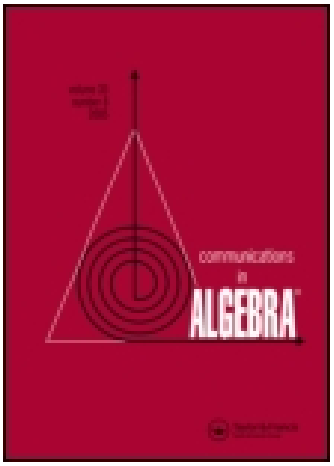


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# A General Theory of Almost Splitting Sets

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# A GENERAL THEORY OF ALMOST SPLITTING SETS

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***Let $*$ be a star-operation of finite type on an integral domain $D$. In this paper, we generalize and study the concept of almost splitting sets. We define a saturated multiplicative subset $S$ of $D$ to be an almost g$^*$-splitting set of $D$ if for each $0 \neq d \in D$, there exists an integer $n = n(d) \geq 1$ such that $d^n = st$ for some $s \in S$ and $t \in D$ with $(t, s')_* = D$ for all $s' \in S$. Among other things, we prove that every saturated multiplicative subset of $D$ is an almost g$^*$-splitting set if and only if $D$ is an almost weakly factorial domain (AWFD) with $*$-dim$(D) = 1$. We also give an example of an almost g$^*$-splitting set which is not a g$^*$-splitting set.***

***Key Words:*** Almost g$^*$-splitting set; Almost weakly factorial domain; $*$-Complement; Star-operation of finite type.

***2010 Mathematics Subject Classification:*** 13A05; 13A15; 13F05; 13G05.



## INTRODUCTION

Throughout this paper, $D$ denotes an integral domain with quotient field $K$, $U(D)$ means the group of units of $D$, and $S$ is a saturated multiplicative subset of $D$ (except for Proposition 2.5). Let $N(S) = \{0 \neq x \in D \mid (x, s)_v = D \text{ for all } s \in S\}$. Then $N(S)$, called the *m-complement of* $S$, is also a saturated multiplicative subset of $D$. We say that $S$ is a *splitting set* if for each $0 \neq d \in D$, we have $d = st$ for some $s \in S$ and $t \in N(S)$. It is well known that if $S$ is a splitting set, then $N(S)$ is also a splitting set and $N(N(S)) = S$. Also it is easy to see that $S$ is a splitting set if and only if $SN(S) = D\backslash\{0\}$. In [12], Gilmer and Parker first introduced this concept to generalize the Nagata theorem that if $S$ is a splitting set generated by prime elements, then $D$ is a UFD if (and only if) $D_S$ is a UFD. In [6], Anderson et al. gave a generalized version of splitting sets by using a star-operation of finite type. For a star-operation $*$ of finite type on $D$, they say that $S$ is a g$^*$-*splitting set* if for each $0 \neq d \in D$, we can write $d = st$ for some $s \in S$ and $t \in N_{*,D}(S)$, where $N_{*,D}(S) = \{0 \neq x \in D \mid (x, s')_* = D \text{ for all } s' \in S\}$. (For the sake of convenience, if the context is clear, then we shall use the notation $N_*(S)$ instead of $N_{*,D}(S)$.) It is easy to show that $N_*(S)$ is also a saturated multiplicative subset of $D$; we called $N_*(S)$ the $*$-*complement* of $S$. It is clear that $S$ is a g$^*$-splitting set if and only if $SN_*(S) = D\backslash\{0\}$, and if $S$ is a

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g*-splitting set, then $N_*(S)$ is also a g*-splitting set. Also, it is true that a g*-splitting set is a splitting set, but the converse is false [6, Example 2.8].

Motivated by the approach to g*-splitting sets from splitting sets, we study a general theory of almost splitting sets. In this article, we introduce the notion of an almost g*-splitting set that is a generalization of almost splitting sets, and investigate several properties. As in [7, Definition 2.1], a saturated multiplicative subset $S$ of $D$ is an *almost splitting set* of $D$ if for each $0 \neq d \in D$, there is an integer $n = n(d) \geq 1$ such that $d^n = st$ for some $s \in S$ and $t \in N(S)$. This notion was first utilized to characterize when the composite polynomial ring $D + XE[X]$ is an integrally closed AGCD-domain, where $D \subsetneq E$ is an extension of integral domains [11, Theorem 3.1]. Clearly, a splitting set is an almost splitting set, but the converse is not true. Let $*$ be a star-operation of finite type on $D$. We call $S$ an *almost* g*-*splitting set* of $D$ if for each $0 \neq d \in D$, there exists an integer $n = n(d) \geq 1$ such that $d^n = st$ for some $s \in S$ and $t \in N_*(S)$. It is obvious that if $* = t$, then the concept of almost g*-splitting sets is precisely the same as that of almost splitting sets. Therefore we can regard an almost g*-splitting set as a star-operation analogue (or a generalization) of almost splitting sets. Since $I \subseteq I_* \subseteq I_t$ for all nonzero fractional ideals $I$ of $D$, an almost g*-splitting set is always an almost splitting set, but the converse does not hold. Also, since an almost splitting set need not be a splitting set, an almost g*-splitting set also need not be a g*-splitting set. (This is the case when $* = t$.) More generally, we give an example of an almost g*-splitting set which is not a g*-splitting set for any star-operation $*$ of finite type (see Proposition 2.5).

This paper consists of three sections including introduction. In Section 1, we study the $*$-complements of saturated multiplicative subsets. We show that for a given star-operation $*$ of finite type on $D$, if $P$ is a prime $*$-ideal of $D$ and $S = D\backslash P$, then $N_*(N_*(S)) = S$ if and only if $P$ is a maximal $*$-ideal containing a nonzero element $d \in D$ which does not belong to any maximal $*$-ideal distinct from $P$. In Section 2, we introduce the concept of almost g*-splitting sets. We show that for a star-operation $*$ of finite type on $D$, every saturated multiplicative subset of $D$ is an almost g*-splitting set if and only if $D$ is an AWFD with $*$-dim$(D) = 1$.

Now, we review some preliminaries. Let $\mathbf{F}(D)$ (resp. $\mathbf{f}(D)$) be the set of nonzero (resp., nonzero finitely generated) fractional ideals of $D$. A *star-operation* on $D$ is a mapping $I \mapsto I_*$ of $\mathbf{F}(D)$ into $\mathbf{F}(D)$ such that the following three properties hold for all $0 \neq x \in D$ and all $I, J \in \mathbf{F}(D)$:

(1) $(x)_* = (x)$ and $(xI)_* = xI_*$;
(2) $I \subseteq I_*$, and if $I \subseteq J$, then $I_* \subseteq J_*$;
(3) $(I_*)_* = I_*$.

The $d$-, $v$-, $t$-, and $w$-operations are well-known examples of star-operations. The *d-operation* is the identity mapping on $\mathbf{F}(D)$, i.e., $I_d = I$ for all $I \in \mathbf{F}(D)$. The *v-operation* is defined by $I_v = (I^{-1})^{-1}$, where $I^{-1} := \{x \in K \mid xI \subseteq D\}$ and the *t-operation* is defined by $I_t = \bigcup\{J_v \mid J \subseteq I \text{ and } J \in \mathbf{f}(D)\}$. The *w-operation* is the mapping on $\mathbf{F}(D)$ defined by $I \mapsto I_w = \{x \in K \mid xJ \subseteq I \text{ for some } J \in \mathbf{f}(D) \text{ with } J^{-1} = D\}$.

It is easy to see that if $I \in \mathbf{f}(D)$, then $I_t = I_v$. An $I \in \mathbf{F}(D)$ is called a $*$-*ideal* if $I_* = I$. It is well known that for a given star-operation $*$ on $D$, the mapping $I \mapsto I_{*_f} = \bigcup\{J_* \mid J \subseteq I \text{ and } J \in \mathbf{f}(D)\}$ is a star-operation on $D$, called the star-operation of finite type associated to $*$ if $* = *_f$. A star-operation $*$ on $D$ is said to be of *finite type* if $* = *_f$. Recall that each prime ideal minimal over a $*_f$-ideal is a $*_f$-ideal, and hence each height-one prime ideal is a $*_f$-ideal. Moreover, if $I$ is a $*_f$-ideal, then $\sqrt{I}$ is also a $*_f$-ideal. Let $*$-Max$(D)$ denote the set of $*$-ideals maximal among proper integral $*$-ideals of $D$. A member in $*$-Max$(D)$ is called a *maximal $*$-ideal* of $D$. It is well known that a maximal $*$-ideal is a prime ideal, and if $D$ is not a field, then each integral $*_f$-ideal is contained in a maximal $*_f$-ideal. The *$*$-dimension* of $D$, denoted by $*$-dim$(D)$, is defined by the supremum of $\{n \mid P_1 \subsetneq \cdots \subsetneq P_n$ is a chain of prime $*$-ideals of $D\}$. Thus $*$-dim$(D) = 1$ if and only if each maximal $*$-ideal of $D$ has height-one. An $I \in \mathbf{F}(D)$ is said to be *$t$-invertible* if $(II^{-1})_t = D$ (or equivalently, $II^{-1} \nsubseteq M$ for all maximal $t$-ideals $M$ of $D$. Let $\mathrm{T}(D)$ be the abelian group of $t$-invertible fractional $t$-ideals of $D$ under the $t$-multiplication $I * J = (IJ)_t$ and $\mathrm{Prin}(D)$ be the subgroup of $\mathrm{T}(D)$ of principal fractional ideals of $D$. Then the *$t$-class group* of $D$ is defined as $\mathrm{Cl}(D) = \mathrm{T}(D)/\mathrm{Prin}(D)$.

Let $*_1$ and $*_2$ be star-operations of finite type on $D$. Following [1], we say that $*_1$ is *coarser* than $*_2$ (denoted by $*_1 \leq *_2$) if $I_{*_1} \subseteq I_{*_2}$ for all $I \in \mathbf{F}(D)$ (or equivalently, if each $*_2$-ideal is a $*_1$-ideal). Then $\leq$ is a partial order on the star-operations on $D$. It is well known that $d \leq *_f \leq * \leq v$ for all star-operations $*$ on $D$, and $d \leq * \leq t \leq v$ if $*$ is of finite type.



## 1. THE $*$-COMPLEMENTS

This section is devoted to study of the $*$-complements of multiplicative subsets. We begin with a lemma collecting elementary properties. The first nine assertions appear in [6, Lemma 2.1] and the remaining two assertions are straightforward.

**Lemma 1.1.** *Let $*$, $*_1$ and $*_2$ be star-operations of finite type on $D$, $\mathcal{S} = \{S_\alpha\}_{\alpha \in \Lambda}$ be a family of saturated multiplicative subsets of $D$, and let $S$, $S_1$ and $S_2 \in \mathcal{S}$. Then the following statements hold*:

(1) *$N_*(S)$ is a saturated multiplicative subset of $D$*;
(2) $S \cap N_*(S) = U(D)$;
(3) $S \subseteq N_*(N_*(S))$;
(4) *If $S_1 \subseteq S_2$, then $N_*(S_2) \subseteq N_*(S_1)$*;
(5) $N_*(N_*(N_*(S))) = N_*(S)$;
(6) *If $*_1 \leq *_2$, then $N_{*_1}(S) \subseteq N_{*_2}(S) \subseteq N(S)$*;
(7) *Let $P$ be a prime $*$-ideal of $D$. Then either $P \cap S = \emptyset$ or $P \cap N_*(S) = \emptyset$*;
(8) *Let $I$ be a $*$-ideal of $D$. Then $I = ID_S \cap ID_{N_*(S)}$. In particular, $D = D_S \cap D_{N_*(S)}$*;
(9) *Let $I$ and $J$ be $*$-ideals of $D$. Then $I = J$ if and only if $ID_S = JD_S$ and $ID_{N_*(S)} = JD_{N_*(S)}$*;
(10) *For any nonempty subset $\Delta$ of $\Lambda$, $N_*(\bigcup_{\alpha \in \Delta} S_\alpha) = \bigcap_{\alpha \in \Delta} N_*(S_\alpha)$.*
(11) $N_*(S_1S_2) = N_*(S_1) \cap N_*(S_2)$.

Next, we give an equivalent condition to have $N_*(N_*(S)) = S$ for a saturated multiplicative subset $S$ of $D$. To do this, we need the following lemma.

**Lemma 1.2.** *Let $P$ be a prime $*$-ideal of $D$, and let $S := D \backslash P$. Then either $N_*(N_*(S)) = S$ or $N_*(N_*(S)) = D \backslash \{0\}$.*

*Proof.* Assume that $S \subsetneq N_*(N_*(S))$, and choose any $a \in N_*(N_*(S))\backslash S$. Then $a \in P$ and $(a, b)_* = D$ for all $b \in N_*(S)$. Hence $N_*(S) \subseteq D\backslash P = S$. By Lemma 1.1(2), $N_*(S) = U(D)$, and thus $N_*(N_*(S)) = D\backslash\{0\}$. □

**Theorem 1.3.** *Let $*$ be a star-operation of finite type on $D$, $P$ be a prime $*$-ideal of $D$, and let $S := D\backslash P$. Then the following assertions are equivalent*:

(1) $N_*(N_*(S)) = S$;
(2) *$P$ is a maximal $*$-ideal containing a nonzero element $d \in D$ which does not belong to any maximal $*$-ideal distinct from $P$.*

*Proof.* (1) $\Rightarrow$ (2) We first claim that $P$ is a maximal $*$-ideal of $D$. If $P$ is not a maximal $*$-ideal, then there exists a maximal $*$-ideal $M$ such that $P \subsetneq M$. Choose an element $x \in M\backslash P$. Then $x \in S$. If $U(D) \subsetneq N_*(S)$, then $N_*(S)\backslash U(D) \subseteq P$ by Lemma 1.1(2). Hence $D = (x, r)_* \subseteq M$ for any $r \in N_*(S)\backslash U(D)$. This contradicts the fact that $M$ is a maximal $*$-ideal. Therefore, $N_*(S) = U(D)$, and hence $S = N_*(N_*(S)) = D\backslash\{0\}$, which is impossible. Thus $P$ is a maximal $*$-ideal of $D$. Next, we show the existence of $d$. Note that $U(D) \subsetneq N_*(S)$, because $N_*(N_*(S)) = S \neq D\backslash\{0\}$. Hence $N_*(S) \cap P \neq \emptyset$. Let $d \in N_*(S) \cap P$. If $d$ belongs to a maximal $*$-ideal $Q$ of $D$ which is distinct from $P$, then we have $Q \cap S \neq \emptyset$ and $Q \cap N_*(S) \neq \emptyset$. However, this is absurd by Lemma 1.1(7). Thus $P$ is the unique maximal $*$-ideal containing $d$.

(2) $\Rightarrow$ (1) If $P$ is the unique maximal $*$-ideal containing $d$, then $(d, s)_* = D$ for all $s \in S$. Hence $d \in N_*(S)$. Since $d$ is a nonunit in $D$, $d \notin N_*(N_*(S))$. Thus $N_*(N_*(S)) = S$ by Lemma 1.2. □

When $* = t$, we recover the following corollary.

**Corollary 1.4** ([4, Proposition 2.7]). *Let $P$ be a prime $t$-ideal of $D$, and let $S := D\backslash P$. Then $N(N(S)) = S$ if and only if $P$ is a maximal $t$-ideal and there exists an element $d \in D$ such that $P$ is the unique maximal $t$-ideal containing $d$.*

Following [5], we say that $D$ is a *generalized weakly factorial domain* (GWFD) if every nonzero prime ideal of $D$ contains a primary element (Recall that a nonzero nonunit $x \in D$ is *primary* if $(x)$ is a primary ideal.). It is known that if $D$ is not a field, then $D$ is a GWFD if and only if $t$-dim$(D) = 1$ and for each $P \in X^1(D)$, $P$ is the radical of a principal ideal, where $X^1(D)$ is the set of height-one prime ideals of $D$ [5, Theorem 2.2]. In [4, Proposition 2.5], Anderson and Chang showed that $D$ is a GWFD if and only if $N(N(S)) = S$ for each saturated multiplicative subset $S$ of $D$. Now, we generalize this result to the $*$-complements of multiplicative subsets.

**Corollary 1.5.** *Let $*$ be a star-operation of finite type on $D$. Then the following statements are equivalent*:

(1) *$D$ is a GWFD with $*$-dim$(D) = 1$*;
(2) *$N_*(N_*(S)) = S$ for each saturated multiplicative subset $S$ of $D$*;
(3) *$N_*(N_*(S)) = S$ for each $S = D\backslash P$, where $P$ is a prime $*$-ideal of $D$.*

*Proof.* (1) $\Rightarrow$ (2) Let $S$ be a saturated multiplicative subset of a GWFD $D$ with $*$-dim$(D) = 1$ and $\Gamma = \{\alpha \mid P_\alpha \in X^1(D)$ and $P_\alpha \cap S \neq \emptyset\}$. Since a $t$-ideal is a $*$-ideal,

$t$-dim$(D) = 1$; so $S = D\setminus\bigcup_{\alpha\in\Gamma} P_\alpha$. Since $D$ is a GWFD, for each $\alpha \in \Gamma$, there exists an element $x_\alpha \in D$ such that $P_\alpha = \sqrt{(x_\alpha)}$ [5, Theorem 2.2]. Set $T = \{ux_{\alpha_1}\cdots x_{\alpha_n} \mid u \in U(D),\ n \geq 0$ and $x_{\alpha_i}$ is an element of $D$ such that $\sqrt{(x_{\alpha_i})} = P_{\alpha_i}$ for some $\alpha_i \in \Gamma\}$. We claim that $N_*(T) = S$. If $a \in N_*(T)$, then $(a, x_\alpha)_* = D$ for all $x_\alpha \in T$, and hence $a \notin P_\alpha$ for all $\alpha \in \Gamma$. Therefore, $a \in S$. For the reverse containment, let $s \in S$. Then $(x_\alpha, s)_* = D$ for all $x_\alpha \in T$, because $*$-dim$(D) = 1$. Hence $s \in N_*(T)$, which proves our claim. Thus $N_*(N_*(S)) = N_*(N_*(N_*(T))) = N_*(T) = S$ by Lemma 1.1(5).

(2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) We first show that $*$-dim$(D) = 1$. Suppose that $*$-dim$(D) \neq 1$, and take a prime $*$-ideal $P$ of $D$ which is not a maximal $*$-ideal. Choose any $x \in P\setminus\{0\}$. By the assumption, $x \notin N_*(N_*(D\setminus P))$, i.e., there exists an element $t \in N_*(D\setminus P)$ such that $(x, t)_* \subsetneq D$. Let $M$ be a maximal $*$-ideal of $D$ such that $(x, t)_* \subseteq M$. Since $P$ is not a maximal $*$-ideal of $D$, we can find an element $\alpha \in M\setminus P$. Keeping in mind that $t \in N_*(D\setminus P)$, it follows immediately that $D = (\alpha, t)_* \subseteq M$, a contradiction. Thus $*$-dim$(D) = 1$.

Next, we show that $D$ is a GWFD. Note that $t$-dim$(D) \leq *$-dim$(D)$; so $t$-dim$(D) = 1$. Therefore, it remains to show that each height-one prime ideal is the radical of a principal ideal. Let $Q$ be a height-one prime ideal of $D$, and set $S := D\setminus Q$. Then $N_*(N_*(S)) = S$ by (3). Hence by Theorem 1.3, there exists a nonzero element $d \in D$ such that $Q$ is the only prime $*$-ideal containing $d$. Note that $\sqrt{(d)} = \bigcap_\alpha P_\alpha$, where $P_\alpha$'s are prime ideals of $D$ containing $d$. By shrinking $P_\alpha$'s to prime ideals minimal over $(d)$ [14, Theorem 10], we may assume that $P_\alpha$'s are prime $*$-ideals of $D$. Thus $Q = \sqrt{(d)}$, because $Q$ is the unique prime $*$-ideal containing $d$. $\square$

Let $X$ be an indeterminate over $D$, $D[X]$ be the polynomial ring over $D$, and $*$ be a star-operation on $D[X]$. Following [15, Proposition 2.1], the mapping $\bar{*}$ on $\mathbf{F}(D)$ defined by $I_{\bar{*}} = (I[X])_* \cap D$ is a star-operation on $D$. The author also showed in [15, Proposition 2.1] that $I[X]_* = (I_{\bar{*}}[X])_*$ for each $I \in \mathbf{F}(D)$ and if $*$ is of finite type, then so is $\bar{*}$. Clearly, if $*$ is the $d$-operation on $D[X]$, then $\bar{*}$ is the $d$-operation on $D$. Moreover, it is known that if $*$ is the $t$-operation (resp., $v$-operation) on $D[X]$, then $\bar{*}$ is the $t$-operation (resp., $v$-operation) on $D$ [13, Proposition 4.3]. By the definition, it can be easily shown that $N_{\bar{*},D}(S) \subseteq N_{*,D[X]}(S)$ and $N_{\bar{*},D}(S) = N_{*,D[X]}(S) \cap D$. We end this section by characterizing the $*$-complements of $S$ in the polynomial ring extension via the induced star-operation $\bar{*}$ on $D$.

**Proposition 1.6.** *Let $*$ be a star-operation of finite type on $D[X]$, $\bar{*}$ be the induced star-operation on D, and S be a saturated multiplicative subset of D. If $Q = (Q \cap D)[X]$ for each maximal $*$-ideal $Q$ of $D[X]$ with $Q \cap D \neq (0)$, then $N_{*,D[X]}(S) = \{0 \neq f \in D[X] \mid (c_D(f), s)_{\bar{*}} = D$ for all $s \in S\}$, where $c_D(f)$ is the ideal of D generated by the coefficients of $f$.*

*Proof.* Let $f \in N_{*,D[X]}(S)$. Then $(f, s)_* = D[X]$ for all $s \in S$. Note that $(f, s)_* \subseteq ((c_D(f), s)D[X])_* \subseteq D[X]$; so $((c_D(f), s)D[X])_* = D[X]$. Hence $(c_D(f), s)_{\bar{*}} = ((c_D(f), s)D[X])_* \cap D = D$. Conversely, if $g$ is a nonzero element of $D[X]$ such that $(g, s)_* \subsetneq D[X]$ for some $s \in S$, then there exists a maximal $*$-ideal $Q$ of $D[X]$ containing $(g, s)_*$. Since $Q \cap D \neq (0)$, $Q = (Q \cap D)[X]$ by the assumption; so

$(c_D(g), s)_{\overline{*}} \subseteq Q \cap D$ (note that $Q \cap D$ is a prime $\overline{*}$-ideal of $D$). This completes the proof. □

**Remark 1.7.**

(1) In Proposition 1.6, the assumption that each maximal $*$-ideal $Q$ of $D[X]$ with $Q \cap D \neq (0)$ is extended from $D$ is essential. Let $\mathbb{Z}$ be the ring of integers, $S := \{\pm 2^n \mid n \geq 0\}$, and $X$ be an indeterminate over $\mathbb{Z}$. Then it is easy to see that $X \in \{0 \neq f \in \mathbb{Z}[X] \mid (c_{\mathbb{Z}}(f), s) = \mathbb{Z}$ for all $s \in S\} \backslash N_{d,\mathbb{Z}[X]}(S)$. Indeed, $(2, X)$ is a maximal ideal of $\mathbb{Z}[X]$ whose contraction to $\mathbb{Z}$ is $2\mathbb{Z}$, but $(2, X) \neq 2\mathbb{Z}[X]$.

(2) If $*$ is the $t$-operation (resp., $w$-operation) on $D[X]$, then $\overline{*}$ is the $t$-operation (resp., $w$-operation) on $D$ [13, Proposition 4.3] (or [15, Remark 2.2]); so it satisfies the assumption of Proposition 1.6.

## 2. ALMOST g*-SPLITTING SETS

As mentioned in the introduction, a saturated multiplicative subset $S$ of $D$ is an almost g*-splitting set if for each $0 \neq d \in D$, there exists a positive integer $n = n(d)$ such that $d^n = st$ for some $s \in S$ and $t \in N_*(S)$. In this section, we study an almost g*-splitting set which is a generalization of almost splitting sets. Our first result gives the relationship between almost splitting sets and almost g*-splitting sets for a given star-operation $*$ of finite type on $D$.

**Theorem 2.1.** *Let $*_1 \leq *_2$ be star-operations of finite type on D, and let S be an almost* g$^{*_1}$*-splitting set of D. Then the following statements hold*:

(1) *$S$ is an almost* g$^{*_2}$*-splitting set of $D$*;
(2) *$S$ is an almost splitting set of $D$*;
(3) *$N_{*_1}(S) = N_{*_2}(S)$. In particular, $N_{*_1}(S) = N(S)$*;
(4) *$N_{*_1}(N_{*_1}(S)) = S$*;
(5) *$N_{*_1}(S)$ is an almost* g$^{*_1}$*-splitting set of $D$.*

*Proof.* (1) Let $0 \neq d \in D$. Then there is an integer $n = n(d) \geq 1$ such that $d^n = st$ for some $s \in S$ and $t \in N_{*_1}(S)$. By Lemma 1.1(6), $t \in N_{*_2}(S)$. Thus $S$ is an almost g$^{*_2}$-splitting set of $D$.

(2) This follows directly from (1) by taking $*_2 = t$.

(3) By Lemma 1.1(6), we have $N_{*_1}(S) \subseteq N_{*_2}(S)$. For the reverse containment, let $d \in N_{*_2}(S)$. Since $S$ is an almost g$^{*_1}$-splitting set, there exists a positive integer $n = n(d)$ such that $d^n = st$ for some $s \in S$ and $t \in N_{*_1}(S)$. Then $(s) = (st, s)_{*_2} = (d^n, s)_{*_2} = D$; so $s$ is a unit of $D$. Since $N_{*_1}(S)$ is saturated, $d \in N_{*_1}(S)$. Hence $N_{*_2}(S) \subseteq N_{*_1}(S)$, and thus $N_{*_1}(S) = N_{*_2}(S)$. The second assertion is the case when $*_2 = t$.

(4) By Lemma 1.1(3), $S \subseteq N_{*_1}(N_{*_1}(S))$. Let $a \in N_{*_1}(N_{*_1}(S))$. Since $S$ is an almost g$^{*_1}$-splitting set of $D$, there exists a positive integer $n = n(a)$ such that $a^n = st$ for some $s \in S$ and $t \in N_{*_1}(S)$. Then $(t) = t(s, t)_{*_1} = (st, t^2)_{*_1} = (a^n, t^2)_{*_1} = D$, because $N_{*_1}(S)$ and $N_{*_1}(N_{*_1}(S))$ are multiplicatively closed. Hence $t$ is a unit

of $D$, and thus $a^n \in S$. Since $S$ is saturated, $a \in S$. Therefore, $N_{*_1}(N_{*_1}(S)) \subseteq S$. Thus $N_{*_1}(N_{*_1}(S)) = S$.

(5) This is an immediate consequence of (4). $\square$

**Remark 2.2.** Let $*$ be a star-operation of finite type on $D$. Note that $N(D\backslash\{0\}) = N_*(D\backslash\{0\}) = U(D)$. Thus for any star-operation $*$ of finite type on $D$, $D\backslash\{0\}$ is both almost splitting and almost g$^*$-splitting in $D$. By Theorem 2.1(5), the same situation occurs for $U(D)$. This also shows that the converse of Theorem 2.1(3) does not hold, i.e., for a saturated multiplicative subset $S$ of $D$ and star-operations $*_1$ and $*_2$ of finite type of $D$, $N_{*_1}(S) = N_{*_2}(S)$ need not imply that $*_1$ and $*_2$ have an order relationship under $\leq$.

**Theorem 2.3.** *Let $*_1$ and $*_2$ be star-operations of finite type on D with $*_1 \leq *_2$, and S be an almost* g$^{*_2}$*-splitting set of D. Then the following statements are equivalent*:

(1) *S is an almost* g$^{*_1}$*-splitting set of D*;
(2) $N_{*_1}(S) = N_{*_2}(S)$;
(3) *For all prime $*_1$-ideals P of D, either $P \cap S = \emptyset$ or $P \cap N_{*_2}(S) = \emptyset$*;
(4) *For all maximal $*_1$-ideals M of D, either $M \cap S = \emptyset$ or $M \cap N_{*_2}(S) = \emptyset$.*

*Proof.* (1) $\Rightarrow$ (2) This implication was already shown in Theorem 2.1(3).

(2) $\Rightarrow$ (3) Suppose to the contrary that there exist two elements $s \in P \cap S$ and $t \in P \cap N_{*_2}(S)$ for some prime $*_1$-ideal $P$ of $D$. Since $N_{*_1}(S) = N_{*_2}(S)$, $t \in N_{*_1}(S)$; so $D = (s, t)_{*_1} \subseteq P_{*_1} = P$, which is a contradiction.

(3) $\Rightarrow$ (4) It suffices to note that each maximal $*_1$-ideal is a prime ideal.

(4) $\Rightarrow$ (1) Assume that $S$ is an almost g$^{*_2}$-splitting set of $D$, and let $0 \neq x \in D$. Then there exists an integer $n = n(x) \geq 1$ such that $x^n = st$ for some $s \in S$ and $t \in N_{*_2}(S)$. If $t \notin N_{*_1}(S)$, then there exists an element $s' \in S$ such that $(s', t)_{*_1} \subsetneq D$. Let $M$ be a maximal $*_1$-ideal containing $(s', t)_{*_1}$. Then we have neither $M \cap S = \emptyset$ nor $M \cap N_{*_2}(S) = \emptyset$, which is absurd. Therefore, $t \in N_{*_1}(S)$, and thus $S$ is an almost g$^{*_1}$-splitting set of $D$. $\square$

When $*_2 = t$, we have the following corollary.

**Corollary 2.4.** *Let $*$ be a star-operation of finite type on D, and let S be an almost splitting set of D. Then the following assertions are equivalent*:

(1) *S is an almost* g$^*$*-splitting set of D*;
(2) $N_*(S) = N(S)$;
(3) *Let P be a prime $*$-ideal of D. Then either $P \cap S = \emptyset$ or $P \cap N(S) = \emptyset$*;
(4) *Let M be a maximal $*$-ideal of D. Then either $M \cap S = \emptyset$ or $M \cap N(S) = \emptyset$.*

Let $\mathbb{N}_0$ (resp., $\mathbb{Z}$) be the set of nonnegative integers (resp., integers). A semigroup $\Gamma$ is called a *numerical semigroup* if $\Gamma$ is a subset of $\mathbb{N}_0$ containing 0 and generates $\mathbb{Z}$ as a group. It is known that if $\Gamma$ is a numerical semigroup, then $\Gamma$ is finitely generated and $\mathbb{N}_0\backslash\Gamma$ is a finite set. Hence there exists the largest nonnegative integer which is not contained in $\Gamma$. This number is called the *Frobenius number* of

$\Gamma$ and is denoted by $F(\Gamma)$. Let $D[\Gamma]$ be a numerical semigroup ring of $\Gamma$ over $D$ and $\Gamma^* = \Gamma\backslash\{0\}$.

For any star-operation $*$ of finite type, we give an example of almost g$^*$-splitting sets that is not a g$^*$-splitting set.

**Proposition 2.5** (cf. [10, Proposition 2.7]). *Let $\Gamma$ be a proper numerical semigroup, $\Gamma^* = \Gamma\backslash\{0\}$, $*$ be a star-operation of finite type on $D[\Gamma]$, $X$ be an indeterminate over $D$, and $S = \{uX^n \mid u \in U(D)$ and $n \in \Gamma\}$. If char$(D) \neq 0$, then the following conditions hold*:

(1) *$N(S) = D[\Gamma]\backslash D[\Gamma^*]$*;
(2) *$S$ is an almost splitting set of $D[\Gamma]$*;
(3) *$S$ is not a $g^*$-splitting set of $D[\Gamma]$*;
(4) *If $D$ is not a field, then $S$ is not an almost $g^d$-splitting set of $D[\Gamma]$*;
(5) *If $D$ is a field, then $S$ is an almost $g^*$-splitting set of $D[\Gamma]$.*

*Proof.* Let $p =$char$(D)$ and $\Gamma = \{0 = \alpha_0, \alpha_1, \ldots, \alpha_n\} \cup \{k \in \mathbb{N}_0 \mid k \geq F(\Gamma) + 1\}$ with $\alpha_i < \alpha_j$ for $i < j$.

(1) Suppose that there exists an $f \in N(S)$ with $f(0) = 0$. Then $fX^{F(\Gamma)} \in D[\Gamma]$. Note that $X^{F(\Gamma)}X^{\alpha} \in D[\Gamma]$ for any $\alpha \in \Gamma^*$. But $X^{F(\Gamma)} \notin D[\Gamma]$, which shows that $D[\Gamma] \subsetneq (f, X^{\alpha})^{-1}$ for any $\alpha \in \Gamma^*$. Hence $f(0) \neq 0$. Conversely, let $\alpha \in \Gamma$ and $g = \sum_{i=0}^{n} g_{\alpha_i} X^{\alpha_i} + \sum_{i=F(\Gamma)+1}^{l} g_i X^i \in D[\Gamma]$ with $g_0 \neq 0$. We claim that $(g, X^{\alpha})_v = D[\Gamma]$; equivalently, $(g, X^{\alpha})^{-1} = D[\Gamma]$. The containment $D[\Gamma] \subseteq (g, X^{\alpha})^{-1}$ is obvious. For the reverse inclusion, let $h \in (g, X^{\alpha})^{-1}$. Then $X^{\alpha}h \in D[\Gamma]$; so $h = \frac{1}{X^{\alpha}}h'$ for some $h' \in D[\Gamma]$. Since $gh \in D[\Gamma]$ and $g_0 \neq 0$, the initial term of $h'$ should have degree at least $\alpha$; so $h \in D[X]$. Now, we write $h = \sum_{i=0}^{m} h_i X^i$. Note that

$$gh = g_0h_0 + g_0 \sum_{i=1}^{\alpha_1 - 1} h_iX^i + (g_0h_{\alpha_1} + g_{\alpha_1}h_0)X^{\alpha_1} + X^{\alpha_1+1}h_1$$

for some $h_1 \in D[X]$. Since $gh \in D[\Gamma]$ and $g_0 \neq 0$, $h_i = 0$ for all $i = 1, \ldots, \alpha_1 - 1$; so $h = h_0 + \sum_{i=\alpha_1}^{m} h_iX^i$. Note that $2\alpha_1 \in \Gamma^*$; so $2\alpha_1 \geq F(\Gamma) + 1$ or $2\alpha_1 = \alpha_p$ for some $p = 2, \ldots, n$. If $2\alpha_1 \geq F(\Gamma) + 1$, then we have

$$gh = g_0h_0 + (g_0h_{\alpha_1} + h_0g_{\alpha_1})X^{\alpha_1} + g_0 \sum_{i=\alpha_1+1}^{\alpha_2 - 1} h_iX^i + (g_0h_{\alpha_2} + h_0g_{\alpha_2})X^{\alpha_2} + X^{\alpha_2+1}h_2$$

for some $h_2 \in D[X]$. Again, since $fg \in D[\Gamma]$ and $g_0 \neq 0$, $h_{\alpha_1+1} = \cdots = h_{\alpha_2-1} = 0$. By repeating this process, we have $h_i = 0$ for all $i \in \mathbb{N}_0\backslash\Gamma$, and hence $h \in D[\Gamma]$. Therefore $(g, X^{\alpha})^{-1} = D[\Gamma]$. If $2\alpha_1 = \alpha_p$ for some $p = 2, \ldots, n$, then a simple modification of the proof of the previous case leads to the same conclusion because $2\alpha_q \geq F(\Gamma) + 1$ for some $q \leq n$. Thus $g \in N(S)$.

(2) Clearly, $S$ is a saturated multiplicative subset of $D[\Gamma]$. Let $f \in D[\Gamma]$. Then $f = X^m g$ for some $g \in D[X]$ with $g(0) \neq 0$. Since char$(D) = p$, $g^{p^l} \in D[\Gamma]$ for some positive integer $l$ with $p^l \geq F(\Gamma) + 1$. Now, we claim that $f^{p^l}D[\Gamma]_S \cap D[\Gamma]$ is principal. Note that $f^{p^l}D[\Gamma]_S \cap D[\Gamma] = g^{p^l}D[\Gamma]_S \cap D[\Gamma]$. Hence it suffices to show that $g^{p^l}D[\Gamma]_S \cap D[\Gamma] = g^{p^l}D[\Gamma]$. The containment $g^{p^l}D[\Gamma] \subseteq g^{p^l}D[\Gamma]_S \cap D[\Gamma]$ is clear.

For the converse, let $h = \sum_{i=0}^{n} h_{\alpha_i} X^{\alpha_i} + \sum_{i=F(\Gamma)+1}^{l} h_i X^i \in g^{p^l} D[\Gamma]_S \cap D[\Gamma]$. Then $X^\alpha h \in g^{p^l} D[\Gamma]$ for some $\alpha \in \Gamma$; so $h = \frac{1}{X^\alpha} g h_1$ for some $h_1 \in D[\Gamma]$. Since $h \in D[\Gamma]$ and $g(0) \neq 0$, $\frac{1}{X^\alpha} h_1 \in D[X]$. Let $\frac{1}{X^\alpha} h_1 = \sum_{i=0}^{p} d_i X^i$. Then we have

$$\begin{aligned} \sum_{i=0}^{n} h_{\alpha_i} X^{\alpha_i} + \sum_{i=F(\Gamma)+1}^{l} h_i X^i &= g^{p^l} \sum_{i=0}^{p} d_i X^i \\ &= g(0)^{p^l} d_0 + g(0)^{p^l} \sum_{i=1}^{F(\Gamma)} d_i X^i + X^{F(\Gamma)+1} h_2 \end{aligned}$$

for some $h_2 \in D[X]$. Hence $d_i = 0$ for all $i \in \{1, \ldots, F(\Gamma)\} \setminus \Gamma$. Therefore $\frac{1}{X^\alpha} h_1 \in D[\Gamma]$, and hence $h \in g^{p^l} D[\Gamma]$. Thus $S$ is an almost splitting set of $D[\Gamma]$ [7, Proposition 2.7].

(3) Since $d \leq *$, it is enough to show that $S$ is not a splitting set of $D[\Gamma]$. Let $f = X^{F(\Gamma)+1}(1+X) \in D[\Gamma]$. Then $fD[\Gamma]_S \cap D[\Gamma] = (1+X)D[\Gamma]_S \cap D[\Gamma]$. If $(1+X)D[\Gamma]_S \cap D[\Gamma] = gD[\Gamma]$ for some $g \in D[\Gamma]$, then $1 + (-1)^{\alpha_1} X^{\alpha_1} = gh$ for some $h \in D[\Gamma]$. Note that $g$ is not a unit in $D[\Gamma]$ (for if $g$ is a unit of $D[\Gamma]$, then $1 \in (1+X)D[\Gamma]_S \cap D[\Gamma]$; so $X^\alpha = (1+X)g_1$ for some $\alpha \in \Gamma$ and $g_1 \in D[\Gamma]$, which is impossible). Hence $g = u(1 + (-1)^{\alpha_1} X^{\alpha_1})$ for some $u \in U(D[\Gamma])$. Let $m \in \Gamma^*$ such that $m$ is not a multiple of $\alpha_1$. Then an easy calculation shows that $g$ cannot divide $1 + (-1)^m X^m$ in $D[\Gamma]$, which is a contradiction. Hence $(1+X)D[\Gamma]_S \cap D[\Gamma]$ is not principal, and thus $S$ is not a splitting set of $D[\Gamma]$ [2, Theorem 2.2]. Note that if $\{\alpha_1, \ldots, \alpha_n\} = \emptyset$, then we can deduce the same conclusion with $F(\Gamma) + 1$ instead of $\alpha_1$.

(4) Let $a$ be a nonzero nonunit of $D$ and $0 \neq f \in D[\Gamma^*]$. If $(a + f, X^{F(\Gamma)+1}) = D[\Gamma]$, then $X^{F(\Gamma)+2} = (a+f)g + X^{F(\Gamma)+1} h$ for some $g, h \in D[\Gamma]$. Since $a \neq 0$ and the degree of the initial term of $X^{F(\Gamma)+1} h$ is at least $F(\Gamma) + 1$, we can write $g = X^{F(\Gamma)+1} g_1$ for some $g_1 \in D[X]$. Also, since $1 \notin \Gamma$, by comparing the coefficients of $X^{F(\Gamma)+2}$ in both sides, we have $1 = ab$, where $b$ is the coefficient of $X^{F(\Gamma)+2}$ in $g$. This is absurd, because $a$ is nonunit. Therefore, $(a + f, X^{F(\Gamma)+1}) \subsetneq D[\Gamma]$, which indicates that $a + f \notin N_d(S)$. Hence $N_d(S) \subsetneq N(S)$ by (1). Thus by Corollary 2.4, $S$ is not an almost $\mathrm{g}^d$-splitting set of $D[\Gamma]$.

(5) We first claim that $S$ is an almost $\mathrm{g}^d$-splitting set of $D[\Gamma]$. By Corollary 2.4 and (2), it suffices to show that $N_d(S) = N(S)$. Let $f = \sum_{i=0}^{m} f_i X^i \in N(S)$ and fix an integer $n \in \Gamma^*$. Note that $(f, X^n) = (\sum_{i=0}^{n+F(\Gamma)} f_i X^i, X^n)$; so we may assume that $f = \sum_{i=0}^{n+F(\Gamma)} f_i X^i$. Now we find polynomials $g = \sum_{i=0}^{n+F(\Gamma)} g_i X^i$ and $h = \sum_{i=\alpha_1}^{n+2F(\Gamma)} h_i X^i$ in $D[\Gamma]$ such that $fg + X^n h = 1$, i.e., we solve a system of equations

$$\begin{cases} f_0 g_0 = 1 & \\ \sum\limits_{i+j=k} f_i g_j = 0 & \text{if } 1 \leq k \leq n + \alpha_1 - 1 \\ \sum\limits_{i+j=k} f_i g_j + h_{k-n} = 0 & \text{if } n + \alpha_1 \leq k \leq 2n + 2F(\Gamma). \end{cases}$$

To do this, take $g_0 = \frac{1}{f_0}$ (note that $f_0$ is a unit by (1)). If we have appropriate $g_0, g_1, \ldots, g_k$ for $k \leq n + \alpha_1 - 2$, then we set $g_{k+1} = -\frac{\sum_{i=1}^{k+1} f_i g_{k+1-i}}{f_0}$. Choose $g_p = 1$

for all $p \in \{n+\alpha_1, \ldots, n+F(\Gamma)\} \cap \Gamma$. Hence the existence of $g$ is proved, and consequently, we have the element $h$ by defining $h_k = -\sum_{i+j=k+n} f_i g_j$ for each $k = \alpha_1, \ldots, n+2F(\Gamma)$. This means that $f \in N_d(S)$, which forces $S$ to be an almost g$^d$-splitting set. Thus by Theorem 2.1(1), $S$ is an almost g$^*$-splitting set of $D[\Gamma]$, because $d \leq *$. □

We say that $D$ is a *weakly Krull domain* if $D = \bigcap_{P \in X^1(D)} D_P$ and this intersection has finite character [8]; $D$ is a *weakly factorial domain* (WFD) if each nonzero nonunit of $D$ is a product of primary elements; and $D$ is an *almost weakly factorial domain* (AWFD) if for each nonzero nonunit $d \in D$, there exists a positive integer $n = n(d)$ such that $d^n$ is a product of primary elements. It is known that $D$ is a weakly Krull domain (resp., WFD) if and only if every saturated multiplicative subset of $D$ is a $t$-splitting set (resp., splitting set) of $D$ [3, p. 8] (resp., [9, Theorem]). (Recall that a multiplicative subset $S$ of $D$ is a *$t$-splitting set* of $D$ if for each nonzero element $d \in D$, $(d) = (AB)_t$ for some integral ideals $A$ and $B$ of $D$, where $A_t \cap sD = sA_t$ (or equivalently, $(A, s)_t = D$) for all $s \in S$ and $B_t \cap S \neq \emptyset$.) Now, we give the almost splitting set analogue of these results.

**Lemma 2.6.** *The following conditions are equivalent*:

(1) *Every saturated multiplicative subset of $D$ is an almost splitting set of $D$*;
(2) *$D$ is a weakly Krull domain and* $\mathrm{Cl}(D)$ *is torsion*;
(3) *$D$ is an AWFD*;
(4) *Let $d$ be a nonzero nonunit of $D$ and let $P$ be a prime ideal of $D$ minimal over $(d)$. Then $P \in X^1(D)$ and there exists a positive integer $n = n(d)$ such that $d^n D_P \cap D$ is principal.*

*Proof.* (2) ⇒ (1) Note that if $\mathrm{Cl}(D)$ is torsion, then a $t$-splitting set of $D$ is an almost splitting set [10, Proposition 2.3]. Thus this implication is an immediate consequence of [2, p. 8].

(1) ⇒ (4) Assume that every saturated multiplicative subset of $D$ is an almost splitting set of $D$. Since an almost splitting set is $t$-splitting, $D$ is a weakly Krull domain; so $t$-$\dim(D) = 1$ [8, Lemma 2.1]. Let $d$ be a nonzero nonunit of $D$ and let $P$ be a prime ideal of $D$ minimal over $(d)$. Then $P$ is a $t$-ideal of $D$, and hence $P \in X^1(D)$. Let $S := D \backslash P$. Then by the assumption, $S$ is an almost splitting set of $D$. Thus there exists a positive integer $n = n(d)$ such that $d^n D_P \cap D = d^n D_S \cap D$ is principal [7, Proposition 2.7].

(2) ⇔ (3) ⇔ (4) [8, Theorem 3.4]. □

It was shown that every saturated multiplicative subset of $D$ is a g$^*$-splitting set if and only if $D$ is a WFD and $*$-$\dim(D) = 1$ [6, Theorem 2.6]. We give the almost g$^*$-splitting set analogue of this result.

**Theorem 2.7.** *Let $*$ be a star-operation of finite type on $D$. Then the following statements are equivalent*:

(1) *Every saturated multiplicative subset of $D$ is an almost* g$^*$*-splitting set*;
(2) *$D$ is an AWFD and* $*$-$\dim(D) = 1$.

*Proof.* (1) $\Rightarrow$ (2) Assume that every saturated multiplicative subset of $D$ is an almost g$^*$-splitting set. Since an almost g$^*$-splitting set is an almost splitting set by Theorem 2.1(2), it follows from Lemma 2.6 that $D$ is an AWFD. Suppose to the contrary that $*$-dim$(D) \geq 2$. Let $(0) \neq P \subsetneq Q$ be prime $*$-ideals of $D$ and let $S = D \backslash P$. Take $a \in P \backslash \{0\}$ and $b \in Q \backslash P$. Note that $S$ is a saturated multiplicative subset of $D$; so $S$ is an almost g$^*$-splitting set of $D$ by the assumption. Hence there exists an integer $n = n(a) \geq 1$ such that $a^n = st$ for some $s \in S$ and $t \in N_*(S)$. Note that $t \in P$ because $s \notin P$. Therefore $D = (b, t)_* \subseteq Q_* = Q$, which is a contradiction. Thus $*$-dim$(D) = 1$.

(2) $\Rightarrow$ (1) Assume that $D$ is an AWFD and $*$-dim$(D) = 1$. Then by Lemma 2.6, every saturated multiplicative subset of $D$ is an almost splitting set of $D$. Let $S$ be a saturated multiplicative subset of $D$. To show that $S$ is an almost g$^*$-splitting set of $D$, it is enough to prove that $N_*(S) = N(S)$ by Corollary 2.4. If $N_*(S) \subsetneq N(S)$, then $(s, t)_* \subsetneq D$ for some $s \in S$ and $t \in N(S)$. Let $M$ be a maximal $*$-ideal of $D$ containing $(s, t)_*$. Since $*$-dim$(D) = 1$, $M$ is a height-one prime ideal, and hence $M$ is a $t$-ideal. Hence $D = (s, t)_v \subseteq M_t = M$, which is impossible. Thus $N_*(S) = N(S)$. □

We have already observed in Theorem 2.1(2) that an almost g$^*$-splitting set is an almost splitting set, but the converse does not hold (Proposition 2.5). However, the proof of (2) $\Rightarrow$ (1) in Theorem 2.7 shows that if $*$-dim$(D) = 1$, then an almost splitting set of $D$ is almost g$^*$-splitting. Thus we have the following corollary.

**Corollary 2.8.** *Let $*$ be a star-operation of finite type on $D$ with $*$-dim$(D) = 1$. Then a saturated multiplicative subset $S$ of $D$ is an almost g$^*$-splitting set of $D$ if and only if $S$ is an almost splitting set of $D$.*

Let $*$ be a star-operation of finite type on an AWFD $D$ with $*$-dim$(D) \geq 2$. Then by Lemma 2.6, every saturated multiplicative subset $S$ of $D$ is an almost splitting set but by Theorem 2.7, some of them are not almost g$^*$-splitting sets. The next example shows that this holds for $D$ a quasi-local UFD with dim$(D) \geq 2$ and $* = d$.

**Example 2.9.** Let $D$ be a quasi-local UFD with dim$(D) \geq 2$. Let $p \in D$ be a prime and $S := \{up^n \mid u \in U(D) \text{ and } n \geq 0\}$. Then $S$ is an almost splitting set in $D$. Note that $N_d(S) = \{0 \neq d \in D \mid (d, up^n) = D \text{ for all } up^n \in S\} = U(D)$, and hence $N_d(N_d(S)) = D \backslash \{0\} \neq S$. Thus by Theorem 2.1, $S$ is not an almost g$^d$-splitting set of $D$.

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