with the proof of Main Theorem we see that the Reeb vector field  $\xi$  belongs either to the distribution  $\mathfrak{D}$  or the distribution  $\mathfrak{D}^{\perp}$ . For  $\xi \in \mathfrak{D}^{\perp}$ , we will prove that any Hopf hypersurface M in  $G_2(\mathbb{C}^{m+2})$  always has a GTW Reeb-parallel normal Jacobi operator.

**Proposition 2.1.** Let M be a Hopf hypersurface in  $G_2(\mathbb{C}^{m+2})$ ,  $m \ge 3$ , such that  $\xi \in \mathfrak{D}^{\perp}$ . Then the normal Jacobi operator  $\overline{R}_N$  is GTW Reeb-parallel.

Proof. Putting  $X = \xi$  and  $\xi = \xi_1$  in (1.3), it becomes

$$\begin{aligned} (\widehat{\nabla}_{\xi}^{(k)}\overline{R}_{N})Y &= \sum_{\nu=1}^{3} \{ 3g(\varphi_{\nu}A\xi,Y)\xi_{\nu} + 3\eta_{\nu}(Y)\varphi_{\nu}A\xi - 2\eta_{\nu}(\varphi A\xi)\varphi_{\nu}\varphi Y \\ &+ 5\eta_{\nu}(\varphi A\xi)\eta(Y)\xi_{\nu} + g(\varphi_{\nu}A\xi,\varphi Y)\varphi_{\nu}\xi + \eta_{\nu}(\varphi Y)\varphi_{\nu}\varphi A\xi \\ &- \alpha\eta(\xi)\eta_{\nu}(\varphi Y)\xi_{\nu} + 3\eta_{\nu}(\varphi A\xi)\eta_{\nu}(Y)\xi - \eta_{\nu}(\xi)g(\varphi A\xi,\varphi_{\nu}\varphi Y)\xi \\ &+ \eta_{\nu}(\xi)\eta_{\nu}(\varphi A\xi)\eta(Y)\xi - \alpha\eta_{\nu}(\xi)\eta(\xi)\eta_{\nu}(\varphi Y)\xi + \eta_{\nu}(A\xi)\eta_{\nu}(\varphi Y)\xi \\ &- 4\eta_{\nu}(\xi)\eta_{\nu}(Y)\varphi A\xi - 4k\eta(\xi)\eta_{\nu}(Y)\varphi_{\nu}\xi + k\eta_{\nu}(\xi)\eta(\xi)\varphi\varphi_{\nu}\varphi Y \\ &- k\eta_{\nu}(\xi)\eta(\xi)\eta(Y)\varphi_{\nu}\xi - k\eta_{\nu}(\xi)\eta(\xi)\eta_{\nu}(\varphi Y)\xi + 4k\eta(\xi)\eta_{\nu}(\varphi Y)\xi_{\nu} \\ &- 4\eta_{\nu}(\xi)g(\varphi A\xi,Y)\xi_{\nu} + \eta_{\nu}(\xi)\eta(Y)\varphi_{\nu}A\xi + k\eta_{\nu}(\xi)\eta(\xi)\varphi_{\nu}Y \} \end{aligned}$$

for any tangent vector field Y on M. Together with the fact that M is Hopf, it can be written as

$$(\widehat{\nabla}_{\xi}^{(k)}\overline{R}_N)Y = \sum_{\nu=1}^3 \{3\alpha g(\varphi_{\nu}\xi, Y)\xi_{\nu} + 3\alpha\eta_{\nu}(Y)\varphi_{\nu}\xi - 2\alpha\eta_{\nu}(\varphi\xi)\varphi_{\nu}\varphi Y\}$$

$$\begin{split} &+ 5\alpha\eta_{\nu}(\varphi\xi)\eta(Y)\xi_{\nu} + \alpha g(\varphi_{\nu}\xi,\varphi Y)\varphi_{\nu}\xi + \alpha\eta_{\nu}(\varphi Y)\varphi_{\nu}\varphi\xi \\ &- \alpha\eta(\xi)\eta_{\nu}(\varphi Y)\xi_{\nu} + 3\alpha\eta_{\nu}(\varphi\xi)\eta_{\nu}(Y)\xi - \alpha\eta_{\nu}(\xi)g(\varphi\xi,\varphi_{\nu}\varphi Y)\xi \\ &+ \alpha\eta_{\nu}(\xi)\eta_{\nu}(\varphi\xi)\eta(Y)\xi - \alpha\eta_{\nu}(\xi)\eta(\xi)\eta_{\nu}(\varphi Y)\xi + \alpha\eta_{\nu}(\xi)\eta_{\nu}(\varphi Y)\xi \\ &- 4\alpha\eta_{\nu}(\xi)\eta_{\nu}(Y)\varphi\xi - 4k\eta(\xi)\eta_{\nu}(Y)\varphi_{\nu}\xi + k\eta_{\nu}(\xi)\eta(\xi)\varphi\varphi_{\nu}\varphi Y \\ &- k\eta_{\nu}(\xi)\eta(\xi)\eta(Y)\varphi_{\nu}\xi - k\eta_{\nu}(\xi)\eta(\xi)\eta_{\nu}(\varphi Y)\xi + 4k\eta(\xi)\eta_{\nu}(\varphi Y)\xi_{\nu} \\ &- 4\alpha\eta_{\nu}(\xi)g(\varphi\xi,Y)\xi_{\nu} + \alpha\eta_{\nu}(\xi)\eta(Y)\varphi_{\nu}\xi + k\eta_{\nu}(\xi)\eta(\xi)\varphi_{\nu}Y \} \\ &= \sum_{\nu=1}^{3} \{3\alpha g(\varphi_{\nu}\xi,Y)\xi_{\nu} - 4k\eta_{\nu}(Y)\varphi_{\nu}\xi + k\eta_{\nu}(\xi)\varphi\varphi_{\nu}\varphi Y \\ &- \alpha\eta_{\nu}(\varphi Y)\xi_{\nu} - 4k\eta_{\nu}(Y)\varphi_{\nu}\xi + k\eta_{\nu}(\xi)\varphi\varphi_{\nu}\varphi Y \\ &- k\eta_{\nu}(\xi)\eta(Y)\varphi_{\nu}\xi - k\eta_{\nu}(\xi)\eta_{\nu}(\varphi Y)\xi + 4k\eta_{\nu}(\varphi Y)\xi_{\nu} \\ &+ \alpha\eta_{\nu}(\xi)\eta(Y)\varphi_{\nu}\xi + k\eta_{\nu}(\xi)\varphi_{\nu}Y \} \end{split}$$

for any tangent vector field Y on M.

By using (2.1) and (2.8) in [11], Section 2, we have

$$(\widehat{\nabla}_{\xi}^{(k)}\overline{R}_N)Y = \sum_{\nu=1}^3 \{-4\alpha\eta_{\nu}(\varphi Y)\xi_{\nu} + 4\alpha\eta_{\nu}(Y)\varphi_{\nu}\xi - 4k\eta_{\nu}(Y)\varphi_{\nu}\xi + 4k\eta_{\nu}(\varphi Y)\xi_{\nu}\}$$

for any tangent vector field Y on M.

Because of (2.3) in [11], Section 2, we get

$$(\widehat{\nabla}_{\xi}^{(k)}\overline{R}_N)Y = -4(\alpha-k)\{\eta_1(\varphi Y)\xi_1 + \eta_2(\varphi Y)\xi_2 + \eta_3(\varphi Y)\xi_3 + \eta_1(Y)\varphi_1\xi + \eta_2(Y)\varphi_2\xi + \eta_3(Y)\varphi_3\xi\} = 0$$

for any tangent vector field Y on M. Thus from (2.1), the normal Jacobi operator  $\overline{R}_N$  is GTW Reeb-parallel.

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