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Transitivity and structure of operator algebras with a metric property

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This paper is dedicated to the memory of Israel Gohberg, one of the giants of linear analysis and a wonderful human being

Abstract

In this paper we discuss a new metric property that some operator algebras on Hilbert space possess and some resulting consequences concerning transitivity and structure theory of such algebras.

Keywords: Transitive algebra; Invariant subspace; K-spectral set

1. Introduction

In this paper $\mathcal{H}$ will always denote a separable, infinite dimensional, complex Hilbert space, and as usual we write $\mathcal{L}(\mathcal{H})$ for the algebra of all (bounded, linear) operators on $\mathcal{H}$. We also write $\mathbf{K}$ for the ideal of compact operators in $\mathcal{L}(\mathcal{H})$ and denote the quotient (Calkin) map $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathbf{K}$ by π. For each $T \in \mathcal{L}(\mathcal{H})$ we employ the notation $\sigma(T)$ and $\sigma_e(T) := \sigma(\pi(T))$ for the spectrum and essential spectrum of T, respectively, and we write $\|T\|_e := \|\pi(T)\|$.

In what follows, $\mathbb{A}$ will always denote a unital, norm-closed, subalgebra of $\mathcal{L}(\mathcal{H})$ and $\mathbb{A}^{-W}$ the closure of $\mathbb{A}$ in the weak (equivalently, strong) operator topology (herein denoted WOT and SOT,

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respectively). Recall that a subalgebra $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$ is called *transitive* if the only subspaces left invariant by every $A \in \mathbb{A}$ are (0) and $\mathcal{H}$, and recall also that long ago, motivated by Burnside's theorem for finite dimensional spaces, R. Kadison in [9] raised the (still open) problem whether every transitive subalgebra $\mathbb{A}$ of $\mathcal{L}(\mathcal{H})$ satisfies $\mathbb{A}^{-W} = \mathcal{L}(\mathcal{H})$. The present authors, in the summer of 2005, as a consequence of their study of the construction in [1], which was itself a modification of the original constructions of Lomonosov [11,12], became interested in the following variant of the Kadison problem.

Problem 1.1. If there exists a transitive subalgebra $\mathbb{A}$ of $\mathcal{L}(\mathcal{H})$ such that $\mathbb{A}^{-W} \neq \mathcal{L}(\mathcal{H})$, is it necessarily true that $\| \ \|$ and $\| \ \|_e$ are equivalent norms on $\mathbb{A}$?

Of course, since no such transitive algebra $\mathbb{A}$ with $\mathbb{A}^{-W} \neq \mathcal{L}(\mathcal{H})$ is presently known to exist, it would certainly be difficult to give a negative answer to Problem 1.1. On the other hand, as mentioned above, the present authors, while making a detailed, in-depth, study of the construction in [1] which eventually resulted in the production of this paper, thought they saw a path to an affirmative answer to Problem 1.1. This study, over time, produced the concept of the sets $\Gamma_\alpha(y)$ defined at the beginning of Section 2 and also the concept of an algebra $\mathbb{A}$ having Property (P) defined below.

We were strongly motivated to solve Problem 1.1 affirmatively because such a result would yield immediately the existence of nontrivial invariant subspaces for a large class of operators including all operators of the form $S + K$, where S is subnormal and $K \in \mathbf{K}$ (see Section 5).

Although we have thus far failed to solve Problem 1.1 affirmatively, we have obtained some weaker results in this direction (see Theorem 6.4 and Corollary 6.5), and moreover, taking into consideration the difficulty in finding useful new consequences of positive solutions to various invariant subspace problems, we suggest that the structure theory of the class of algebras studied herein is interesting independent of the existence or non-existence of invariant subspaces. For example, our theorems show that the transitive algebras studied in [1,11,12] have the metric property (P).

As usual, we write for each $y_0 \in \mathcal{H}$ and each $\delta > 0$,

$$B(y_0, \delta) := \{y \in \mathcal{H} : \|y - y_0\| < \delta\},$$

i.e., $B(y_0, \delta)$ is the open ball in $\mathcal{H}$ centered at y_0 and having radius δ. We can now introduce the metric property referred to in the title.

Definition 1.2. With $\mathbb{A}$ and $\alpha > 0$ given and y arbitrary in $\mathcal{H} \setminus (0)$, we define

$$\Gamma_\alpha(y) := \{Ay : A \in \mathbb{A}, \ \ \|A\|_e \le \alpha\}, \tag{1.1}$$

and say of a (unital, norm closed) subalgebra $\mathbb{A}$ of $\mathcal{L}(\mathcal{H})$ that $\mathbb{A}$ has *Property* (P) if there exists a quadruple $(y_0, \alpha, \delta, \delta_0)$, called an *implementing quadruple*, such that

(1) $y_0 \in \mathcal{H} \setminus (0)$,
(2) $\alpha \in (0, 1/2)$,
(3) $\delta \in (0, \|y_0\|)$,
(4) $\delta_0 \in (0, (1 - 2\alpha)\delta)$,
and such that the sets $\Gamma_\alpha(y)$ have the property that
(5) for every $y \in B(y_0, \delta)^-$, $\Gamma_\alpha(y) \cap B(y_0, \delta_0)^- \neq \varnothing$, (i.e., for every y satisfying $\|y_0 - y\| \le \delta$, there exists $A_y \in \mathbb{A}$ with $\|A_y\|_e \le \alpha$ and A_y moves y into the smaller closed ball centered at y_0 with radius $\delta_0 < \delta$).

Remark 1.3. For brevity, throughout this article we shall write simply "$\mathbb{A}$ has (P)" in place of the longer "$\mathbb{A}$ has Property (P) ". It may be of interest, when studying algebras $\mathbb{A}$ with (P), to ask about the plenitude of quadruples $\sigma = (y_0, \alpha, \delta, \delta_0)$ satisfying the appropriate requirements of Definition 1.2 so that σ implements (P). In this connection, we note two facts.

(i) Suppose $(y_0, \alpha, \delta, \delta_0)$ implements (P) for an algebra $\mathbb{A}$. Then every quadruple $(r y_0, \alpha, r\delta, r\delta_0)$, $r > 0$, also implements (P) for $\mathbb{A}$.

(ii) Trivial arithmetic shows that if $(y_0, \alpha, \delta, \delta_0)$ implements (P) for $\mathbb{A}$, then so does every quadruple $(y_0', \alpha, \delta' = \delta - \rho, \delta_0' = \delta_0 + \rho)$, where ρ, satisfying

$$\|y_0' - y_0\| < \rho < ((1 - 2\alpha)\delta - \delta_0)/(2(1 - \alpha)),$$

is fixed. Thus the set of vectors y_0 appearing in an implementing quadruple σ for $\mathbb{A}$ is an open cone in the norm topology on $\mathcal{H}$.

The following theorem and its corollaries, which arose from a further distillation and refinement of ideas from [11,1], are the primary justifications for studying the class of algebras $\mathbb{A}$ having (P).

Theorem 1.4. *Every $\mathbb{A}$ with (P) contains a nonzero idempotent of finite rank.*

This result has some immediate corollaries.

Corollary 1.5. (i) *If $\mathbb{A}$ has (P), then $\mathbb{A}'$ (the commutant of $\mathbb{A}$) is intransitive.*
(ii) *If $\mathbb{A}$ is transitive, then $\mathbb{A}'$ does not have (P).*

Corollary 1.6. *Every transitive algebra $\mathbb{A}$ with (P) satisfies $\mathbb{A}^{-W} = \mathcal{L}(\mathcal{H})$.*

Indeed, $\mathbb{A}$ contains a nonzero idempotent of finite rank by Theorem 1.4, and thus $\mathbb{A}^{-W} = \mathcal{L}(\mathcal{H})$ by [17, Theorem 8.12].

Corollary 1.7. *Suppose $\mathbb{A}$ is a transitive algebra and $\mathbb{A}^{-W} \neq \mathcal{L}(\mathcal{H})$. Then $\mathbb{A}$ does not have (P) and therefore, for every quadruple $(y_0, \alpha, \delta, \delta_0)$ satisfying* (1)–(4) *of Definition* 1.2*, there exists (a nonzero) $y = y(y_0, \alpha, \delta, \delta_0)$ satisfying $\|y - y_0\| \leq \delta$ and*

$$\|Ay - y_0\| > \delta_0, \quad \forall A \in \mathbb{A} \text{ such that } \|A\|_e \leq \alpha. \tag{1.2}$$

We begin with the following.

Proof of Theorem 1.4. Let $(y_0, \alpha, \delta, \delta_0)$ be an implementing quadruple with respect to a fixed algebra $\mathbb{A}$ having (P), and note from (4) that $0 < \alpha < (\delta - \delta_0)/2\delta$. Now fix $\beta \in (\alpha, (\delta - \delta_0)/2\delta)$, and note that this implies that

$$0 < 2\beta\delta + \delta_0 < \delta. \tag{1.3}$$

Thus, for every $y \in B(y_0, \delta)^-$, there exists $A_y \in \mathbb{A}$ such that

$$\left\|A_y\right\|_e \leq \alpha, \qquad \left\|A_y y - y_0\right\| \leq \delta_0, \tag{1.4}$$

and, since $\beta > \alpha$, we may write

$$A_y = T_y + K_y, \qquad \left\|T_y\right\| < \beta, \quad K_y \in \mathbf{K}. \tag{1.5}$$

Since K_y is compact, it is a continuous map from $\mathcal{H}$ with its weak topology to $\mathcal{H}$ with its norm topology, and this implies that, with $\delta_1 := \delta - (\delta_0 + 2\beta\delta) > 0$, for every $y \in B(y_0, \delta)^-$,

$$\mathcal{V}_y(\delta_1) := \{w \in B(y_0, \delta)^- : \|K_y(w - y)\| < \delta_1\} \tag{1.6}$$

is a (relatively) weakly open neighborhood of y in $B(y_0, \delta)^-$. Since $B(y_0, \delta)^-$ is weakly compact, there exist sets $\mathcal{V}_{y_1}(\delta_1), \mathcal{V}_{y_2}(\delta_1), \ldots, \mathcal{V}_{y_n}(\delta_1)$ as in (1.6) that cover $B(y_0, \delta)^-$. Let $\{f_1, f_2, \ldots, f_n\}$ be a partition of unity (formed by weakly continuous functions) subordinate to the above open covering of $B(y_0, \delta)^-$, so, by definition, the support supp $f_j \subset \mathcal{V}_{y_j}(\delta_1)$.

Observe that, by the weak compactness of $B(y_0, \delta)^-$, we have

$$\max\{\|K_{y_j}(w - y_j)\| : w \in \operatorname{supp} f_j\} = \theta_j \delta_1, \quad j = 1, \ldots, n,$$

where each $\theta_j < 1$. Hence $\theta := \max\{\theta_j : 1 \le j \le n\} < 1$. Upon defining

$$g(w) := \sum_{j=1}^{n} f_j(w) A_{y_j} w, \quad w \in B(y_0, \delta)^-,$$

we obtain a weakly continuous map from $B(y_0, \delta)^-$ into $\mathcal{H}$. Moreover, since for each $w \in B(y_0, \delta)^-$ we have

$$\begin{aligned}
\|g(w) - y_0\| &\le \sum_{j=1}^{n} f_j(w) \left\|A_{y_j} w - y_0\right\| \\
&\le \sum_{j=1}^{n} f_j(w) \left(\left\|A_{y_j} w - A_{y_j} y_j\right\| + \left\|A_{y_j} y_j - y_0\right\|\right) \\
&\le \delta_0 + \sum_{j=1}^{n} f_j(w) \left(\left\|T_{y_j}\left(w - y_j\right)\right\| + \left\|K_{y_j}\left(w - y_j\right)\right\|\right) \\
&\le \delta_0 + \sum_{j=1}^{n} f_j(w) \left(2\beta\delta + \theta_j \delta_1\right) \\
&\le 2\delta\beta + \theta\delta_1 + \delta_0 \\
&= 2\delta\beta + \theta(\delta - \delta_0 - 2\beta\delta) + \delta_0 \\
&= \theta\delta + (1 - \theta)(\delta_0 + 2\beta\delta) \\
&= \delta - (1 - \theta)(\delta - \delta_0 - 2\beta\delta), \quad \forall w \in B(y_0, \delta)^-,
\end{aligned}$$

so we obtain

$$\|g(w) - y_0\| \le \delta - (1 - \theta)(\delta - \delta_0 - 2\beta\delta), \quad \forall w \in B(y_0, \delta)^-;$$

hence by (1.3), g maps $B(y_0, \delta)^-$ into $B(y_0, \delta)$, and thus by the Schauder–Tychonoff fixed point theorem for locally convex spaces [4, Vol. I, p. 456], g has a (nonzero) fixed point $w_0 \in B(y_0, \delta)$. Define

$$A_0 = \sum_{j=1}^{n} f_j(w_0) A_{y_j},$$

and note that $A_0 \in \mathbb{A}$, $\|A_0\|_e \le \alpha < 1/2$, and $A_0 w_0 = w_0$. Thus $1 \in \sigma_p(A_0) \setminus \sigma_e(A_0)$, and from the Fredholm theory one knows that 1 must be an isolated eigenvalue of A_0 of finite multiplicity. Thus the root space corresponding to the eigenvalue 1 of A_0 is finite dimensional also, and it results easily from the Cauchy integral formula and the fact that $\mathbb{A}$ is norm closed that the Riesz idempotent associated with this finite dimensional space belongs to $\mathbb{A}$. □

2. The sets $\Gamma_\alpha(y)$

We now discuss some properties of the sets $\Gamma_\alpha(y)$ defined as in (1.1).

Proposition 2.1. *For arbitrary* $\mathbb{A}, \alpha > 0$, *and* $y \in \mathcal{H} \setminus (0)$, *the set* $\Gamma_\alpha(y)$ *has the following properties.*

(i) $\Gamma_\alpha(y)$ *is convex and balanced, so is absolutely convex, as is* $\Gamma_\alpha(y)^-$.
(ii) $\Gamma_\alpha(y) = \alpha\Gamma_1(y)$.
(iii) *For every* $\lambda \in \mathbb{C}$ *such that* $|\lambda| \le \alpha$, $\lambda y \in \Gamma_\alpha(y)$.
(iv) $A(\Gamma_\alpha(y)) \subset \Gamma_\alpha(y)$ *for every* $A \in \mathbb{A}$ *with* $\|A\|_e \le 1$.
(v) *Let* $\vee\{\Gamma_\alpha(y)\}$ *denote, as usual, the subspace of* $\mathcal{H}$ *generated by* $\Gamma_\alpha(y)$. *Then* $\mathbb{A}(\vee\{\Gamma_\alpha(y)\}) \subset \vee\{\Gamma_\alpha(y)\}$. *In particular, if* $\mathbb{A}$ *is transitive, then* $\vee\{\Gamma_\alpha(y)\} = \mathcal{H}$.
(vi) *If* $\lambda y \in \Gamma_\alpha(y)$ *for some* $\lambda \in \mathbb{C}$ *with* $|\lambda| > \alpha$, *then the algebra* $\mathbb{A}$ *contains a nonzero idempotent of finite rank.*
(vii) *If* $\Gamma_\alpha(y)$ *contains any real line, i.e., any set of the form* $\{x + rz : r \in \mathbb{R}\}$, *where* $x \in \Gamma_\alpha(y)$ *and* z *is a fixed nonzero vector, then* $\mathbb{A}z \subset \Gamma_\alpha(y)$, *so* $\Gamma_\alpha(y)$ *contains the line* $\mathbb{R}z$ *through the origin. Moreover, if* z *is a cyclic vector for* $\mathbb{A}$, *then* $\Gamma_\alpha(y)^- = (\mathbb{A}z)^- = \mathcal{H}$.
(viii) *If* $\Gamma_\alpha(y)^-$ *contains some* βy *with* $|\beta| > \alpha$, *then* $\mathbb{A}y \subset \Gamma_\alpha(y)^-$. *Thus if* y *is a cyclic vector for* $\mathbb{A}$, $\Gamma_\alpha(y)^- = \mathcal{H}$.
(ix) *If* $\mathbb{A}$ *has* (P) *and* $E \in \mathbb{A}$ *is any nonzero idempotent of finite rank in* $\mathbb{A}$ *(via Theorem 1.4), then* $\mathbb{A}E\mathbb{A}y \subset \Gamma_\alpha(y)$.
(x) *An algebra* $\mathbb{A} = \mathbb{A}^{-W}$ *has the property that there exists* $\alpha > 0$ *such that for every* $y \in \mathcal{H} \setminus (0)$, $\Gamma_\alpha(y) = \mathcal{H}$ *if and only if* $\mathbb{A} = \mathcal{L}(\mathcal{H})$.

Proof. That (i)–(v) are true is elementary, and that (vi) is true follows as at the end of the proof of Theorem 1.4 since $\lambda y = Ay$ for some $A \in \mathbb{A}$ satisfying $\|A\|_e \le \alpha$ and hence $\lambda \in \sigma(A) \setminus \sigma_e(A)$.

With respect to (vii), in the notation as above, since $\Gamma_\alpha(y)$ is closed under multiplication by -1, it follows that $rz \in \Gamma_\alpha(y)$ for every $r \in \mathbb{R}$, so for each such r, there exists $A_r \in \mathbb{A}$ with $\|A_r\|_e \le \alpha$ such that $A_r y = rz$. Now let B be arbitrary in $\mathbb{A} \setminus \mathbf{K}$, and define $r_0 = \|B\|_e$. Then $Bz = (B/\|B\|_e)A_{r_0}y \in \Gamma_\alpha(y)$, and since obviously $Kz \in \Gamma_\alpha(y)$ for $K \in \mathbf{K} \cap \mathbb{A}$, we get, finally, that $\mathbb{A}z \subset \Gamma_\alpha(y)$, as desired.

Concerning (viii), suppose first that $A \in \mathbb{A}$ and $0 < \|A\|_e \le \beta$. Choose $\{B_n\} \subset \mathbb{A}$ such that $B_n y \to \|A\|_e y$ and $\|B_n\|_e \le \alpha$ for all $n \in \mathbb{N}$. Then $(A/\|A\|_e)B_n y \in \Gamma_\alpha(y)$ and $(A/\|A_e\|)B_n y \to Ay$, so $Ay \in \Gamma_\alpha(y)^-$. For $A \in \mathbb{A}$ and $\|A\|_e = 0$, we define $A_\varepsilon = A + \varepsilon I$ for $\beta \ge \varepsilon > 0$. The above argument shows that $A_\varepsilon y \in \Gamma_\alpha(y)^-$ for $\varepsilon > 0$, and thus $Ay \in \Gamma_\alpha(y)^-$ too. This shows that $\beta\Gamma_1(y)^- = \Gamma_\beta(y)^- = \Gamma_\alpha(y)^- = \alpha\Gamma_1(y)^-$. Thus $(\beta/\alpha)\Gamma_1(y)^- = \Gamma_1(y)^-$, and by iteration we get that $(\beta/\alpha)^n\Gamma_1(y)^- = \Gamma_1(y)^-$ for $n \in \mathbb{N}$. Thus $\mathbb{R}\Gamma_1(y)^- = \Gamma_1(y)^-$ and $\mathbb{A}y \subset (\alpha\mathbb{R})\,\Gamma_1(y)^- = \alpha\Gamma_1(y)^- = \Gamma_\alpha(y)^-$ as promised.

That (ix) is true follows immediately from (iii) and (iv) upon noting that $\|E\|_e = 0$. Half of (x) is trivial and the other half follows from the fact that the only WOT-closed $\mathbb{A}$ having every nonzero vector as a strictly cyclic vector is $\mathcal{L}(\mathcal{H})$ [17, Corollary 9.10]. □

Remark 2.2. (a) For every algebra $\mathbb{A}$ and every associated $\Gamma_\alpha(y)$, it is trivial that $\Gamma_\alpha(y)^- \subset (\mathbb{A}y)^-$ and that $(\mathbb{A}y)^-$ is an invariant subspace for $\mathbb{A}$. Thus if $y \ne 0$ is not cyclic for $\mathbb{A}$, all of the results herein can be applied to the algebra $\mathbb{A}_y := \{A|_{(\mathbb{A}y)^-} : A \in \mathbb{A}\}$, and $\Gamma_\alpha(y)$ relative to $\mathbb{A}_y$ may be different from $\Gamma_\alpha(y)$ relative to $\mathbb{A}$.

(b) It is obvious that if $\mathbb{A}_1 \subset \mathbb{A}_2$ and $\mathbb{A}_1$ has (P), then so does $\mathbb{A}_2$. Not so obvious is the fact illustrated in the example below, that there exist algebras $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$ such that $\mathbb{A}^{-W} = \mathcal{L}(\mathcal{H})$ (so $\mathbb{A}$ is transitive) but $\mathbb{A}$ *does not have* (P). In other words, the converse of Corollary 1.6 is not true.

3. Examples of algebras $\mathbb{A}$ that do (do not) have (P)

As the reader will see, there is a close relation between an algebra $\mathbb{A}$ having (P) and $\mathbb{A}$ having several other familiar properties such as

(a) having a cyclic (strictly cyclic) vector,
(b) containing a compact operator,
(c) containing a nonzero idempotent of finite rank,
(d) being transitive.

Nevertheless, as can be seen from the examples given below, for an algebra $\mathbb{A}$ to have (P) is not equivalent, in general, to any one of (a)–(d) above.

Example 3.1. Let $\mathbb{A}$ be the C^*-algebra acting on $L^2(\mathbb{T})$ (with $\mathbb{T}$ the unit circle in $\mathbb{C}$) generated by $\{M_\varphi : \varphi \in L^\infty(\mathbb{T})\}$ and $R_\alpha \in \mathcal{L}(L^2(\mathbb{T}))$ (a rotation) defined by $(R_\alpha f)(e^{i\theta}) = f(e^{i(\theta+\alpha)})$, where α (mod 2π) is irrational. Then $\mathbb{A}$ has some, but not all, of the properties listed above, as we will now show. If we prove that $\mathbb{A}' = \mathbb{C}1_{\mathcal{H}}$, then $(\mathbb{A}^{-W})'' = (\mathbb{A}')' = \mathcal{L}(\mathcal{H})$, which implies that $\mathbb{A}^{-W} = \mathcal{L}(\mathcal{H})$ by the double commutant theorem, and shows also that $\mathbb{A}$ is transitive and irreducible (since no nontrivial projection belongs to $\mathbb{A}'$). So let $T \in \mathbb{A}'$. Then $T \in \{M_\varphi : \varphi \in L^\infty(\mathbb{T})\}'$, a maximal abelian algebra, so $T = M_{\varphi_1}$ for some $\varphi_1 \in L^\infty$. But also $TR_\alpha = R_\alpha T$ so for each $f \in L^2(\mathbb{T})$, $TR_\alpha(f) = R_\alpha T(f)$. Thus

$$\begin{aligned}\varphi_1(e^{i(\theta+\alpha)})f(e^{i(\theta+\alpha)}) &= R_\alpha(\varphi_1 f)(e^{i\theta}) = ((R_\alpha T)f)(e^{i\theta}) \\ &= (TR_\alpha(f))(e^{i\theta}) = \varphi_1(e^{i\theta})f(e^{i(\theta+\alpha)}).\end{aligned}$$

Thus $\varphi_1(e^{i(\theta+\alpha)}) = \varphi_1(e^{i\theta})$ for almost all θ in $[0, 2\pi]$, and since R_α is ergodic, φ_1 is constant a.e., so $\mathbb{A}' = \mathbb{C}1_{\mathcal{H}}$ as asserted above. To see that $\mathbb{A} \cap \mathbf{K} = (0)$ and thus that $\mathbb{A}$ does not have (P) (via Theorem 1.4), it suffices to observe that if $\mathbb{A} \cap \mathbf{K} \neq (0)$, then $\mathbb{A} \cap \mathbf{K}$ is a nontrivial ideal in $\mathbb{A}$, and it is well-known that $\mathbb{A}$ is simple (cf., e.g., [2, Theorem VI 1.4]). (That this algebra $\mathbb{A}$ has the above properties was pointed out to us by Ron Douglas.)

Remark 3.2. Note that in the preceding example, $\mathbb{A} \cap \mathbf{K} = (0)$, from which one deduces easily that $\| \|_e$ is a C^*-norm on (the C^*-algebra) $\mathbb{A}$. But it is well-known that a C^*-algebra carries a unique C^*-norm, so we conclude that $\|A\|_e = \|A\|$ for every $A \in \mathbb{A}$. Also, $\mathbb{A}^{-W} = \mathcal{L}(\mathcal{H})$, which shows that, in general, the equivalence of $\| \|_e$ and $\| \|$ on a norm-closed algebra $\mathbb{A}$ does not extend to $\mathbb{A}^{-W}$.

Example 3.3. *If* $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$ is transitive, then $\mathbb{A}$ *has* (P) *if and only if* $\mathbb{A}$ *contains a nonzero idempotent* E *of finite rank*. Indeed, by virtue of Theorem 1.4 it suffices to show that if $\mathbb{A}$ is transitive and contains a nonzero idempotent E of finite rank, then $\mathbb{A}$ has (P). Thus our task is to find a suitable quadruple $(y_0, \alpha, \delta, \delta_0)$ with the properties as in Definition 1.2. Let $0 \neq y_0 \in E\mathcal{H}$, let $0 < \alpha < 1/4$ be arbitrary, and set $\delta = \|y_0\|/2$ and $\delta_0 = \delta/2$. Let now y be arbitrary in $B(y_0, \delta)^-$, and observe that since $\mathbb{A}$ is transitive, there is a sequence $\{A_n\} \subset \mathbb{A}$ such that $\|A_n y - y_0\| \to 0$. Thus $\|EA_n y - y_0\| \to 0$ also, and since $EA_n \in \mathbb{A}$ and $\|EA_n\|_e = 0$, the result follows.

Example 3.4. *If $\mathbb{A}$ is abelian and transitive, then $\mathbb{A}$ does not have (P).* Indeed, if $\mathbb{A}$ has (P), then by Corollary 1.6, $\mathbb{A}^{-W} = \mathcal{L}(\mathcal{H})$, and this contradicts the fact that $\mathbb{A}^{-W}$ is abelian with $\mathbb{A}$.

The thrust of Example 3.4 is that in the future study of Kadison's transitive algebra problem, one may assume that not only is $\mathbb{A}$ abelian and transitive, but that $\mathbb{A}$ does not have (P).

Example 3.5. Let $D \in \mathcal{L}(\mathcal{H})$ be any Donoghue shift operator (i.e., let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis for $\mathcal{H}$, and let D be defined by $De_n = \alpha_n e_{n+1},\ n = 0, 1, \ldots,$ where $\{\alpha_n\}_{n=0}^{\infty}$ is a strictly decreasing sequence of positive numbers such that $\{\alpha_n\}_{n=0}^{\infty} \in l^p$ for some $1 \le p < \infty$). Moreover, let $\mathbb{A}_D$ be the (unital, norm-closed) subalgebra of $\mathcal{L}(\mathcal{H})$ generated by D. Then $D \in \mathbf{K}$ and $\mathbb{A}_D$ has a strictly cyclic vector e_0 (cf. [18, p. 98] or [10]), but $\mathbb{A}_D$ does not have (P) because $\mathbb{A}_D$ is abelian and unicellular and therefore contains no nontrivial idempotent (cf. Theorem 1.4).

The same argument shows that if $V \in \mathcal{L}(L^2([0, 1]))$ is the usual Volterra integral operator defined by

$$(Vf)(x) = \int_0^x f(t)\,dt, \quad x \in [0, 1],\ f \in L^2([0, 1]),$$

which is also compact and unicellular, then $\mathbb{A}_V$ does not have (P).

On the other hand, if K is any nonzero unicellular compact operator in $\mathcal{L}(\mathcal{H})$, and $\widetilde{\mathbb{A}}_K$ is the (unital) C^*-algebra generated by K, then $\widetilde{\mathbb{A}}_K$ is transitive and $(\widetilde{\mathbb{A}}_k)^{-W} = \mathcal{L}(\mathcal{H})$ by Example 3.3, since K^*K is a nonzero positive compact operator in $\widetilde{\mathbb{A}}_K$ and the nontrivial finite-rank spectral projections of K^*K belong to $(\widetilde{\mathbb{A}}_K)^{-W}$.

More examples of algebras $\mathbb{A}$ with (P) will be found in Section 8.

Proposition 3.6. *If $\mathbb{A}$ is a transitive algebra with (P), then there exists a nonzero idempotent F of finite rank in $\mathbb{A}$ such that the algebra $F\mathbb{A}F|_{\mathcal{F}} = \mathcal{L}(\mathcal{F})$ (the algebra of all operators on the finite dimensional space $\mathcal{F} = F\mathcal{H}$).*

Proof. Let F be the finite-rank idempotent in $\mathbb{A}$ given by Theorem 1.4. Then the algebra $F\mathbb{A}F|_{\mathcal{F}}$ is a subalgebra of $\mathcal{L}(\mathcal{F})$, and by Burnside's theorem, it suffices to show that this algebra is transitive. Thus let $x, y \in \mathcal{F}$ with $x \neq 0$. Then $Fx = x$ and since $\mathbb{A}$ is transitive, there exists a sequence $\{A_n\}_{n=1}^{\infty} \subset \mathbb{A}$ such that $\|A_n Fx - y\| \to 0$ and thus $\|FA_n Fx - y\| \to 0$, which shows that $F\mathbb{A}F|_{\mathcal{F}}$ is transitive. □

Corollary 3.7. *If $\mathbb{A}$ is a transitive algebra with (P) and $F \neq 0$ is as in Proposition 3.6, then $\mathbb{A}$ contains every $B \in \mathcal{L}(\mathcal{H})$ such that* range $B \subset$ range F *and* ker $B \supset$ ker F. *In particular, $\mathbb{A}$ contains idempotents of rank one.*

Proof. The algebra $F\mathbb{A}F$ is a subalgebra of $\mathbb{A}$ and $BF = FB = B$, so $B|_{\mathcal{F}} \in \mathcal{L}(\mathcal{F}) = F\mathbb{A}F|_{\mathcal{F}}$. Thus there exists $A_B \in \mathbb{A}$ such that $FA_BF|_{\mathcal{F}} = B|_{\mathcal{F}}$. Since both FA_BF and B vanish on $\ker F$, $B = FA_BF \in \mathbb{A}$. □

4. More properties of algebras with (P)

The proof of Theorem 1.4 can be used to find some new features of subalgebras $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$ with (P) and an implementing quadruple $(y_0, \alpha, \delta, \delta_0)$. With the notation as in that proof, let

$$A(w) := \sum_{j=1}^{n} f_j(w) A_{y_j}, \quad \forall w \in B(y_0, \delta)^- \tag{4.1}$$

and note that the function $w \to A(w)$ is continuous when $B(y_0, \delta)^-$ is given its weak topology and $\mathbb{A}$ its norm topology. Thus the map $w \to g(w) = A(w)w$ is also continuous from $(B(y_0, \delta)^-$, weak) to ($\mathcal{H}$, weak) and satisfies

$$\|g(w) - y_0\| \leq \delta - (1-\theta)\delta_1, \quad \forall w \in B(y_0, \delta)^-,$$

where $\delta_1 = \delta - \delta_0 - 2\delta\beta \in (0, \delta)$. Set

$$\rho := \frac{(1-\theta)\delta_1}{\|y_0\| + \delta}, \tag{4.2}$$

and observe that for $\lambda \in D(1, \rho)^- := \{\lambda \in \mathbb{C} : |1 - \lambda| \leq \rho\}$, by using (4.2) and the inequality

$$\begin{aligned} \|\lambda g(w) - y_0\| &= \|g(w) - y_0 - (1-\lambda)g(w)\| \\ &\leq \delta - (1-\theta)\delta_1 + \rho\|g(w)\| \\ &\leq \delta - (1-\theta)\delta_1 + \rho(\|y_0\| + \delta) \\ &= \delta, \end{aligned}$$

we get that for all pairs (λ, w) with $\lambda \in D(1, \rho)^-$ and $w \in B(y_0, \delta)^-$, we have $\|\lambda g(w) - y_0\| \leq \delta$, so the function $w \to \lambda g(w)$ is weakly continuous and maps $B(y_0, \delta)^-$ into itself; consequently, again by the Schauder–Tychonoff theorem, for every fixed $\lambda \in D(1, \rho)^-$, there exists $w_\lambda \in B(y_0, \delta)^-$ such that $\lambda g(w_\lambda) = w_\lambda$, or, equivalently,

$$A(w_\lambda)w_\lambda = (1/\lambda)w_\lambda. \tag{4.3}$$

But $\rho < 1/2$ and $|\lambda| \leq 1 + \rho < 3/2$, so $(1/|\lambda|) > 2/3 > 1/2 > \alpha$. Hence $1/\lambda \notin \sigma_e(A(w_\lambda))$ and hence, as in the proof of Theorem 1.4, the Riesz–Dunford idempotent $E_\lambda \neq 0$ associated with the root space of the eigenvalue $1/\lambda$ of $A(w_\lambda)$ belongs to $\mathbb{A}$, and the rank of E_λ is finite. In this way we associate with any $\lambda \in D(1, \rho)^-$ a vector $w_\lambda \in B(y_0, \delta)^-$ and an operator $A(w_\lambda) \in \mathbb{A}$ with $\|A(w_\lambda)\|_e \leq \alpha$ satisfying $A(w_\lambda)w_\lambda = (1/\lambda)w_\lambda$, where $1/|\lambda| \geq 1/2 > \alpha$. Due to the above remark we introduce the following.

Definition 4.1. A vector $w \in B(y_0, \delta)^-$ will be called a *Gohberg vector associated with* an algebra $\mathbb{A}$ having (P), an implementing quadruple $\sigma = (y_0, \alpha, \delta, \delta_0)$, and a function $A(\cdot) : D(1, \rho) \to \mathbb{A}$ defined by

$$A(w) := \sum_{j=1}^{n} f_j(w) A_{y_j}, \quad \forall w \in D(1, \rho),$$

(where the f_j and A_j are as in the notation of the proof of Theorem 1.4), if there exists μ_w with $|\mu_w| \geq 1/2$ such that

$$A(w)w = \mu_w w. \tag{4.4}$$

The set of all Gohberg vectors associated with $\mathbb{A}, \sigma$, and the function $A(\cdot)$ will be denoted by $G := G(\mathbb{A}, \sigma, A(\cdot))$.

The discussion preceding the definition is, in fact, the proof of the following.

Proposition 4.2. *With $\mathbb{A}, \sigma$, and $A(\cdot)$ as in Definition 4.1, for all $\lambda \in D(1, \rho)^-$, the vectors w_λ as in (4.3) belong to G and satisfy $\mu_{w_\lambda} = 1/\lambda$. Moreover,* $\mathrm{card}(G)$*, the cardinal number of G, is $2^{\aleph_0}$.*

Proof. If $\lambda_1 \neq \lambda_2$ then (4.3) implies that $w_{\lambda_1} \neq w_{\lambda_2}$. Consequently the map $\lambda \to w_\lambda$ of $D(1,\rho)^-$ into G is injective and the cardinal number of G is at least $2^{\aleph_0}$. Since card $\mathcal{H} = 2^{\aleph_0}$, the result follows. □

Proposition 4.3. *With* $\mathbb{A}, \sigma$*, and* $A(\cdot)$ *as in Definition 4.1,*

$$\operatorname{card}(\{A(w_\lambda) : A(w_\lambda)w_\lambda = (1/\lambda)w_\lambda,\ \lambda \in D(1,\rho)^-\}) = 2^{\aleph_0}. \tag{4.5}$$

Proof. We introduce an equivalence relation $\sim$ on $D(1,\rho)^-$ by declaring that $\lambda_1 \sim \lambda_2$ if $A(w_{\lambda_1}) = A(w_{\lambda_2})$. (Recall from above that $\lambda_1 \neq \lambda_2$ implies $w_{\lambda_1} \neq w_{\lambda_2}$). Obviously the cardinal number of the set in (4.5) is the number of equivalence classes in the corresponding partition of $D(1,\rho)^-$, and thus to show it is $2^{\aleph_0}$ it suffices to show that each equivalence class in this partition contains only finitely many λ's. If we suppose otherwise, then there exists a sequence $\{\lambda_n\}_{n=1}^\infty$ in the same equivalence class with all λ_n distinct. Thus $A(w_{\lambda_n})w_{\lambda_n} = (1/\lambda_n)w_{\lambda n},\ n \in \mathbb{N}$, which gives $A(w_{\lambda_1})$ countably many isolated eigenvalues satisfying $|1/\lambda_1| > 1/2$ and contradicts the fact that $\|A(w_{\lambda_1})\|_e \leq \alpha < 1/2$. □

The next proposition provides supplementary properties of G.

Proposition 4.4. *With* $\mathbb{A}, \sigma$*, and* $A(\cdot)$ *as in Definition 4.1, G also has the following properties*

(i) *the map* $w \mapsto A(w)$ *is continuous from* (G, weak) *to* $(\mathbb{A}, \|\,\|)$,
(ii) *G is a weakly compact subset of* $B(y_0,\delta)^-$,
(iii) $\{A(w) : w \in G\}$ *is compact in the norm topology of* $\mathcal{L}(\mathcal{H})$,
(iv) *for* $w \in G$*, let* E_w *denote the Riesz–Dunford idempotent in* $\mathbb{A}$ *associated with the equation* $A(w)w = \mu_w w$*. Then* rank E_w *is finite and the map* $w \to E_w$ *is continuous from* (G, weak) *into* $(\mathbb{A}, \|\,\|)$,
(v) $\{E_w : w \in G\}$ *is compact in the norm topology of* $\mathbb{A}$,
(vi) *for any set* $W \subset G$ *such that*

$$E_{w_1}E_{w_2} = E_{w_2}E_{w_1}, \quad \forall w_1, w_2 \in W,$$

the set $\{E_w : w \in W\}$ *is finite,*
(vii) *the set* $\{\text{rank } E_w : w \in G\}$ *is finite.*

Proof. Properties (iii) and (v) are direct consequences of Properties (i), (ii), and (iv), and we have already noted that (i) holds. As regards (ii), let $\{w_n\}_{n=1}^\infty \subset G$ satisfy $w_n \to w_0$ weakly. From (i) we know that

$$\|A(w_n) - A(w_0)\| \to 0,$$

and these two facts together imply that

$$\mu_{w_n} \to \mu_{w_0}, \qquad |\mu_{w_0}| \geq 1/2 > \alpha, \tag{4.6}$$

and that $A(w_0)w_0 = \mu_{w_0}w_0$. Thus $w_0 \in G$. This shows that G is weakly closed, and since $B(y_0,\delta)^- \supset G$ is weakly compact, so is G.

Concerning (iv), we note (see Definition 4.1) the useful fact that

$$\|A(w)\| \leq M_0 := \max_j \|A_{y_j}\|, \qquad \|A(w)\|_e \leq \alpha, \quad \forall w \in B(y_0,\delta)^-. \tag{4.7}$$

Let now $w_n \in G$ with $w_n \to w_0 (\in G)$ weakly. Then from (4.6) we know that $\mu_{w_n} \to \mu_{w_0}$ and that $|\mu_{w_0}| \geq 1/2$. Thus, using the upper semi-continuity of the spectrum, we can find a suitable $\varepsilon > 0$ small enough that $\Gamma_\varepsilon = \{\zeta : |\zeta - \mu_{w_0}| = \varepsilon\}$ is disjoint from $\cup_{n=1}^{\infty} \sigma(A_{w_n})$. Then

$$E_{w_n} = \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} (\zeta I - A(w_n))^{-1} d\zeta$$

makes sense for all n. It is now easy to infer from standard properties of the functional calculus that

$$\lim_{n\to\infty} \|E_{w_n} - E_{w_0}\| = 0. \tag{4.8}$$

This establishes (iv) and hence (v) too. Therefore there exists $M_1 < \infty$ such that $M_1 := \sup\{\|E_w\| : w \in G\}$.

Concerning (vi), let W be a set as defined therein. Arguing by contradiction, we assume that there exists a sequence with distinct entries, say $\{E_{w_n}\}$ with $w_n \in W$ for $n \in \mathbb{N}$. Due to (ii), we can also assume, by a change of notation, that there exists $w_0 \in G$ such that $w_n \to w_0$ weakly. Then, by (iv), the sequence $\{E_{w_n}\}$ is a norm-Cauchy sequence and if w_n, w_m are arbitrary in W, then the idempotent $E_{w_n}(1 - E_{w_m})$ is either 0 or

$$1 \leq \|E_{w_n}(1 - E_{w_m})\| = \|E_{w_n}(E_{w_n} - E_{w_m})\| \leq M_1 \|E_{w_n} - E_{w_m}\| \to 0,$$

which is impossible. Thus for n, m sufficiently large we must have $E_{w_n}(1-E_{w_m}) = 0$ and $E_{w_n} = E_{w_n} E_{w_m}$. By reversing the roles of m and n, we see that for such m and n, $E_{w_m} = E_{w_m} E_{w_n}$ so the sequence $\{E_{w_n}\}$ is eventually constant, a contradiction. This concludes the proof of (vi).

To establish (vii), take some $w_1, w_2 \in G$ such that $E_{w_2} E_{w_1} x = 0$, where $x \in \mathcal{H}$. Then

$$\|E_{w_1} x\| \leq \|(E_{w_2} - E_{w_1}) E_{w_1} x\| \leq M_1 \|E_{w_2} - E_{w_1}\| \, \|E_{w_1} x\|. \tag{4.9}$$

Thus if $\|E_{w_2} - E_{w_1}\| < 1/M_1$, we discover from (4.9) that $E_{w_1} x = 0$; this shows that $E_{w_2}|_{E_{w_1}\mathcal{H}}$ is injective, and therefore that rank $E_{w_2} \geq$ rank E_{w_1}; by symmetry we get equality of those ranks. The desired conclusion now follows from the norm-compactness of W. □

Using Proposition 4.4(iv) and the well-known fact that the set of all idempotents in $\mathcal{L}(\mathcal{H})$ with a fixed rank k is relatively open in the norm topology we deduce the following.

Corollary 4.5. *With $\mathbb{A}, \sigma$, and $A(\cdot)$ as in Definition 4.1, for every $k \in \mathbb{N}$, the set $\{w \in G :$ rank $E_w = k\}$ is weakly open and weakly closed in G.*

The properties of the family of idempotents $\{E_w : w \in G\}$, discussed above have some analogs for the family of *all* the Riesz idempotents of $A(w)$ $(w \in G)$ associated with the points in the spectrum of $A(w)$ outside the disc $D(0, 1/2)$. In other words, whereas the above discussion considered exactly one idempotent E_w associated with a point $w \in G$ (namely the one satisfying $w \in E_w \mathcal{H}$), we now turn our attention to (the finite number of) all isolated eigenvalues in $\sigma(A(w)) \cap (\mathbb{C} \setminus D(0, 1/2))$ and their associated idempotents.

To facilitate the exposition we will first introduce some supplementary notation.

For $w \in G$ let $\sigma(w)$ denote $\sigma(A(w)) \setminus D(0, 1/2)$ and $n(w) = \operatorname{card} \sigma(w)$. It is clear that $n(w) < \infty$. We also have

$$N := \sup_{w \in G} n(w) < \infty. \tag{4.10}$$

Indeed, otherwise there exists a sequence $\{w_n\}_{n=1}^{\infty} \subset G$ such that $n(w_j) \to \infty$. Without loss of generality we can assume that $w_j \to w_0$ ($\in G$) weakly; hence $A(w_j) \to A(w_0)$ in norm. Let

$$\delta_0 := \min\{|\mu_1 - \mu_2| : \mu_1, \mu_2 \in \sigma(w_0),\ \mu_1 \neq \mu_2\}$$

and denote by V_δ the set $\cup_{\mu\in\sigma(w_0)} D(\mu, \delta)$, where $\delta \in (0, \delta_0/3)$. By virtue of the norm-upper semicontinuity of the spectrum, there exists $j(\delta)$ such that for $j \geq j(\delta)$ we have $\sigma(w_j) \subset V_\delta$ and that

$$\left\| \frac{1}{2\pi i} \int_{|z-\mu_0|=\delta_0/3} (z - A(w_j))^{-1} dz - \frac{1}{2\pi i} \int_{|z-\mu_0|=\delta_0/3} (z - A(w_0))^{-1} dz \right\| < \varepsilon, \quad (4.11)$$

where ε is sufficiently small, $\mu_0 \in \sigma(w_0)$, and the integrals are well-defined. Let $E_{j,\mu}$ denote the Riesz idempotent of $A(w_j)$ associated with $\mu \in \sigma(w_j)$ $(j = 0, 1, 2, \ldots)$; clearly $1 \leq \operatorname{rank} E_{j,\mu} < \infty$. Define

$$n_0 = n(w_0) \max\{\operatorname{rank} E_{0,\mu} : \mu \in \sigma(w_0)\}.$$

According to (4.11) we will have (for $j \geq j(\delta)$),

$$\left\| E_{0,\mu_0} - \sum_{\mu\in D(\mu_0,\delta_{0/3})} E_{j,\mu} \right\| < \varepsilon$$

for any $\mu_0 \in \sigma(w_0)$. Using the argument in the proof of Proposition 4.4(vii) we obtain

$$\operatorname{rank} E_{0,\mu_0} = \operatorname{rank} \left(\sum_{\mu\in D(\mu_0,\delta/3)} E_{j,\mu} \right) \geq \operatorname{card}(\sigma(w_j) \cap D(\mu_0, \delta_0/3)),$$

and consequently (by summing over $\mu_0 \in \sigma(w_0)$) we get

$$n_0 \geq \operatorname{card} \sigma(w_j), \quad \forall j \geq j(\delta),$$

a contradiction.

The preceding proof (cf. (4.10)) can be easily adapted to yield some supplementary properties of G, given below.

Proposition 4.6. *With $\mathbb{A}, \sigma$, and $A(\cdot)$ as in Definition 4.1, the following properties hold:*
(i) *the function $n(w)$, $w \in G$, is weakly lower semi-continuous;*
(ii) *for $w \in G$, define $\operatorname{rank}(w) := \sum_{\mu\in\sigma(w)} \operatorname{rank} E_\mu(w)$, where $E_\mu(w)$ denotes the Riesz idempotent of $E_\mu(w)$ associated with $\mu \in \sigma(w)$, then*
 (a) *$r(G) := \max\{\operatorname{rank}(w) : w \in G\} < \infty$;*
 (b) *the set $R_k = \{w \in G : \operatorname{rank}(w) = k\}$ is open in the relative weak topology on G, for any $k = 1, 2, \ldots, r(G)$;*
 (c) *$G = \cup_{k=1}^{r(G)} R_k$ is a partition of G into separated parts; so each R_k is also weakly closed (note that some of the R_k may be empty).*

An obvious consequence of properties (i) and (ii)-(b) above and Corollary 4.5 is the following.

Corollary 4.7. *The set $\{w \in G : n(w) = N\}$ (where N is defined in (4.10)) is open in the relative weak topology on G. Moreover,*

$$G = \bigcup_{1\leq k,\ell\leq \operatorname{rank} G} \{w \in G : \operatorname{rank}(w) = k, \operatorname{rank} E_w = \ell\}$$

and each set in this union is both relatively open and closed in the weak topology on G.

Remark 4.8. We note that in the proofs of Proposition 4.6 and Corollary 4.7 we made use only of the following properties of the map $w \to A(w)$; namely, that it is continuous from $(B(y_0, \delta_0)^-, \text{weak})$ to $(\mathbb{A}, \| \|)$ and $\|A(w)\|_e \le \alpha$ for every $w \in B(y_0, \delta)^-$. A Gohberg set associated with such a map can be defined as in Definition 4.1. In case this set G is not empty we say that the algebra $\mathbb{A}$ has *Property* (G) or, more simply, that $\mathbb{A}$ has (G). Note that we have established that for an algebra $\mathbb{A}$, having (P) implies having (G), and if $\mathbb{A}$ has (P), then G is infinite. Moreover, if $\mathbb{A}$ has (G) and the Gohberg set G is infinite, then the conclusions of Proposition 4.6 are valid in this context.

Remark 4.9. We note that the results presented above display the flavor of some of the deep contributions of I. Gohberg to the Fredholm theory of analytic operator valued functions (e.g., [8] and the references of [6,7]).

5. Equivalence of the norms $\| \|$ and $\| \|_e$

In this section we investigate conditions on an algebra $\mathbb{A}$ that imply the equivalence of the norms $\| \|$ and $\| \|_e$ on $\mathbb{A}$, and consequences about the transitivity of $\mathbb{A}$ when this equivalence is present.

Our principal theorem concerning the latter comes after the next definition. For $T \in \mathcal{L}(\mathcal{H})$, we write $\mathcal{R}(\sigma(T))$ for the algebra of all complex-valued rational functions $r(\zeta)$ with poles off $\sigma(T)$, and we recall that if there exists $B \ge 1$ such that

$$\|r(T)\| \le B \sup_{\delta \in \sigma(T)} |r(\zeta)|, \quad \forall r \in \mathcal{R}(\sigma(T)),$$

then $\sigma(T)$ is called a *B-spectral set* for T.

Theorem 5.1. *For $T \in \mathcal{L}(\mathcal{H})$, let $\mathbb{C}[z]$, as usual, denote the algebra of all complex polynomials $p(\zeta)$ and define $\mathbb{A}_T$ as either*

$$\mathbb{A}_{T_p} = \{p(T) : p \in \mathbb{C}[z]\}^{-\| \|} \quad \text{or} \quad \mathbb{A}_{T_r} = \{r(T) : r \in \mathcal{R}(\sigma(T))\}^{-\| \|}.$$

If $\| \|_e$ is a norm on $\mathbb{A}_T$ equivalent to $\| \|$ and there exists $B \ge 1$ such that $\sigma(\pi(T))$ $(=\sigma_e(T))$ is a B-spectral set for $\pi(T)$ for some $B \ge 1$, then the algebra $\mathbb{A}_T$ has a nontrivial invariant subspace. If, in addition, $T \notin \mathbb{C}1_{\mathcal{H}}$ and $\mathcal{R}(\sigma_e(T))$ is norm dense in $C(\sigma_e(T))$, then T has a nontrivial hyperinvariant subspace.

Proof. For the first statement, we give the proof only for the second case; the other is exactly the same. Let $M > 0$ be such that

$$\|r(T)\| \le M\|r(T)\|_e, \quad \forall r \in \mathcal{R}(\sigma(T)).$$

Obviously we may assume that $T \notin \mathbb{C}1_{\mathcal{H}}$ and also that $\sigma(T) = \sigma_e(T)$, for otherwise $\sigma_p(T^*) \cup \sigma_p(T) \ne \emptyset$ and thus $\mathbb{A}_T$ has a nontrivial invariant subspace. Hence

$$\sup_{\zeta \in \sigma_e(T)} |r(\zeta)| \le \|r(T)\| \le M\|r(\pi(T))\| \le MB \sup_{\zeta \in \sigma_e(T)} |r(\zeta)|, \quad \forall r \in \mathcal{R}(\sigma(T)), \tag{5.1}$$

which shows that $\sigma(T) = \sigma_e(T)$ is an MB-spectral set for T, and thus by Stampfli's theorem [19], $\mathbb{A}_T$ has a nontrivial invariant subspace.

Turning now to the proof of the second statement of the theorem, and again with $\sigma_e(T) = \sigma(T)$, we note that (5.1), together with the hypothesis that $\mathcal{R}(\sigma_e(T))$ is norm dense in $C(\sigma_e(T))$, yields immediately that there exists a norm bicontinuous algebra isomorphism φ of $C(\sigma_e(T)) = C(\sigma(T))$ onto $\mathbb{A}_{T_r}$. Thus in the language of Colojoara–Foias, T is decomposable and the algebra $\mathbb{A}_{T_r}$ is a scalar algebra. As is well-known, such operators T have nontrivial hyperinvariant subspaces. $\square$

Corollary 5.2. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is invertible, there exists $B \geq 1$ such that $\sigma_e(T)$ is a B-spectral set for $\pi(T)$, and there exists $M > 0$ such that*

$$\|r(T)\| \leq M\|r(T)\|_e, \quad \forall r \in \mathcal{R}(\sigma(T)).$$

Then T and T^{-1} have a common nontrivial invariant subspace.

Corollary 5.3. *Let S be a subnormal operator in $\mathcal{L}(\mathcal{H})$, let K be arbitrary in $\mathbf{K}$, and define $T = S + K$ and $\mathbb{A}_T$ as either of the algebras in the statement of Theorem 5.1. If $\| \ \|_e$ is a norm on $\mathbb{A}_T$ equivalent to $\| \ \|$, then there exists a nontrivial subspace $\mathcal{M} \subset \mathcal{H}$ such that $\mathbb{A}_T\mathcal{M} \subset \mathcal{M}$.*

Proof. Obviously $\pi(T) = \pi(S)$, which is subnormal in the Calkin representation, and thus $\sigma(\pi(S)) = \sigma_e(T)$ is a spectral set for $\pi(T)$, so the result follows from Theorem 5.1. $\square$

Concerning the hypothesis in Theorem 5.1 that $\mathcal{R}(\sigma_e(T))^{-\| \ \|} = C(\sigma_e(T))$, we note for the interested reader that necessary and sufficient conditions on a compact set $\Sigma \subset \mathbb{C}$ in order that $\mathcal{R}(\Sigma)$ be norm dense in $C(\Sigma)$ are well-known although complicated. This result is essentially due to Vitushkin [20]; cf. also [5].

We now give some conditions on the sets $\Gamma_\alpha(y)$ that imply the equivalence of $\| \ \|_e$ and $\| \ \|$ on an algebra $\mathbb{A}$.

Theorem 5.4. *With $\mathbb{A}$ as always and $\alpha > 0$ fixed, each set $\Gamma_\alpha(y)$ in the collection $\{\Gamma_\alpha(y) : y \in \mathcal{H}\}$ is bounded if and only if $\| \ \|_e$ is a norm on $\mathbb{A}$ equivalent to $\| \ \|$ (i.e., if and only if there exists $M > 0$ such that*

$$\|A\| \leq M\|A\|_e, \quad \forall A \in \mathbb{A}). \tag{5.2}$$

Proof. Suppose first that (5.2) holds, and let y be arbitrary in $\mathcal{H} \setminus (0)$. Then

$$\Gamma_\alpha(y) = \{Ay : A \in \mathbb{A}, \ \|A\|_e \leq \alpha\} \subset B(0, \alpha M\|y\|)^-. \tag{5.3}$$

On the other hand, suppose now that each set $\Gamma_\alpha(y)$, $y \in \mathcal{H}$, is a bounded subset of $\mathcal{H}$, i.e., that $\{Ay : A \in \mathbb{A}, \|A\|_e \leq \alpha\}$ is bounded for each $y \in \mathcal{H}$. (Observe first that this implies that $\mathbb{A} \cap \mathbf{K} = \{0\}$, for if $0 \neq K \in \mathbb{A} \cap \mathbf{K}$, then $nKy \in \Gamma_\alpha(y)$ for every $n \in \mathbb{N}$, so for each $y \notin \ker K$, $\Gamma_\alpha(y)$ is unbounded.) Then by the uniform boundedness principle, the set $\{A \in \mathbb{A} : \|A\|_e \leq \alpha\}$ is bounded in $\mathcal{L}(\mathcal{H})$; say $\|A\| \leq N$ for all such A. Defining $B_A = \alpha A / \|A\|_e$ for each $A \in \mathbb{A}$, we see that $0 \neq \|B_A\|_e = \alpha$ for $A \in \mathbb{A}$, and thus that $\|A\| \leq (N/\alpha)\|A\|_e$ for all $A \in \mathbb{A}$. $\square$

Another sufficient condition for the equivalence of the norms $\| \ \|_e$ and $\| \ \|$ is the following.

Theorem 5.5. *Suppose $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$, $\alpha > 0$, and there exists $y \in \mathcal{H}$ such that $\Gamma_\alpha(y)^-$ is bounded and has nonempty interior. Then,*

(i) *$\| \ \|_e$ and $\| \ \|$ are equivalent norms on $\mathbb{A}$ and the collection of sets*

$$\{\Gamma_\alpha(y)^- : y \in B(0,1)^-\}$$

is uniformly bounded by αM, where M is as in (5.2),

(ii) *y is a strictly cyclic vector for $\mathbb{A}^{-W}$, and*
(iii) *if $\mathbb{A} = \mathbb{A}^{-W}$, then $\mathbb{A}$ has a nontrivial invariant subspace.*

Corollary 5.6. *If $\mathbb{A}$ is a transitive algebra and $\alpha > 0$, then for every $y \in \mathcal{H} \setminus (0)$, either $\Gamma_\alpha(y)$ is unbounded or $\Gamma_\alpha(y)$ is nowhere dense in $\mathcal{H}$.*

Proof of Theorem 5.5. (i) Let $y \in \mathcal{H}$ be such that $\Gamma_\alpha(y)^-$ is bounded by R and has nonempty interior. Then, using the absolute convexity of $\Gamma_\alpha(y)^-$, we conclude that there exists $0 < r < R$ such that

$$B(0,r)^- \subset \Gamma_\alpha(y)^- \subset B(0,R)^-. \tag{5.4}$$

Since for every $A \in \mathbb{A}$ with $\|A\|_e \le 1$, we have $A\Gamma_\alpha(y)^- \subset \Gamma_\alpha(y)^-$, we infer that for such A and for every $x \in \mathcal{H}$ with $\|x\| = 1$, we have $\|A(rx)\| \le R$. Thus $\|A\| \le (R/r)\,\|A\|_e$ for $A \in \mathbb{A}$, and we conclude that $\|\ \|_e$ and $\|\ \|$ are equivalent on $\mathbb{A}$. The second statement follows from (5.3) with $M = R/r$.

(ii) As before, (5.4) holds. Now let z be arbitrary in $B(0,1)^-$. Then $rz \in B(0,r)^-$, so there exists a sequence $\{A_n\} \subset \mathbb{A}$ with $\|A_n\|_e \le \alpha$ such that $\|A_n y - rz\| \to 0$ (where α and y are as in (i)). Since by (i), $\|\ \|_e$ and $\|\ \|$ are equivalent on $\mathbb{A}$, the sequence $\{A_n\}$ is bounded, and thus has a subsequence $\{A_{n_k}\}$ that converges WOT to some $A_0 \in \mathbb{A}^{-W}$. Thus, with $r > 0$ as in (5.4), $(1/r)A_0 y = z$, which shows that $(\mathbb{A}^{-W})y = \mathcal{H}$ as desired.

(iii) If $\mathbb{A}$ were transitive, then by (ii) and [17, Corollary 9.10], $\mathbb{A}$ would be $\mathcal{L}(\mathcal{H})$, but $\|\ \|_e$ and $\|\ \|$ are equivalent on $\mathbb{A}$ by (i), and not equivalent on $\mathcal{L}(\mathcal{H})$, so $\mathbb{A}$ cannot be transitive. □

We now exhibit two illustrative examples which show that for certain abelian algebras $\mathbb{A}$, there are indeed associated sets $\Gamma_\alpha(y)$ (or $\Gamma_\alpha(y)^-$) that have nonvoid interior.

Example 5.7. (i) Let $D \in \mathcal{L}(\mathcal{H})$ and let $\mathbb{A}_D$ be as in Example 3.5; recall that e_0 is a strictly cyclic vector for $\mathbb{A}_D$. Then easy calculations, using general facts about strictly cyclic abelian algebras (cf. [18, p. 92]), show that for every $\alpha > 0$,

$$\Gamma_\alpha(e_0) = \left\{ \sum_{n \in \mathbb{N}_0} \zeta_n e_n \in \mathcal{H} : |\zeta_0| \le \alpha \right\}.$$

Therefore $\Gamma_\alpha(e_0)$ is a closed set, contains the ball $B(0,\alpha)^-$, and is unbounded.

(ii) Let V be the Volterra integral operator from Example 3.5, let $\mathbb{A}_V$ be, as before, the unital, norm closed algebra generated by V, and recall that the constant function $f_0 \equiv 1$ is a cyclic vector for A_V since the polynomials are norm-dense in $L^2([0,1])$. Well-known facts about V, together with easy calculations show that for every $\alpha > 0$, $\Gamma_\alpha(f_0)^- = \mathcal{H}$, the entire Hilbert space.

(The calculations for (i) and (ii) use the fact that $S, V \in \mathbf{K}$ and therefore every element in each algebra has the form $\lambda + K$ for some $\lambda \in \mathbb{C}$ and $K \in \mathbf{K}$, so obviously $\|\lambda + K\|_e = |\lambda|$.)

We close this section by giving a short proof, using only the definition of Property (P) and Corollary 1.6, of V. Lomonosov's strongest theorem concerning transitivity.

Theorem 5.8 (*Lomonosov [12]; cf. also [3,16,1]*)**.** *Suppose $\mathbb{A} = \mathbb{A}^{-W} \ne \mathcal{L}(\mathcal{H})$ and $\mathbb{A}$ contains a net $\{A_\lambda\}$ with the properties that $A_\lambda \to A_0 \ne 0$ in the WOT and $\|A_\lambda\|_e \to 0$. Then $\mathbb{A}$ has a nontrivial invariant subspace.*

Proof. Suppose, to the contrary, that $\mathbb{A}$ is transitive, and using convexity, we can obtain from $\{A_\lambda\}$ a net $\{A_\mu\} \subset \mathbb{A}$ such that $A_\mu \to A_0$ in the SOT. Let y_0 be arbitrary such that $A_0 y_0 \neq 0$, and choose $0 < \delta < (\|A_0 y_0\|/2)\|A_0\|^{-1}$, $0 < \alpha < 1/2$, and $0 < \delta_0 < (1-2\alpha)\delta$. Let $y \in \mathcal{H}$ be any vector satisfying $\|y - y_0\| < \delta$ and observe that $\|A_0 y\| > \|A_0 y_0\|/2 > 0$. Then, via transitivity, choose $A_1 \in \mathbb{A}$ such that $\|A_1(A_0 y) - y_0\| < \delta_0$. Now choose μ_0 sufficiently far out in the directed set $\{\mu\}$ that $\left\|A_1 A_{\mu_0} y - y_0\right\| < \delta_0$ and $\|A_1 A_{\mu_0}\|_e < \alpha$. This shows that $\mathbb{A}$ has (P), and therefore by Corollary 1.6 that $\mathbb{A}^{-W} = \mathcal{L}(\mathcal{H})$, which is a contradiction and shows that $\mathbb{A}$ must be intransitive. □

6. Bounded $\Gamma_\alpha(y)$'s

The results obtained in Theorems 5.4, 5.5, and Corollary 5.6 stress the role that the boundedness of the $\Gamma_\alpha(y)$'s plays in the study of the algebras $\mathbb{A}$. We will now provide some supplementary similar results. For this purpose we recall the well-known fact that if $\mathcal{K} \subset \mathcal{H}$ is nonempty, absolutely convex and (norm-) closed in $\mathcal{H}$, then for every $x \in \mathcal{H}$ there exists a unique w_x satisfying the property

$$\|x - w_x\| = \min\{\|z - x\| : z \in \mathcal{K}\},$$

(clearly, $w_x = x$ if $x \in \mathcal{K}$; otherwise $\|x - w_x\| > 0$). Moreover, the map $x \mapsto \Phi_{\mathcal{K}}(x) := x - w_x$ satisfies the relations

$$\begin{aligned} |\langle z, \Phi_{\mathcal{K}}(x)\rangle| &\le \operatorname{Re}\langle w_x, \Phi_{\mathcal{K}}(x)\rangle = \langle w_x, \Phi_{\mathcal{K}}(x)\rangle \\ &= \langle x, \Phi_{\mathcal{K}}(x)\rangle - \|\Phi_{\mathcal{K}}(x)\|^2, \quad \forall z \in \mathcal{K}. \end{aligned} \tag{6.1}$$

Perhaps less widely known (but easy to prove by geometric arguments in a Euclidean plane) are the Lipschitz property

$$\|\Phi_{\mathcal{K}}(x_1) - \Phi_{\mathcal{K}}(x_2)\| \le \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathcal{H}, \tag{6.2}$$

and the monotonicity property

$$\langle \Phi_{\mathcal{K}}(x_1) - \Phi_{\mathcal{K}}(x_2), x_1 - x_2\rangle \ge 0, \quad \forall x_1, x_2 \in \mathcal{H}, \tag{6.3}$$

of the map $\Phi_{\mathcal{K}}$.

In connection with the above two concepts, one of the end results of G. Minty's ideas (e.g., [13,14]) yields the following property of the map $\Phi_{\mathcal{K}}$.

Proposition 6.1 (*See [15, Corollary 5.1.8]*). *With $\mathcal{K}$ and $\Phi_{\mathcal{K}}$ as above, if*

$$\lim_{\|x\|\to+\infty} \|\Phi_{\mathcal{K}}(x)\| = +\infty, \tag{6.4}$$

then $\Phi_{\mathcal{K}}(\mathcal{H}) = \mathcal{H}$.

Remark 6.2. In case $\mathcal{K}$ is bounded then obviously (6.4) is satisfied, and therefore $\Phi_{\mathcal{K}}(\mathcal{H}) = \mathcal{H}$. Conversely, if (6.4) holds, then $\mathcal{K}$ must be bounded (since otherwise there exists $\{x_n\} \subset \mathcal{K}$ with $\|x_n\| \to +\infty$ but $\Phi_{\mathcal{K}}(x_n) \equiv 0$).

Some direct consequences of the properties of $\Phi_{\mathcal{K}}$ are the following.

Lemma 6.3. *Let $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$, y_0, $y \in \mathcal{H}\setminus(0)$, and $\alpha > 0$ be such that $y_0 \notin \Gamma_\alpha(y)^-$. Then setting $\mathcal{K} = \Gamma_\alpha(y)^-$, we have*

$$|\langle Aw, \Phi_{\mathcal{K}}(y_0)\rangle| \le \|A\|_e(\langle y_0, \Phi_{\mathcal{K}}(y_0)\rangle - \|\Phi_{\mathcal{K}}(y_0)\|^2), \quad \forall w \in \Gamma_\alpha(y)^-, \ A \in \mathbb{A}. \tag{6.5}$$

Proof. Indeed, in this case

$$(1/(\|A\|_e+\varepsilon))A(\Gamma_\alpha(y)^-)\subset\Gamma_\alpha(y)^-,\quad \forall\varepsilon>0,\ A\in\mathbb{A};$$

consequently

$$|\langle Aw,\Phi_{\mathcal{K}}(y_0)\rangle|\le(\|A\|_e+\varepsilon)(\langle y_0,\Phi_{\mathcal{K}}(y_0)\rangle-\|\Phi_{\mathcal{K}}(y_0)\|^2),$$
$$\forall\varepsilon>0,\ w\in\Gamma_\alpha(y)^-,\ A\in\mathbb{A},$$

and (6.5) follows by letting $\varepsilon\searrow 0$. □

Theorem 6.4. *Let $\mathbb{A}\subset\mathcal{L}(\mathcal{H})$, y_0, $y\in\mathcal{H}\setminus(0)$, and $\alpha>0$ be such that $y_0\notin\Gamma_\alpha(y)^-$. Suppose also that $\mathbb{A}$ is transitive and let $\{A_n\}\subset\mathbb{A}$ be any sequence satisfying $\|A_n\|_e\to 0$ and $\sup_n\|A_n\|<\infty$. Then $A_n\to 0$ in the WOT.*

Proof. We replace A in (6.5) by BA_nC with $B,C\in\mathbb{A}$ and w by αy to deduce that

$$\left\langle A_nCy,B^*\Phi_{\mathcal{K}}(y_0)\right\rangle\to 0,\quad B\in\mathbb{A}.$$

It follows that $\langle A_nCy,B^*\Phi_{\mathcal{K}}(y_0)\rangle\to 0$ for all $C,B\in\mathbb{A}$. Using the boundedness of $\{A_n\}$ and the fact that y and $\Phi_{\mathcal{K}}(y_0)\neq 0$ are cyclic vectors for $\mathbb{A}$ and $\mathbb{A}^*$, respectively, it is straightforward to conclude that $A_n\to 0$ in the WOT. □

Corollary 6.5. *Let $\mathbb{A}$ be as in Theorem 6.4. Then $\|\ \|_e$ is a norm on $\mathbb{A}$ and the resulting norm topology is stronger than the WOT on bounded subsets of $\mathbb{A}$; moreover $\mathbb{A}$ does not have (P).*

Proof. That $\|\ \|_e$ is a norm on $\mathbb{A}$ follows from Theorem 6.4, and the remainder of the proof is routine. □

By using Proposition 6.1, we can obtain the following variant of Theorem 6.4.

Theorem 6.6. *Let $\mathbb{A}$, $\mathcal{K}$, y_0, and $y\in\mathcal{H}\setminus(0)$ be as in Lemma 6.3 with $\mathcal{K}$ $(=\Gamma_\alpha(y)^-)$ bounded; moreover, let $\{A_n\}\subset\mathbb{A}$ satisfy $\|A_n\|_e\to 0$. Then the following statements are valid.*

(i) *If y is a cyclic vector for $\mathbb{A}$ and $\sup_n\|A_n\|<\infty$, then $A_n\to 0$ in the WOT.*
(ii) *If y is a strictly cyclic vector for $\mathbb{A}$, then $A_n\to 0$ in the WOT.*

Proof. According to Proposition 6.1 and Remark 6.2 we have $\Phi_{\mathcal{K}}(\mathcal{H})=\mathcal{H}$. Since (6.5) is trivially valid when $y_0\in\Gamma_\alpha(y)$ (i.e., (6.5) reduces to $0\le 0$), (6.5) can be written as

$$|\langle Aw,x\rangle|\le\|A\|_e(\langle x',x\rangle-\|x'\|^2)\le\|A\|_e\|x\|\,\|x'\|,\quad\forall A\in\mathbb{A},\ w\in\mathcal{K},$$

where $x\in\mathcal{H}$ and $x'\in\Phi_{\mathcal{K}}^{-1}(\{x\})$ are arbitrary. Let

$$M(x)=\inf\{\|x'\|:x'\in\Phi_{\mathcal{K}}^{-1}(\{x\})\}.$$

Then the preceding inequalities yield

$$|\langle Aw,x\rangle|\le\|A\|_e\,\|x\|\,M(x),\quad\forall x\in\mathcal{H},w\in\mathcal{K},\ A\in\mathbb{A}.\tag{6.6}$$

For $w=\alpha Cy/(\|C\|+1)$ with $C\in\mathbb{A}$ and $A=A_n$, from (6.6) we infer that $|\langle A_nCy,x\rangle|\to 0$ for all $C\in\mathbb{A}$. Thus

$$A_n|_{\mathbb{A}y}\to 0 \text{ in the (WOT).}\tag{6.7}$$

Under the hypothesis in (i), we have $(\mathbb{A}y)^-=\mathcal{H}$ and consequently $A_n\to 0$ in the WOT. Under the hypothesis in (ii), we have $\mathbb{A}y=\mathcal{H}$ and (6.7) implies again that $A_n\to 0$ in the WOT. □

In terms of $\Phi_{\mathcal{K}}$, for $\mathcal{K} = \Gamma_\alpha(y)^-$ the boundedness of $\mathcal{K}$ is obtained under a slightly better condition, as shown below.

Proposition 6.7. *Let $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$, $y \in \mathcal{H} \setminus (0)$, and $\mathcal{K} = \Gamma_\alpha(y)^-$ be given. Then there exists a hyperplane $\mathcal{M}$ (i.e., a translation of a subspace of codimension 1) such that the following statements are equivalent:*

(i) *$\mathcal{K}$ (equivalently, $\Gamma_\alpha(y)$) is bounded,*
(ii) $\lim_{\substack{\|x\|\to+\infty \\ x\in\mathcal{M}}} \|\Phi_{\mathcal{K}}(x)\| = +\infty$.

Proof. The implication (i) $\Rightarrow$(ii) is a trivial consequence of Remark 6.2. Now let

$$\mathcal{M} = 2y + \{z \in \mathcal{H} : \langle z, \Phi_{\mathcal{K}}(y)\rangle = 0\}$$

and assume that it has the property stated in (ii). If $\Gamma_\alpha(y)$ were unbounded, there would exist $\{w_n\} \subset \Gamma_\alpha(y)$ such that $\|w_n\| \to +\infty$. Then (see (6.1)),

$$|\langle w_n, \Phi_{\mathcal{K}}(y)\rangle| \le \langle y, \Phi_{\mathcal{K}}(y)\rangle - \|\Phi_{\mathcal{K}}(y)\|^2 \le (\|y\| - \|\Phi_{\mathcal{K}}(y)\|)\,\|\Phi_{\mathcal{K}}(y)\|, \quad \forall n \in \mathbb{N},$$

whence

$$\langle w_n, \Phi_{\mathcal{K}}(y)/\,\|\Phi_{\mathcal{K}}(y)\|\rangle \le \|y\|, \quad \forall n \in \mathbb{N}. \tag{6.8}$$

Taking

$$x_n = 2y + w_n - \left\langle w_n, \Phi_{\mathcal{K}}(y)/\,\|\Phi_{\mathcal{K}}(y)\|^2\right\rangle \Phi_{\mathcal{K}}(y)$$

and using (6.8), we have

$$\|\Phi_{\mathcal{K}}(x_n)\| \le \|2y\| + |\langle w_n, \Phi_{\mathcal{K}}(y)\rangle| \,/\, \|\Phi_{\mathcal{K}}(y)\| \le 3\|y\|, \quad \forall n \in \mathbb{N},$$

as well as

$$\begin{aligned} \|x_n\| &= \left\|2y - \left(\langle w_n, \Phi_{\mathcal{K}}(y)\rangle \,/\, \|\Phi_{\mathcal{K}}(y)\|^2\right) \Phi_{\mathcal{K}}(y)\right\| \\ &\ge \|w_n\| - \left\|2y - \left\langle w_n, \Phi_{\mathcal{K}}(y)/\,\|\Phi_{\mathcal{K}}(y)\|^2\right\rangle \Phi_{\mathcal{K}}(y)\right\| \\ &\ge \|w_n\| - 3\|y\| \to +\infty, \quad \forall n \in \mathbb{N}, \end{aligned}$$

a contradiction. □

Remark 6.8. It is worth mentioning that a set $\mathcal{K}$ as in Proposition 6.7 can have many of the diverse properties of the sets $\Gamma_\alpha(y)$ and still be unbounded. Here is an illustrative example.

Let $\alpha \in (0, 1)$ be fixed and define

$$\mathcal{K} = \{Q^{1/2}v : v \in \mathcal{H},\ \|(1 - Q)^{1/2}v\| \le \alpha\},$$

where Q is the multiplication operator M_x on $\mathcal{H} = L^2([0, 1])$ (i.e., defined by $(M_x f)(x) = xf(x)$ for $f \in L^2([0, 1])$). Then $\mathcal{K}$ has the following properties:

(1) $\mathcal{K}$ is absolute convex,
(2) $\mathcal{K}$ is weakly closed,
(3) $\mathcal{K}^\circ = \varnothing$,
(4) $\mathcal{K} \ne \mathcal{H}$,
(5) no ray $\{\beta x : \beta > 0\}$, where $x \in \mathcal{H} \setminus \{0\}$, is included in $\mathcal{K}$,
(6) $(\cup_{r\ge 0}\, r\mathcal{K})^- = \mathcal{H}$,
(7) $\mathcal{K}$ is unbounded.

Indeed, note first that (1) and (5) are obvious and that

$$\mathcal{K} \subset Q^{1/2}\mathcal{H} \quad \text{and} \quad \bigcup_{r \geq 0} r\mathcal{K} = Q^{1/2}\mathcal{H}.$$

Thus (3) and respectively (4) follow directly from (2), respectively (3). For (7) just consider the sequence $v_n = n\chi_{(1-\varepsilon_n,1)}$ $(n = 1, 2, \ldots)$, where $\varepsilon_n = \alpha\sqrt{2}/n$ and take $f_n = Q^{1/2}v_n$ $(n = 1, 2, \ldots)$. Then $\{f_n\} \subset \mathcal{K}$ and

$$\|f_n\|^2 = (n\alpha\sqrt{2}/2)(2 - \alpha\sqrt{2}/n) \to \infty \quad \text{for } n \to \infty.$$

It remains to prove (2). For this let $\{v_n\} \subset \mathcal{H}$ satisfy $Q^{1/2}v_n \to u$ weakly. Since $\left\|(1 - Q)^{1/2}v_n\right\| \leq \alpha$ for all v_n, we can assume (by replacing the original sequence with an adequate subsequence) that also $(1 - Q)^{1/2}v_n \to w$ weakly. Consequently

$$v_n = Qv_n + (1 - Q)v_n \to v := Q^{1/2}u + (1 - Q)^{1/2}w \text{ weakly}.$$

Thus $Q^{1/2}v_n \to Q^{1/2}v$, $(1 - Q)^{1/2}v_n \to (1 - Q)^{1/2}v$ weakly. It follows that $u = Q^{1/2}v$ and $\left\|(1 - Q)^{1/2}v\right\| \leq \alpha$, that is, $u \in \mathcal{K}$.

7. Subalgebras without (P)

Our next aim is to continue to investigate the (unital, norm closed) subalgebras $\mathbb{A}$ which do not have (P).

Let $\mathbb{A}$ be such an algebra. Then given $y_0 \in \mathcal{H} \setminus (0), \alpha \in (0, 1/2), \delta \in (0, \|y_0\|)$ and $\delta_0 \in (0, (1 - 2\alpha)\delta)$, there exists $y = y(y_0, \alpha, \delta, \delta_0)$ with the properties

$$y \in B(y_0, \delta)^-, \quad \Gamma_\alpha(y) \cap B(y_0, \delta_0) = \varnothing.$$

Hence

$$\|\Phi_{\mathcal{K}}(y_0)\| \geq \delta_0 \quad \text{for } \mathcal{K} = \Gamma_\alpha(y)^-,$$

and (see (6.5))

$$|\langle w, \Phi_{\mathcal{K}}(y_0)\rangle| \leq \langle y_0, \Phi_{\mathcal{K}}(y_0)\rangle - \|\Phi_{\mathcal{K}}(y_0)\|^2, \quad w \in \mathcal{K},$$
$$\|y_0 - \Phi_{\mathcal{K}}(y_0)\|^2 = \|y_0\|^2 - \|\Phi_{\mathcal{K}}(y_0)\|^2 - 2\left(\langle y_0, \Phi_{\mathcal{K}}(y_0)\rangle - \|\Phi_{\mathcal{K}}(y_0)\|^2\right),$$

whence

$$\delta_0 \leq \|\Phi_{\mathcal{K}}(y_0)\| \leq \|y_0\|,$$
$$|\langle w, \Phi_{\mathcal{K}}(y_0)\rangle| \leq (\|y_0\| - \delta_0)\|y_0\|, \quad w \in \mathcal{K},$$
$$\|y_0 - \Phi_{\mathcal{K}}(y_0)\|^2 \leq \|y_0\|^2 - \delta_0^2.$$

We can now infer that

$$\left|\langle w, C^*y_0\rangle\right| \leq (\|y_0\| - \delta_0)\|y_0\| + \|w\|\left(\|y_0\|^2 - \delta_0^2\right)^{1/2} \tag{7.1}$$

for all $w := Ay(y_0, \alpha, \delta, \delta_0)$, $A, C \in \mathbb{A}$ such that $\|A\|_e \leq \alpha$, $\|C\| \leq 1$. Assume that y_0 is a strictly cyclic vector for $\mathbb{A}^*$. Then apply the open mapping theorem to the map $C^* \mapsto C^*y_0$ from $\mathbb{A}^*$ onto $\mathcal{H}$. It follows that there exists $r > 0$ such that

$$B(0, r)^- \subset \{C^*y_0 : C \in \mathbb{A}, \ \|C\| \leq 1\}.$$

In particular, for $w \in \mathcal{K}$ in (7.1) there exists an operator $C_w \in \mathbb{A}$ with $\|C_w\| \leq 1$ such that $w = r^{-1}\|w\| C_w^* y_0$. Thus from (7.1) we obtain

$$r\|w\| \leq (\|y_0\| - \delta_0)\|y_0\| + \|w\|(\|y_0\|^2 - \delta_0^2)^{1/2},$$

whence for $\|y_0\|^2 - \delta_0^2 \leq (r/2)^2$ we obtain

$$\|w\| \leq (2/r)(\|y_0\| - \delta_0)\|y_0\|, \quad \forall w \in \Gamma_\alpha(y).$$

Thus by choosing $\delta_0 = (1 - 1/n)\|y_0\|$, $\delta = (1 - 1/(2n))\|y_0\|$, we obtain the following fact.

Proposition 7.1. *If $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$ does not have (P), then for every $y_0 \in \mathcal{H} \setminus (0)$ which is strictly cyclic for $\mathbb{A}^*$, there exist sequences $\{\alpha_n\}$ converging to 0 and $\{y_n\} \subset B(y_0, \|y_0\|)$ such that*

$$\sup\{\|w\| : w \in \Gamma_{\alpha_n}(y_n)\} \to 0. \tag{7.2}$$

8. The finite dimensional case

As was made clear at the beginning of Section 1, all of the (unital, norm closed) algebras $\mathbb{A}$ studied heretofore in this article have consisted of operators acting on the Hilbert space $\mathcal{H}$ of dimension $\aleph_0$. But, as the reader will no doubt have noticed, (P) makes perfect sense for unital algebras $\mathbf{A} \subset \mathcal{L}(\mathcal{M})$, where $\mathcal{M}$ is any finite dimensional complex Hilbert space. Furthermore, as we show later in this section, there are intimate connections between an algebra $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$ having (P) and certain of its restrictions or compressions to finite-dimensional spaces having (P). This motivates the study of Property (P) in finite matrix algebras. Of course, since $\mathcal{L}(\mathcal{M})$ itself is finite dimensional, there is only one linear topology on $\mathcal{L}(\mathcal{M})$ (so the WOT, SOT, and norm topologies on $\mathcal{L}(\mathcal{M})$ coincide), and all algebras $\mathbf{A} \subset \mathcal{L}(\mathcal{M})$ are automatically closed in this topology. Thus it makes sense to ask which (unital) algebras $\mathbf{A} \subset \mathcal{L}(\mathcal{M})$ have (P), with the agreed upon convention that all $A \in \mathcal{L}(\mathcal{M})$ satisfy $\|A\|_e = 0$.

We begin with two elementary facts about algebras $\mathbf{A} \subset \mathcal{L}(\mathcal{M})$ and Property (P) which the reader can easily verify for himself.

Proposition 8.1. *If $\dim \mathcal{M} < \aleph_0$, then $\mathcal{L}(\mathcal{M})$ has (P) and the algebra $\mathbb{C}1_{\mathcal{M}}$ of scalars has (P) if and only if $\dim \mathcal{M} = 1$.*

Proposition 8.2. *If $\mathbf{A} \subset \mathcal{L}(\mathcal{M})$ has a cyclic (equivalently, strictly cyclic) vector, then $\mathbf{A}$ has (P), and moreover, if y_0 is any strictly cyclic vector for $\mathbf{A}$ (i.e., $\mathbf{A}y_0 = \mathcal{M}$), then there exists $\delta \in (0, \|y_0\|)$ such that every vector y satisfying $\|y - y_0\| < \delta$ is also a strictly cyclic vector for $\mathbf{A}$. Thus for each such y, there exists $A_y \in \mathbf{A}$ such that $A_y y = y_0$.*

Proof. It is well known [10] that an algebra with a strictly cyclic vector has the property that the set of strictly cyclic vectors is open and the result follows immediately. □

Remark 8.3. Note that this proposition can be paraphrased by saying that if $\mathbf{A} \subset \mathcal{L}(\mathcal{M})$ and y_0 is any strictly cyclic vector for $\mathbf{A}$, then every quadruple $(y_0, \alpha, \delta, \delta_0)$, where $\alpha \in [0, 1/2)$, δ is sufficiently small, and $\delta_0 \in [0, (1 - 2\alpha)\delta)$ implements Property (P) for $\mathbf{A}$.

Recall also from Example 3.5 that the algebra $\mathbb{A}_0 \subset \mathcal{L}(\mathcal{H})$ has a strictly cyclic vector e_0 but does not have (P). Thus the counterpart of Proposition 8.2 for algebras $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$ is not true without some additional hypotheses.

Proposition 8.4. *If* $\mathbf{A} \subset \mathcal{L}(\mathcal{M})$ *does not have* (P)*, then* $\mathbf{A}$ *is not transitive and* $\mathbf{A}^*$ *has no cyclic vector.*

Proof. Since $\mathbf{A}$ does not have (P), by Proposition 8.1, $\mathbf{A} \neq \mathcal{L}(\mathcal{M})$, and by Burnside's theorem, $\mathbf{A}$ is not transitive. Note also that for $\alpha > 0$ and $y \in \mathcal{M}$, $\Gamma_\alpha(y) = \mathbf{A}y$. Now with $y_0 \neq 0$ arbitrarily given, choose sequences $\{\delta_0^{(n)}\}$ and $\{\delta^{(n)}\}$ such that for $n \in \mathbb{N}$, $0 < \delta_0^{(n)} < \delta^{(n)} < \|y_0\|$ and $\lim \delta_o^{(n)} = \|y_0\|$. Since $\mathbf{A}$ does not have (P), for each $n \in \mathbb{N}$, there exists $y_n \in B(y_0, \delta^{(n)})^-$ such that $\operatorname{dist}(\mathbf{A}y_n, y_0) > \delta_0^{(n)}$. Now fix an arbitrary A in $\mathbf{A}$. Then, since for $n \in \mathbb{N}$,

$$\left\langle y_0, Ay_n/\|Ay_n\|^2 \right\rangle Ay_n \in \mathbf{A}y_n \quad (= \Gamma_\alpha(y_n)),$$

we have, by some trivial arithmetic,

$$\|y_0\|^2 - |\langle y_0, Ay_n/\|Ay_n\|\rangle|^2 \geq (\delta_0^{(n)})^2,$$

which becomes

$$|\langle y_0, Ay_n/\|Ay_n\|\rangle| \leq (\|y_0\|^2 - (\delta_0^{(n)})^2)^{1/2},$$

and clearly implies that $\langle y_0, Ay_n/\|Ay_n\|\rangle \to 0$. Since $\mathcal{M}$ is finite dimensional, without loss of generality, we may suppose, by dropping to a subsequence, that $Ay_n/\|Ay_n\| \to y_\infty$ in norm, and thus $\|y_\infty\| = 1$. Since the range of A is closed, $y_\infty = Ay'_\infty$ for some $y'_\infty \neq 0$. Thus $\langle Ay'_\infty, y_0\rangle = 0 = \langle y'_\infty, A^* y_0\rangle$, and since A was arbitrary in $\mathbf{A}$, y_0 is not cyclic for $\mathbf{A}^*$. But y_0 was arbitrary in $\mathcal{M}$, so $\mathbf{A}^*$ has no cyclic vector. □

A natural question that Propositions 8.2 and 8.4 bring to mind is whether for subalgebras $\mathbf{A}$ of $\mathcal{L}(\mathcal{M})$ having a cyclic vector is equivalent to having (P). The following example shows that this is not case.

Example 8.5. Let $\dim \mathcal{M} = 5$ and let $A \in \mathcal{L}(\mathcal{M})$ be the operator whose matrix relative to some orthonormal basis $\{e_j\}_{j=1}^5$ for $\mathcal{M}$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that A is in Jordan canonical form and that the minimal polynomial for A is $\lambda^2(\lambda - 1)$. Therefore the algebra $\mathbf{A}_A = \{p(A) : p \in \mathbb{C}[z]\}$ is 3-dimensional, and since $\dim \mathcal{M} = 5$, A cannot have a cyclic vector. On the other hand, it is equally obvious that $\mathbf{A}_A = \mathbf{A}_{A_1} \oplus \mathbf{A}_{A_2}$ relative to the orthogonal decomposition of $\mathcal{M}$ as $\mathcal{M}_1 \oplus \mathcal{M}_2$, where $\mathcal{M}_1 = \mathbb{C}e_1$, $\mathcal{M}_2 = \vee\{e_j\}_{j=2}^5$ and $A_1 = 1_{\mathcal{M}_1}$. Thus $\mathbf{A}_{A_1}$ has (P) by Proposition 8.1 and $\mathbf{A}_A$ has (P) by Theorem 8.7. Summarizing, we see that $\mathbf{A}_A$ has (P) but no cyclic vector. It is also not hard to show that $\mathbf{A}_{A_2}$ does not have (P).

Another natural question that Propositions 8.2 and 8.4 bring to mind is whether for algebras $\mathbf{A} \subset \mathcal{L}(\mathcal{M})$ (with $\dim \mathcal{M} < \aleph_0$) it is the case that $\mathbf{A}$ has a cyclic vector if and only if $\mathbf{A}^*$ does. Recall that this is certainly the case if $\mathbf{A}$ is singly generated (consider Jordan forms). But that this is, in general, false even for commutative algebras, was pointed out to us by Hari Bercovici, who kindly supplied the following example.

Example 8.6. Let $\dim \mathcal{M} = 3$, and let $\mathcal{E} = \{e_1, e_2, e_3\}$ be an orthonormal basis for $\mathcal{M}$. Consider the algebra $\mathbf{A} \subset \mathcal{L}(\mathcal{M})$ given matricially with respect to $\mathcal{E}$ as

$$\mathbf{A} = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \alpha & 0 \\ \gamma & 0 & \alpha \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\}.$$

It is clear that e_1 is a cyclic vector for $\mathbf{A}$ and easy calculations show that for any $f = \delta e_1 + \varepsilon e_2 + \varphi e_3$ in $\mathcal{M}$, we have that $\mathbf{A}^* f \subset \vee\{e_1, \varepsilon e_2 + \varphi e_3\}$, so f is not cyclic for $\mathbf{A}^*$.

This next result gives several connections between algebras $\mathbf{A} \subset \mathcal{L}(\mathcal{M})$ having (P) and algebras $\mathbb{A} \subset \mathcal{L}(\mathcal{H} \oplus \mathcal{M})$ having (P).

Theorem 8.7. *Suppose $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$ and contains a nonzero idempotent E of finite rank.*

(i) *If the algebra $E\mathbb{A}E|_{E\mathcal{H}}$ has (P) and has an implementing quadruple $(y_0, \alpha, \delta, \delta_0)$ with $\delta_0 \le \delta / \|E\|$, then $\mathbb{A}$ has (P) with implementing quadruple $(y_0, \alpha, \delta/\|E\|, \delta_0)$.*
(ii) *If the algebra $E\mathbb{A}E|_{E\mathcal{H}}$ has a cyclic vector, then $\mathbb{A}$ has (P).*
(iii) *If the algebra $E\mathbb{A}E|_{E\mathcal{H}}$ has (P) and $\|E\| = 1$ (i.e., $E = E^*$), then $\mathbb{A}$ has (P).*

Proof. (i) Let $\mathbb{A}$ and E satisfy the hypotheses of (i). We construct an implementing quadruple $(y_0', \alpha', \delta', \delta_0')$ to show that $\mathbb{A}$ has (P) as follows. Set $y_0' = y_0 (\in E\mathcal{H})$, $\alpha' = \alpha$, $\delta' = \delta/\|E\|$, and $\delta_0' = \delta_0$. Let $y' \in \mathcal{H}$ and satisfy $\left\| y' - y_0' \right\| \le \delta' = \delta/\|E\|$ and let $y = Ey'$. Then

$$\|y - y_0\| = \|E(y' - y_0)\| \le \|E\| \, \|y' - y_0\| \le \delta,$$

so there exists $Ay \in \mathbb{A}$ such that $\left\| EA_yE|_{E\mathcal{H}} y - y_0 \right\| \le \delta_0$. Define $A_{y'} = EA_yE \in \mathbb{A}$ and observe that

$$\left\| A_{y'} y' - y_0 \right\| = \left\| EA_yEy' - y_0 \right\| = \left\| EA_yE|_{E\mathcal{H}} y - y_0 \right\| \le \delta_0,$$

which proves that $\mathbb{A}$ has (P) with the asked for implementing quadruple.

(ii) This is immediate from (i) and Proposition 8.2 (see Remark 8.3).

(iii) This is immediate from (i). □

Before stating our last result we recall that if $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$ has (P) with implementing quadruple $(y_0, \alpha, \delta, \delta_0)$, then the proof of Theorem 1.4 yields a vector $w \in B(y_0, \delta)$ and an operator $A_w \in \mathbb{A}$ such that $A_w w = w$ and $\|A_w\|_e \le \alpha < 1$. Moreover, the Riesz idempotent E_w associated with A_w and the isolated eigenvalue of A_w belongs to $\mathbb{A}$.

Theorem 8.8. *Suppose $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$ is abelian and has (P), and $(y_0, \alpha, \delta, \delta_0)$, w, Aw, and E_w are as just mentioned above. If $\|E_w\| = 1$ (i.e., $E_w = E_w^*$), then the algebra $E_w\mathbb{A}E_w|_{E_w\mathcal{H}}$ also has (P).*

Proof. We have

$$\|w - E_w y_0\| = \|E_w(w - y_0)\| \le \delta_2 := \|w - y_0\| < \delta,$$
$$\|(1 - E_w)y_0\| = \|(1 - E_w)(w - y_0)\| \le \|w - y_0\| = \delta_2.$$

Hence

$$\widetilde{\delta} := (\delta^2 - \|(1 - E_w)y_0\|^2)^{1/2} > 0.$$

Therefore if $x = E_w x$ with $x \in B(E_w, y_0, \widetilde{\delta}_0)^-$, then

$$\|x - y_0\|^2 = \|x - E_w y_0\|^2 + \|(1 - E_w)y_0\|^2 \le \widetilde{\delta}^2 + \|(1 - E_w)y_0\|^2 = \delta^2,$$

and consequently, there exists an operator $A_x \in \mathbb{A}$ satisfying $\|A_x x - y_0\| \leq \delta_0$. Therefore, using also that $E_w A_x = A_x E_w$ we have

$$\begin{aligned}\delta_0^2 &\geq \|E_w A_x E_w x - E_w y_0\|^2 + \|(1 - E_w)(A_x x - y_0)\|^2 \\ &= \|E_w A_x E_w x - E_w y_0\|^2 + \|(1 - E_w) y_0\|^2,\end{aligned}$$

or equivalently,

$$\|E_w A_x E_w x - E_w y_0\|^2 \leq \widetilde{\delta}_0 := \left(\delta_0^2 - \|(1 - E_w) y_0\|^2\right)^{1/2} > 0,$$

where also $\widetilde{\delta}_0 < \widetilde{\delta}$. Thus, using the fact that $\dim E_w \mathcal{H} < \aleph_0$, we can choose a number $\widetilde{\alpha} > 0$ satisfying $\widetilde{\delta}_0 < (1 - 2\widetilde{\alpha})\widetilde{\delta}$. Thus we can conclude that $(E_w, y_0, \widetilde{\alpha}, \widetilde{\delta}, \widetilde{\delta}_0)$ is an implementing quadruple for $E_w \mathbb{A} E_w|_{E_w \mathcal{H}}$. $\square$

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