J. Math. Anal. Appl. 408 (2013) 298–305

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Quadratically hyponormal weighted shifts with recursive tail

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ARTICLE INFO

Article history:
Received 11 August 2011
Available online 7 June 2013
Submitted by Lawrence A Fialkow

Keywords:
Weighted shift
Quadratic hyponormality
Positive quadratic hyponormality

ABSTRACT

We characterize positive quadratic hyponormality of the weighted shift $W_{\alpha(x)}$ associated to the weight sequence $\alpha(x) : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^{\wedge}$ with Stampfli recursive tail, and produce an interval in x with non-empty interior in the positive real line for quadratic hyponormality but not positive quadratic hyponormality for such a shift.

1. Introduction

There has been considerable recent study of classes of operators on Hilbert spaces "weaker than subnormal" in some sense. Often these classes have been related in some way to the concept of hyponormality as defined in [18]. Let $\mathcal{H}$ be a separable complex Hilbert space, and let $L(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator T in $L(\mathcal{H})$ is *normal* if it commutes with its adjoint, *subnormal* if it is (unitary equivalent to) the restriction of a normal operator to an invariant subspace, and *hyponormal* if $T^*T \geq TT^*$. For $A, B \in L(\mathcal{H})$, we set $[A, B] := AB - BA$. A k-tuple $\mathbf{T} = (T_1, \ldots, T_k)$ of operators in $L(\mathcal{H})$ is called *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^k$ is positive on the direct sum of k copies of $\mathcal{H}$. For $k \in \mathbb{N}$ and $T \in L(\mathcal{H})$, T is said to be *k-hyponormal* if $(I, T, \ldots, T^k)$ is hyponormal. It is well known that $T \in L(\mathcal{H})$ is subnormal if and only if T is k-hyponormal for all $k \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers [3,18]. A k-tuple $\mathbf{T} = (T_1, \ldots, T_k)$ is *weakly hyponormal* if $\lambda_1 T_1 + \cdots + \lambda_k T_k$ is hyponormal for every $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, k$, where $\mathbb{C}$ is the set of complex numbers. An operator T is *weakly k-hyponormal* if $(T, T^2, \ldots, T^k)$ is weakly hyponormal; equivalently, for every complex polynomial p of degree k or less, $p(T)$ is hyponormal [5]. An operator T is *polynomially hyponormal* if, for every polynomial p with complex coefficients, $p(T)$ is hyponormal. In [5], Curto initiated the study of classes (actually or potentially) between hyponormal and subnormal with the following implications: "subnormal $\Rightarrow \cdots \Rightarrow$ 2-hyponormal $\Rightarrow$ hyponormal"; the converse implications are not always true [6]. Also it holds obviously that "subnormal $\Rightarrow$ polynomially hyponormal $\Rightarrow \cdots \Rightarrow$ weakly 2-hyponormal $\Rightarrow$ hyponormal"; but the converse implications are not developed completely yet except for weak 2- and 3-hyponormalities [2,12,19] and the foundational (negative) result that polynomial hyponormality does not imply subnormality or even 2-hyponormality (see [13,14]). In particular, the weak k-hyponormality case in which $k = 2$ has received considerable attention (see, for example, [6,10,11,15,17,16,20]), and operators in this class are usually called *quadratically hyponormal*.

On the other hand, the study of flatness for weighted shifts is a good approach to detect operator gaps between subnormality and hyponormality [4,6,10,17,21]. It was shown in [6] that, if W_α is a 2-hyponormal weighted shift with $\alpha_n = \alpha_{n+1}$

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0022-247X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmaa.2013.05.058

for some $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, then $\alpha_1 = \alpha_2 = \cdots$. But in general this flatness need not hold in the case of quadratic hyponormality; for example, if $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{n+1}{n+2}}$ $(n \geq 2)$, then W_α is quadratically hyponormal (see [6]). And so the following problem was posed in [7]: describe all quadratically hyponormal weighted shifts with first two weights equal. In [9,10,20], the recursively generated positively quadratically hyponormal weighted shifts with weight sequence $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$ were described, and it was proved that quadratic hyponormality of the weighted shift W_α with $\alpha : \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ is equivalent to positive quadratic hyponormality, where $(\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ is the Stampfli subnormal completion of three values $\sqrt{u}, \sqrt{v}, \sqrt{w}$ (see [22]). And also, in [15,21], the recursively generated positively quadratically hyponormal weighted shifts with weight sequence $\alpha : \sqrt{x}, \sqrt{y}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with Stampfli's tail were described, and it was proved that there exists a quadratically hyponormal but not positively quadratically hyponormal weighted shift W_α. In this note, we obtain a sufficient condition for quadratic hyponormality of weighted shifts, and compare positive quadratic hyponormality and quadratic hyponormality: in particular, for the weight sequence $\alpha(x) : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ for some u, v, w, we obtain an interval J in x with non-empty interior in the positive real line such that a weighted shift $W_{\alpha(x)}$ is quadratically hyponormal but not positively quadratically hyponormal on J.

Some of the calculations in this paper were aided by use of the software tool *Mathematica* [23].

2. Preliminaries and notation

The usual examples for the class of quadratically hyponormal operators on Hilbert spaces have come from weighted shifts, so we recall some notation that is by now standard. Let α denote a weight sequence, $\alpha : \alpha_0, \alpha_1, \alpha_2, \ldots$, where it is without loss of generality to assume that these are all positive. The weighted shift W_α acting on $\ell^2(\mathbb{N}_0)$, with standard basis $e_0, e_1, \ldots$, is defined by $W_\alpha(e_j) = \alpha_j e_{j+1}$ for all $j \in \mathbb{N}_0$. It is standard that W_α is quadratically hyponormal if and only if $W_\alpha + sW_\alpha^2$ is hyponormal for any $s \in \mathbb{C}$. Let P_n denote the orthogonal projection onto $\vee_{k=0}^n \{e_k\}$. For $s \in \mathbb{C}$ and $n \geq 0$, define D_n by

$$D_n := D_n(s) = P_n[(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]P_n$$

$$= \begin{pmatrix} q_0 & \bar{r}_0 & 0 & \cdots & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \cdots & 0 & 0 \\ 0 & r_1 & q_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q_{n-1} & \bar{r}_{n-1} \\ 0 & 0 & 0 & \cdots & r_{n-1} & q_n \end{pmatrix}, \tag{2.1}$$

where $\alpha_{-1} = \alpha_{-2} := 0$ and, for $k \geq 0$,

$$q_k = u_k + |s|^2 v_k, \qquad r_k = s\sqrt{w_k}, \qquad u_k = \alpha_k^2 - \alpha_{k-1}^2,$$
$$v_k = \alpha_k^2 \alpha_{k+1}^2 - \alpha_{k-1}^2 \alpha_{k-2}^2, \qquad w_k = \alpha_k^2(\alpha_{k+1}^2 - \alpha_{k-1}^2)^2, \quad k \geq 0.$$

Since W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for every $s \in \mathbb{C}$ and every $n \geq 0$, in anticipation of the usual use of Sylvester's criterion (which is sometimes called the *Nested Determinant Test*; for example, see [8, p. 213]) for positivity of a matrix, we consider $d_n(\cdot) := \det(D_n(\cdot))$; d_n is actually a polynomial in $t := |s|^2$ of degree $n+1$, having Maclaurin's expansion $d_n(t) := \sum_{i=0}^{n+1} c(n, i)t^i$. Since we usually work with these determinants, we abuse notation slightly to regard D_n as a function of $t = |s|^2$ and write, henceforth, $D_n(t)$.

By some standard computations (see [9]), it is known that

$$d_0 = q_0, \qquad d_1 = q_0 q_1 - |r_0|^2, \qquad d_{n+2} = q_{n+2} d_{n+1} - |r_{n+1}|^2 d_n, \quad n \geq 0, \tag{2.2}$$

and that also

$$\begin{aligned} &c(0, 0) = u_0, \qquad c(0, 1) = v_0, \qquad c(1, 0) = u_1 u_0, \\ &c(1, 1) = u_1 v_0 + u_0 v_1 - w_0, \qquad c(1, 2) = v_1 v_0, \\ &c(n+2, i) = u_{n+2} c(n+1, i) + v_{n+2} c(n+1, i-1) - w_{n+1} c(n, i-1), \quad n \geq 0, \\ &c(n, n+1) = v_0 v_1 \cdots v_n \geq 0, \quad n \geq 0. \end{aligned} \tag{2.3}$$

Note also for future use that u_n, v_n, and w_n are non-negative for all $n \geq 0$, at least if we assume (as we shall discuss shortly) that the weights are strictly increasing. We will also have frequent occasion to use $z_n := \frac{v_n}{u_n}$, $n \geq 0$. If we want all the d_n to be non-negative, surely there is an easy way this may occur: in [9], a weighted shift W_α is defined to be *positively quadratically hyponormal* if $c(n, n+1) > 0$ and $c(n, i) \geq 0$, $0 \leq i \leq n$, $n \in \mathbb{N}_0$. This class of operators has been studied in, for example, [1,6,17,20].

We specialize to a class of shifts due to Stampfli and arising in his consideration of the problem of completing an initial finite sequence of weights to yield the weight sequence for a subnormal shift [22]. Given weights $0 < \alpha_0 < \alpha_1 < \alpha_2$, there is a canonical way to generate a satisfactory completion recursively: define

$$\widehat{\alpha}_n = \left(\Psi_1 + \frac{\Psi_0}{\alpha_{n-1}^2}\right)^{1/2}, \quad n \geq 3, \tag{2.4}$$

where $\Psi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2-\alpha_1^2)}{\alpha_1^2-\alpha_0^2}$ and $\Psi_1 = \frac{\alpha_1^2(\alpha_2^2-\alpha_0^2)}{\alpha_1^2-\alpha_0^2}$. This produces a bounded sequence $\widehat{\alpha} := \{\widehat{\alpha}_i\}_{i=0}^{\infty}$, for which $\widehat{\alpha}_i = \alpha_i$ ($0 \leq i \leq 2$). We usually write the weight sequence $(\alpha_0, \alpha_1, \alpha_2)^\wedge$ and the resulting weighted shift as $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$.

Throughout this paper, $\mathbb{R}_0$ is the set of non-negative real numbers.

3. Positive quadratic hyponormality

Let W_α be the unilateral weighted shift with a weight sequence $\alpha := \{\alpha_i\}_{i\in\mathbb{N}_0}$. Suppose that $\alpha_k = \alpha_{k+j}$ for some $k, j \in \mathbb{N}$. Then W_α is quadratically hyponormal if and only if $\alpha_0 \leq \alpha_1 = \alpha_2 = \cdots$. Hence we assume throughout that $\alpha_k < \alpha_{k+1}$ for all $k \in \mathbb{N}$ to avoid the trivial case.

Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 < x < u < v < w$. For our convenience, we recall from (2.4) that $\alpha_n^2 = \Psi_1 + \frac{\Psi_0}{\alpha_{n-1}^2}$ ($n \geq 6$), where $\Psi_1 = \frac{v(w-u)}{v-u}$ and $\Psi_0 = -\frac{uv(w-v)}{v-u}$.

We now begin our work with the following lemma.

Lemma 3.1. *Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 < x < u < v < w$. Then, for $n \geq 5$, we have*

$$c(n,i) = \begin{cases} v_n \cdots v_4 c(3,4), & i = n+1, \\ u_n c(n-1,n) + v_n \cdots v_5 \eta_2, & i = n, \\ u_n c(n-1,n-1) + v_n \cdots v_5 \eta_3, & i = n-1, \\ u_n \cdots u_{i+2} c(i+1,i), & 3 \leq i \leq n-2, \\ u_n \cdots u_5 c(4,i), & 0 \leq i \leq 2, \end{cases}$$

where $\eta_2 = v_4 c(3,3) - w_3 c(2,3)$ and $\eta_3 = v_4 c(3,2) - w_3 c(2,2)$.

Proof. We shall claim that, for $n \geq 5$ and $0 \leq i \leq n+1$,

$$c(n,i) = u_n c(n-1,i) + v_n \cdots v_5 [v_4 c(3, i-n+3) - w_3 c(2, i-n+3)]. \tag{3.1}$$

And we now recall from [10, Lemma 2.1] that

$$w_n = u_n v_{n+1} (n \geq 4). \tag{3.2}$$

Then, for $n = 5$ and $0 \leq i \leq 6$, we obtain

$$\begin{aligned} c(5,i) &= u_5 c(4,i) + v_5 c(4,i-1) - w_4 c(3,i-1) \\ &= u_5 c(4,i) + v_5 [v_4 c(3,i-2) - w_3 c(2,i-2)]. \end{aligned} \tag{3.3}$$

Also, by the inductive hypothesis, (3.2) and (3.3), we have

$$\begin{aligned} c(n+1,i) &= u_{n+1} c(n,i) + v_{n+1} c(n,i-1) - w_n c(n-1,i-1) \\ &= u_{n+1} c(n,i) + (v_{n+1} u_n - w_n)\, c(n-1,i-1) + v_{n+1} \cdots v_5 [v_4 c(3, i-(n+1)+3) \\ &\quad - w_3 c(2, i-(n+1)+3)] \\ &= u_{n+1} c(n,i) + v_{n+1} \cdots v_5 [v_4 c(3, i-(n+1)+3) - w_3 c(2, i-(n+1)+3)]. \end{aligned}$$

According to (3.1), we have that, for $n \geq 5$ and $0 \leq i \leq n+1$,

$$c(n,i) = \begin{cases} v_n \cdots v_4 c(3,4), & i = n+1, \\ u_n c(n-1,n) + v_n \cdots v_5 \eta_2, & i = n, \\ u_n c(n-1,n-1) + v_n \cdots v_5 \eta_3, & i = n-1, \\ u_n c(n-1,i), & 0 \leq i \leq n-2. \end{cases}$$

Finally, observe that

$$\begin{aligned} c(n,i) &= u_n c(n-1,i) = u_n u_{n-1} c(n-2,i) \quad (0 \leq i \leq n-2) \\ &= u_n \cdots u_{i+2} c(i+1,i) \quad (3 \leq i \leq n-2) \end{aligned}$$

and $c(n,i) = u_n c(n-1,i) = u_n \cdots u_5 c(4,i)$, $i = 0, 1, 2$. Hence the proof is complete. $\square$

Lemma 3.2. *Under the same notation in the previous section, we have that*

$$K := -\frac{\Psi_1^2}{\Psi_0} L^2 = \lim_{n\to\infty} \frac{v_n}{u_n} \quad \textit{with } L^2 = \frac{\Psi_1 + \sqrt{\Psi_1^2 + 4\Psi_0}}{2},$$

and more precisely that $K = \frac{A+\sqrt{B}v(w-u)^2}{C}$, *where* $A = v^2(w-u)^3$, $B = v^2(w-u)^2 - 4uv(w-v)(v-u)$, *and* $C = 2u(v-u)^2(w-v)$.

Proof. See [9, Theorem 4.3]. □

Lemma 3.3. *Using the notation above, we have that the expression*

$$g_n := z_{n-1} + (z_n - z_{n-1}) + z_{n-1}\left(\frac{z_{n+1} - z_n}{z_n - z_{n-1}}\right) + (z_{n+1} - z_n)$$

is constant in n for $n \geq 6$, *with value*

$$g_n = \frac{v^2(u-w)^3}{u(u-v)^2(v-w)}, \quad n \geq 6. \tag{3.4}$$

Proof. The proof is a straightforward adaptation of the proof of [17, Lemma 4.3]. □

Lemma 3.4. *Using the notation above, the function*

$$F(z_{n-1}, z_n) := \eta_1 + z_{n-1}\eta_2 + z_{n-1}z_n\eta_3 \quad (n \geq 6) \tag{3.5}$$

is either all increasing in n for $n \geq 6$ *or is all decreasing in n for* $n \geq 6$.

Proof. By computation in the proof of [17, Lemma 4.4], we obtain that, for $n \geq 6$,

$$F(z_n, z_{n+1}) - F(z_{n-1}, z_n) = (z_n - z_{n-1})\,(\eta_2 + \eta_3 g_n)\,,$$

where g_n is as in Lemma 3.3. The latter term is constant in n for $n \geq 6$, from Lemma 3.3, since η_2, η_3 have no ns. It is known that, for $n \geq 6$, $z_n \nearrow K$ [9, p. 397]. Obviously $F(z_{n-1}, z_n)$ is increasing ($n \geq 6$) if $\eta_2 + \eta_3 g_n > 0$ and decreasing ($n \geq 6$) if $\eta_2 + \eta_3 g_n < 0$. □

Theorem 3.5. *Let* $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$, *with* $1 < x < u < v < w$. *Then* W_α *is positively quadratically hyponormal if and only if*

(i) $c(3, 2) \geq 0$ *(i.e.,* $x \leq 2 - \frac{1}{u}$*);*
(ii) $c(4, 4) \geq 0$, $c(4, 3) \geq 0$*;*
(iii) $c(5, 5) \geq 0$, $c(5, 4) \geq 0$*;*
(iv) *it holds that*
  1° $\eta_2 \geq 0$, *or*
  2° $(\eta_2 < 0)$ *and* $K \leq \frac{\eta_1}{|\eta_2|}$,
  where η_2, η_3 *are as in Lemma 3.1, and* $\eta_1 := v_4 c(3, 4)$*;*
(v) *it holds that*
  1° *if* $\eta_2 + \frac{v^2(u-w)^3}{u(u-v)^2(v-w)}\eta_3 \geq 0$, *then* $\eta_1 + z_5\eta_2 + z_5 z_6 \eta_3 \geq 0$ *or, equivalently,* $c(6, 5) \geq 0$,
  2° *if* $\eta_2 + \frac{v^2(u-w)^3}{u(u-v)^2(v-w)}\eta_3 < 0$, *then* $\eta_1 + \eta_2 K + \eta_3 K^2 \geq 0$.

Proof. According to Lemma 3.1, we have that W_α is positively quadratically hyponormal if and only if it satisfies the following three conditions:

$$\begin{aligned} &c(n, i) \geq 0 \quad (0 \leq n \leq 4,\ 2 \leq i \leq n+1), \\ &c(n, n) \geq 0, \qquad c(n, n-1) \geq 0 \quad (n \geq 5). \end{aligned}$$

It is obvious from the above coefficients that $c(n, i) \geq 0$ $(0 \leq n \leq 4,\ 0 \leq i \leq n+1)$ if and only if (i) and (ii) hold.

Claim 1. $c(n, n) \geq 0$ $(n \geq 5)$ if and only if $c(5, 5) \geq 0$ and (iv) hold.

By Lemma 3.1, we have that

$$c(n, n) = u_n c(n-1, n) + v_n \cdots v_5 \eta_2 \quad (n \geq 5)$$

$$\begin{aligned} c(n, n) &= u_n v_{n-1} \cdots v_4 c(3, 4) + v_n \cdots v_5 \eta_2 \quad (n \geq 6) \\ &= v_{n-1} \cdots v_5 (u_n \eta_1 + v_n \eta_2) \quad (n \geq 6). \end{aligned}$$

Since $v_{n-1}\cdots v_5 > 0$ $(n \geq 6)$, $c(n,n) \geq 0$ $(n \geq 6)$ is equivalent to $u_n\eta_1 + v_n\eta_2 \geq 0$ $(n \geq 6)$. If $\eta_2 \geq 0$, then $u_n\eta_1 + v_n\eta_2 \geq 0$. If $\eta_2 < 0$, then

$$z_n = \frac{v_n}{u_n} \leq \frac{\eta_1}{|\eta_2|} \quad (n \geq 6). \tag{3.6}$$

Recalling $z_n \nearrow K$ [9, p. 397], the inequality (3.6) is equivalent to $K \leq \frac{\eta_1}{|\eta_2|}$.

Claim 2. $c(n, n-1) \geq 0$ $(n \geq 5)$ if and only if $c(5,4) \geq 0$, $c(6,5) \geq 0$ and (v) holds.

Using Lemma 3.1, we have that

$$\begin{aligned} c(6,5) &= u_6 c(5,5) + v_6 v_5 \eta_3 = u_6 u_5 c(4,5) + u_6 v_5 \eta_2 + v_6 v_5 \eta_3 \\ &= u_6 u_5 v_4 c(3,4) + u_6 v_5 \eta_2 + v_6 v_5 \eta_3 = u_6 u_5 \eta_1 + u_6 v_5 \eta_2 + v_6 v_5 \eta_3 \\ &= u_6 u_5 (\eta_1 + z_5 \eta_2 + z_5 z_6 \eta_3), \end{aligned}$$

using the definition of z_i, which easily implies that $\eta_1 + z_5\eta_2 + z_5 z_6 \eta_3 \geq 0$ is equivalent to $c(6,5) \geq 0$.

For $n \geq 7$, by Lemma 3.1, we have that

$$\begin{aligned} c(n,n-1) &= u_n c(n-1,n-1) + v_n \cdots v_5 \eta_3 \\ &= u_n u_{n-1} c(n-2,n-1) + u_n v_{n-1} \cdots v_5 \eta_2 + v_n \cdots v_5 \eta_3 \\ &= u_n u_{n-1} v_{n-2} \cdots v_4 c(3,4) + u_n v_{n-1} \cdots v_5 \eta_2 + v_n \cdots v_5 \eta_3 \\ &= v_{n-2} \cdots v_5 (u_n u_{n-1} \eta_1 + u_n v_{n-1} \eta_2 + v_n v_{n-1} \eta_3) \\ &= v_{n-2} \cdots v_5 u_n u_{n-1} (\eta_1 + z_{n-1} \eta_2 + z_n z_{n-1} \eta_3). \end{aligned}$$

Since $v_{n-2}\cdots v_5 u_n u_{n-1} > 0$ $(n \geq 7)$, we have that $\eta_1 + z_{n-1}\eta_2 + z_n z_{n-1}\eta_3 \geq 0$ $(n \geq 7)$ if and only if $c(n, n-1) \geq 0$ $(n \geq 7)$. If $\eta_2 + \eta_3 g_6 \geq 0$, it follows from Lemma 3.4 that $F(z_{n-1}, z_n)$ is increasing for $n \geq 6$, where $F(z_{n-1}, z_n)$ is as in (3.5). Thus

$$F(z_{n-1}, z_n) \geq F(z_5, z_6) = \eta_1 + z_5\eta_2 + z_5 z_6 \eta_3 \quad \text{for all } n \geq 6.$$

Also, if $\eta_2 + \eta_3 g_6 < 0$, then $F(z_{n-1}, z_n)$ is decreasing in n for $n \geq 6$. Hence

$$F(K,K) \leq \cdots \leq F(z_{n-1}, z_n) \leq \cdots \leq F(z_5, z_6).$$

Since $\lim_n z_n = K$, obviously we have that

$$F(z_{n-1}, z_n) \geq 0 \quad (n \geq 6) \iff F(K,K) = \eta_1 + \eta_2 K + \eta_3 K^2 \geq 0.$$

Hence the proof is complete. □

We now discuss an example using Theorem 3.5.

Example 3.6. Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$. Take $u = \frac{11}{10}$, $v = \frac{12}{10}$, and $w = \frac{15}{10}$. After some computations, we get the following (with only possibly relevant ranges presented).

(i) $x \leq 2 - \frac{1}{u} \iff 1 < x \leq \frac{12}{11} \approx 1.09090\cdots$.

(ii) $c(4,4) \geq 0, c(4,3) \geq 0 \iff 1.00462\cdots \leq x \leq \frac{841}{770} \approx 1.09220\cdots$.

(iii) $c(5,5) \geq 0, c(5,4) \geq 0 \iff 1.00349\cdots \leq x \leq 1.09629\cdots$.

Note that $c(6,5) \geq 0$ for $1.022\cdots \leq x \leq 1.09312\cdots$ is required for (v).

We now consider the following inequalities restricting our attention to only x with $1.00462\cdots \leq x \leq \frac{12}{11}$.

(iv) Observe that $\eta_2 < 0 \Leftrightarrow 1.00462\cdots \leq x < 1.02095\cdots$.

In this range, we get $\eta_1 + K\eta_2 \geq 0$ for $1.01499\cdots \leq x < 1.02095\cdots$ (and $\eta_1 + K\eta_2 < 0$ for $1.00462\cdots \leq x < 1.01499\cdots$). And observe that $\eta_2 \geq 0 \Leftrightarrow 1.02095\cdots \leq x \leq 1.09090\cdots$. Hence (iv) holds $\Leftrightarrow 1.01499\cdots \leq x \leq 1.09090\cdots$.

(v) Observe that $\eta_2 + \frac{v^2(u-w)^3}{u(u-v)^2(v-w)}\eta_3 \geq 0 \Leftrightarrow 1.00568\cdots \leq x \leq 1.06172\cdots$.

As noted above, $c(6,5) \geq 0$ for $1.022\cdots \leq x \leq 1.09312\cdots$. And observe that, under the range of x with $1.00462\cdots \leq x \leq \frac{12}{11}$, $\eta_2 + \frac{v^2(u-w)^3}{u(u-v)^2(v-w)}\eta_3 < 0 \Leftrightarrow 1.00462\cdots \leq x < 1.00568\cdots$ and $1.06172\cdots < x \leq \frac{12}{11}$. And we obtain $\eta_1 + \eta_2 K + \eta_3 K^2 \geq 0$ for $1.00505\cdots \leq x < 1.06584\cdots$; in our region of interest, the expression is negative to the left and right of these bounds. So (v) holds $\Leftrightarrow 1.00505\cdots \leq x \leq 1.06583\cdots$.

Combining (i)–(v), we have that W_α is positively quadratically hyponormal $\Leftrightarrow \delta_1 = 1.01499\cdots \leq x \leq \delta_2 = 1.06583\cdots$, where δ_1 is the smallest positive root of $\eta_1 + K\eta_2 = 0$, where

$$\begin{aligned} \eta_1 + K\eta_2 = \frac{x}{12\,500}\{&(52\,800\sqrt{5} + 226\,325)x^3 - (226\,176\sqrt{5} + 963\,169)x^2 \\ &+ (304\,512\sqrt{5} + 1\,291\,368)x - 131\,328\sqrt{5} - 555\,012\}, \end{aligned}$$

and δ_2 is the second smallest positive root of $\eta_1+\eta_2K+\eta_3K^2=0$, where

$$\eta_1+\eta_2K+\eta_3K^2 = \frac{x}{7\,562\,500}\{(136\,926\,625+31\,944\,000\sqrt{5})x^3-(1\,349\,119\,805+428\,799\,360\sqrt{5})x^2 + (2\,355\,886\,536+784\,080\,768\sqrt{5})x-(1\,143\,988\,596+387\,341\,568\sqrt{5})\}.$$

Remark 3.7. Using some technical computations, it is possible to characterize the positive quadratic hyponormality of weighted shifts W_α with weight sequence $\alpha : \sqrt{y_m}, \sqrt{y_{m-1}}, \ldots, \sqrt{y_1}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ which generalizes Theorem 3.5.

4. Quadratic hyponormality

Recall that quadratic hyponormality of some weighted shift W_α is equivalent to the positivity of the matrices $D_n(s)$ for all $s\in\mathbb{C}$, as in Section 2. As usual, for each n, this is equivalent to the positivity of the associated complex quadratic form. The consequence of Lemma 3.1 in [20] is that this is equivalent to the positivity of a real quadratic form associated with a slightly different matrix. Define, for $t\geq 0$, and for each n,

$$A_n(t)=\begin{pmatrix} q_0 & -\sqrt{w_0t} & 0 & \cdots & 0 & 0\\ -\sqrt{w_0t} & q_1 & -\sqrt{w_1t} & \cdots & 0 & 0\\ 0 & -\sqrt{w_1t} & q_2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & q_{n-1} & -\sqrt{w_{n-1}t}\\ 0 & 0 & 0 & \cdots & -\sqrt{w_{n-1}t} & q_n \end{pmatrix}, \tag{4.1}$$

where the q_i, u_i, v_i, and w_i are as for D_n as in (2.1). Then, from Lemma 3.1 of [20], one obtains the following proposition.

Proposition 4.1. *Let W_α be a weighted shift. Then, for each n, $D_n(s)$ is a (complex) positive quadratic form over $\mathbb{C}^{n+1}$ for all $s\in\mathbb{C}$ if and only if $A_n(t)$ is a (real) positive quadratic form over $\mathbb{R}^{n+1}$ for all $t\geq 0$.*

Specialize to the case of the 3-length backstep extension of a Stampfli shift with weight sequence $\sqrt{y_3}, \sqrt{y_2}, \sqrt{y_1}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$. We will follow the development in Sections 3 and 4 of [20], suitably generalized from a 1-length backstep extension to a 3-length backstep extension. To test positivity of the quadratic form for $A_n(t)$ at a vector $(x_0,\ldots,x_n)$ in $\mathbb{R}_0^{n+1}$, what arises is

$$\begin{aligned} F_n(x_0,\ldots,x_n,t) &:= \sum_{i=0}^{n}u_ix_i^2-2\sqrt{t}\sum_{i=0}^{n-1}\sqrt{w_i}x_ix_{i+1}+t\sum_{i=0}^{n}v_ix_i^2\\ &= f(x_0,\ldots,x_4,t)+\sum_{i=4}^{n-1}(\sqrt{u_i}x_i-\sqrt{v_{i+1}t}x_{i+1})^2+u_nx_n^2, \end{aligned}\tag{4.2}$$

where

$$f(x_0,\ldots,x_4,t):=\sum_{i=0}^{3}u_ix_i^2-2\sqrt{t}\sum_{i=0}^{2}\sqrt{w_i}x_ix_{i+1}+t\sum_{i=0}^{4}v_ix_i^2;$$

note that the equality in (4.2) is the result of a simple computation and the use of $u_iv_{i+1}=w_i$ for $i\geq 5$.

Recall that $K=\lim_{n\to\infty}z_n$ with explicit expression as in Lemma 3.2. Continuing the argument as in [20], we arrive at the following, which is an analog of their Theorem 4.6.

Proposition 4.2. *For the weight sequence $\sqrt{y_3}, \sqrt{y_2}, \sqrt{y_1}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ and its associated weighted shift W_α, the following assertions are equivalent:*

(i) *$F_n(x_0,\ldots,x_n,t)\geq 0$ for any $x_0,\ldots,x_n,t$ in $\mathbb{R}_0$ and $\frac{1}{K}<t$ for all $n\geq 5$;*
(ii) *$f(x_0,\ldots,x_4,t)\geq 0$ for all $x_0,\ldots,x_4,t$ in $\mathbb{R}_0$ and $\frac{1}{K}<t$.*

We now discuss gaps between the quadratic and positive quadratic hyponormalities. For given u,v,w with $1<u<v<w$, let $\alpha:=\alpha(x):1,1,\sqrt{x},(\sqrt{u},\sqrt{v},\sqrt{w})^\wedge$ with $1<x<u$. Applying this case to Proposition 4.2 and (4.2), we obtain that, if $f(x_0,\ldots,x_4,t)\geq 0$, then W_α is quadratically hyponormal. Observe that the quadratic function of $f(x_0,\ldots,x_4,t)$ corresponds to the symmetric matrix

$$\Delta(t)=\begin{pmatrix} u_0+v_0t & -\sqrt{w_0t} & 0 & 0 & 0\\ -\sqrt{w_0t} & u_1+v_1t & -\sqrt{w_1t} & 0 & 0\\ 0 & -\sqrt{w_1t} & u_2+v_2t & -\sqrt{w_2t} & 0\\ 0 & 0 & -\sqrt{w_2t} & u_3+v_3t & -\sqrt{w_3t}\\ 0 & 0 & 0 & -\sqrt{w_3t} & v_4t \end{pmatrix}; \tag{4.3}$$

also, see (4.1). Hence, it follows from the positivity of $\Delta(t)$ that W_α is quadratically hyponormal. Recall the upper left $n \times n$ proper submatrices of $\Delta(t)$ are exactly $A_{n-1}(t)$, $n = 1, 2, 3, 4$, which are as in (4.1).

Theorem 4.3. *Let $\alpha(x) : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 < x < u < v < w$. If $\phi_1(t) := c(3,4)t^2 + c(3,3)t + c(3,2) > 0$ and $\phi_2(t) := \eta_1 t^2 + \eta_2 t + \eta_3 \geq 0$ for all $t > 0$, then $W_{\alpha(x)}$ is quadratically hyponormal.*

Proof. To show that $\Delta(t) \geq 0$, we use the Nested Determinant Test. Since $\det A_1(t) = ((x-1) + xt)t > 0$ and $\det A_2(t) = (tx(ux-1) + ux(x-1))t^2 > 0$ for $t > 0$, it is sufficient to consider $\det A_3(t) > 0$ and $\det \Delta(t) \geq 0$. By (2.1) and (4.3), $\det A_3(t) = \det D_3(t)$ and

$$\det A_3(t) = (c(3,4)t^2 + c(3,3)t + c(3,2))t^2. \tag{4.4}$$

And, by (4.3) and (4.4), we obtain that

$$\begin{aligned}
\det \Delta(t) &= v_4 t \cdot d_3(t) + \sqrt{w_3 t} \cdot \det \begin{pmatrix} u_0 + v_0 t & -\sqrt{w_0 t} & 0 & 0 \\ -\sqrt{w_0 t} & u_1 + v_1 t & -\sqrt{w_1 t} & 0 \\ 0 & -\sqrt{w_1 t} & u_2 + v_2 t & -\sqrt{w_2 t} \\ 0 & 0 & 0 & -\sqrt{w_3 t} \end{pmatrix} \\
&= v_4 t \cdot d_3(t) + \sqrt{w_3 t}\left(0 - \sqrt{w_3 t} \cdot d_2(t)\right) \\
&= v_4 t (c(3,4)t^4 + c(3,3)t^3 + c(3,2)t^2) - w_3 t (c(2,3)t^3 + c(2,2)t^2) \\
&= t^3 (v_4 c(3,4)t^2 + (v_4 c(3,3) - w_3 c(2,3))t + (v_4 c(3,2) - w_3 c(2,2))) \\
&= t^3 (\eta_1 t^2 + \eta_2 t + \eta_3).
\end{aligned}$$

Thus the proof is complete. $\square$

Example 4.4 (*Continued from* Example 3.6). Let $\alpha(x) : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$. Take $u = \frac{11}{10}$, $v = \frac{12}{10}$, and $w = \frac{15}{10}$. After some direct computations, we obtain that

$$\begin{aligned}
\eta_1 &= \frac{x}{2500}\left(-5940 + 14\,664x - 11\,693x^2 + 3025x^3\right), \\
\eta_2 &= \frac{x}{2500}\left(-7524 + 17\,446x - 12\,958x^2 + 3025x^3\right), \\
\eta_3 &= \frac{x}{2500}\left(-1044 + 2034x - 990x^2\right).
\end{aligned}$$

If one of the following two cases holds:

(i) $\eta_2, \eta_3 \geq 0$ (note: $\eta_1 > 0$),
(ii) $\eta_3 \geq 0$ and the discriminant $\eta_2^2 - 4\eta_1\eta_3 \leq 0$,

then $\phi_2(t) \geq 0$ for all $t \geq 0$ and all x satisfying (i)–(ii). By direct computation, we obtain the range $1.00544\cdots \leq x \leq 1.05454\cdots$ for x satisfying (i)–(ii) (with only possibly relevant ranges presented).

Similarly, applying $\phi_1(t)$ with the same method above, we obtain a range $1 < x < \frac{12}{11}$ for $\phi_1(t) \geq 0$ for all t.

According to Theorem 4.3, if we take x satisfying $1.00544\cdots \leq x \leq 1.05454\cdots$, then W_α is quadratically hyponormal. Hence, with the range of x for the positive quadratic hyponormality in Example 3.6, we obtain a range $1.00544\cdots \leq x \leq 1.06583\cdots$ for the quadratic hyponormality of $W_{\alpha(x)}$. Thus the open interval $\left(\frac{100\,545}{100\,000}, \frac{10149}{10\,000}\right)$ is a range in x for quadratic hyponormality but not positive quadratic hyponormality of $W_{\alpha(x)}$.

Observe finally that the test in Theorem 4.3 is clearly sufficient, but not necessary, for quadratic hyponormality, as evidenced by the interval $1.05454\cdots \leq x \leq 1.06583\cdots$ in the example above.

Acknowledgments

The second and fourth authors were supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education Science and Technology (2012R1A2A2A02008590). The third author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0025830). Some of this work was done when the first author visited Kyungpook National University in summer 2011, and he wishes to thank the Department of Mathematics for its kind hospitality. The authors would like to thank the referee for careful reading of the paper and helpful comments.

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