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S-Noetherian properties on amalgamated algebras along an ideal

Jung Wook Lim [a], Dong Yeol Oh [b,*]

[a] Department of Mathematics, Kyungpook National University, Daegu 702-701, Republic of Korea
[b] Department of Mathematics, Pohang University of Science and Technology, Pohang 790-784, Republic of Korea

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ABSTRACT

Let A and B be commutative rings with identity, $f : A \to B$ a ring homomorphism and J an ideal of B. Then the subring $A \bowtie^f J := \{(a, f(a) + j) \mid a \in A \text{ and } j \in J\}$ of $A \times B$ is called the amalgamation of A with B along with J with respect to f. In this paper, we investigate a general concept of the Noetherian property, called the S-Noetherian property which was introduced by Anderson and Dumitrescu, on the ring $A \bowtie^f J$ for a multiplicative subset S of $A \bowtie^f J$. As particular cases of the amalgamation, we also devote to study the transfers of the S-Noetherian property to the constructions $D + (X_1, \dots, X_n)E[X_1, \dots, X_n]$ and $D + (X_1, \dots, X_n)E[\![X_1, \dots, X_n]\!]$ and Nagata's idealization.

1. Introduction

In the classical ideal theory, pullbacks have for many years been an important tool in the commutative algebra because of their use in producing many examples. A typical example of pullbacks is the $D + M$ construction. This kind of structure was first introduced by Gilmer [9] and mass produced several other instances such as the $D + (X_1, \dots, X_n)E[X_1, \dots, X_n]$ and $D + (X_1, \dots, X_n)E[\![X_1, \dots, X_n]\!]$ constructions. Another important example of pullbacks is Nagata's idealization. While we do not know who first considered the idealization, the idea to use idealization to extend results concerning ideals to modules is due to Nagata [16].

In order to set up a more general setting of the idealization, D'Anna and Fontana introduced and studied the amalgamated duplication of a commutative ring R along an R-module M (resp., an ideal I of R) which is an ideal in some overring of R, and denoted by $R \bowtie M := \{(r, r + m) \mid r \in R \text{ and } m \in M\}$ (resp., $R \bowtie I$) [6,7]. Note that $R \bowtie M$ forms a subring of $R \times T(R)$ under the usual componentwise operations, where $T(R)$ is the total quotient ring of R.

Recently, in [4], the authors developed the concept of the amalgamation equipped with an ideal by using a ring homomorphism, *i.e.*, they introduced the amalgamation algebra along an ideal (shortly amalgamation algebra), denoted by $A \bowtie^f J := \{(a, f(a) + j) \mid a \in A \text{ and } j \in J\}$, where $f : A \to B$ is a ring homomorphism and J is an ideal of B. They showed that the amalgamation algebra can be realized as a pullback [4, Proposition 4.7]. In fact, the amalgamation algebra generalizes not only classical constructions such as the $D + M$ construction and Nagata's idealization but also the amalgamated duplication [4, Examples 2.4 and 2.6 and Remark 2.8]. D'Anna and Fontana also characterized when those pullbacks that arise from the amalgamation are Noetherian [4, Propositions 5.6, 5.7 and 5.8] and when the so-called composite ring extensions $D + XE[X]$ and $D + XE[\![X]\!]$ are Noetherian as a consequence of the amalgamation algebra [4, Corollary 5.9]. Note that the Noetherian properties of the constructions $D + XE[X]$ and $D + XE[\![X]\!]$ were characterized by Hizem and Benhissi with totally different approaches [10,11].

* Corresponding author.
E-mail addresses: jwlim@knu.ac.kr (J.W. Lim), dongyeol70@gmail.com (D.Y. Oh).

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Meanwhile, there were attempts to generalize the concept of Noetherian rings in order to extend the well-known results for Noetherian rings (see [1,2,15,17]). One of them is the notion of S-Noetherian rings. Let R denote a commutative ring with identity, M a unitary R-module and S a multiplicative subset of R. In [2], the authors introduced the concept of "almost finitely generated" to study Querré's characterization of divisorial ideals in integrally closed polynomial rings. Later, Anderson and Dumitrescu [1] abstracted this notion to any commutative ring and defined a general concept of Noetherian rings. They call R an *S-Noetherian ring* if each ideal of R is S-finite, *i.e.*, for each ideal I of R, there exist an $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$. They defined M to be *S-finite* if there exist an $s \in S$ and a finitely generated R-submodule F of M such that $sM \subseteq F$. Also, M is called *S-Noetherian* if each submodule of M is S-finite. It is clear that if S is a subset of units of R, then an S-Noetherian ring is Noetherian and an S-finite R-module is a finitely generated R-module. In [1], the authors gave a number of S-variants of well-known results for Noetherian rings: S-versions of Cohen's result, the Eakin–Nagata theorem and the Hilbert basis theorem under an additional condition. In [17], Zhongkui studied when the ring of generalized power series is S-Noetherian. In [15], the authors completely classified when the composite ring extensions of the forms $D + E[\Gamma^*]$ and $D + [\![E^{\Gamma^*,\leqslant}]\!]$ are S-Noetherian. (Note that in terms of a torsion-free grading monoid Γ, the rings $D + E[\Gamma^*]$ and $D + [\![E^{\Gamma^*,\leqslant}]\!]$ generalize those of $D + XE[X]$ and $D + XE[\![X]\!]$, respectively.)

D'Anna et al. dealt with the amalgamation having the Noetherian property [4]. In [15], they investigated the S-Noetherian properties on the rings $D + XE[X]$ and $D + XE[\![X]\!]$ which are typical examples of the amalgamation. So it is natural to study when the amalgamated algebra has the S-Noetherian property.

In this paper, we investigate the S-Noetherian properties on the amalgamation which completely generalize well-known results for Noetherian properties on the amalgamation by D'Anna et al. [4]. To do this, we begin with a study of the basic S-Noetherian properties. In particular, in Section 2, we briefly review some known results on S-Noetherian rings and give a simple result on S-Noetherian properties on pullbacks. In Section 3, we classify a necessary and sufficient condition for the amalgamated algebra $A \bowtie^f J$ to be S-Noetherian (Theorems 3.2 and 3.4). As a corollary, we provide a simple proof of [15, Theorems 3.6 and 4.4] which represent the S-Noetherian properties of the composite ring extensions $D + XE[X]$ and $D + XE[\![X]\!]$ by using different approaches, while the condition is weakened (Corollaries 3.5 and 3.6). We finally devote to Nagata's idealization (Theorem 3.8).

2. Basic results: S-Noetherian properties on pullbacks

Let R be a commutative ring with identity and S a multiplicative subset of R. It is well known that if $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules, then M is a Noetherian module if and only if so are M' and M''. We extend this result to S-Noetherian R-modules. The proof is standard as in the Noetherian case; so we omit it.

Lemma 2.1. *Let R be a commutative ring with identity and S a multiplicative subset of R. Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules. Then M is S-Noetherian if and only if M' and M'' are S-Noetherian. In particular, if R is S-Noetherian, then every finitely generated R-module is also S-Noetherian.*

Now we study the S-Noetherian property on a pullback. To do this, we recall the pullbacks.

Definition 2.2. If $\alpha : A \to C$ and $\beta : B \to C$ are ring homomorphisms, then the subring $D := \alpha \times_C \beta := \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$ of $A \times B$ is called the *pullback* (or *fiber product*) of α and β.

Let D be a pullback of α and β as in Definition 2.2. We denote the restriction to D of the projection of $A \times B$ onto A (resp., B) by p_A (resp., p_B). Then we have the following diagram:

$$\begin{tikzcd} D := \alpha \times_C \beta \arrow[r, "p_A"] \arrow[d, "p_B"'] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"] & C. \end{tikzcd}$$

Note that if β is surjective, then p_A is surjective. We also note that $\operatorname{Ker}(\beta)$ and A can be regarded as D-modules (with the D-module structures naturally induced by p_B and p_A, respectively) in the following sense:

$$d \cdot b := p_B(d)b \quad \text{and} \quad d \cdot a := p_A(d)a$$

for all $d \in D, a \in A$ and $b \in B$. Clearly, $p_A(S)$ is a multiplicative subset of A whenever S is a multiplicative subset of D.

Proposition 2.3. *Let D be the pullback of α and β as in Definition 2.2, where β is surjective. If S is a multiplicative subset of D, then the following assertions are equivalent.*

(1) *D is an S-Noetherian ring.*
(2) *A is a $p_A(S)$-Noetherian ring and $\operatorname{Ker}(\beta)$ is an S-Noetherian D-module.*

Proof. Note that $\operatorname{Ker}(p_A) = \{0\} \times \operatorname{Ker}(\beta)$. Since β is surjective, p_A is also surjective. Hence we have the following short exact sequence of D-modules:

$$0 \longrightarrow \operatorname{Ker}(\beta) \xrightarrow{\ \iota\ } D = \alpha \times_C \beta \xrightarrow{\ p_A\ } A \longrightarrow 0,$$

where ι is the natural D-module embedding defined by $x \longmapsto (0, x)$ for each $x \in \operatorname{Ker}(\beta)$. By Lemma 2.1, D is an S-Noetherian ring if and only if A is an S-Noetherian D-module and $\operatorname{Ker}(\beta)$ is an S-Noetherian D-module. Since p_A is surjective, the D-submodules of A are exactly the ideals of the ring A. Thus A is an S-Noetherian D-module if and only if A is a $p_A(S)$-Noetherian ring. $\square$

Note that if S is a subset of units of D, then $p_A(S)$ is a subset of units of A. Applying Proposition 2.3 to the case when S is a subset of units of D, we can get the following known result.

Corollary 2.4. *(See [4, Proposition 4.10].) Let D be the pullback of α and β as in Definition 2.2, where β is surjective. Then the following conditions are equivalent.*

(1) *D is a Noetherian ring.*
(2) *A is a Noetherian ring and $\operatorname{Ker}(\beta)$ is a Noetherian D-module.*

3. Main results

In this section, we apply the S-Noetherian properties of a pullback to those of the amalgamated algebra along an ideal with respect to a ring homomorphism. We borrow the definition of the amalgamated algebra introduced in [4].

Definition 3.1. Let $f : A \to B$ be a ring homomorphism and J an ideal of B. The subring $A \bowtie^f J$ of $A \times B$ is defined as follows:

$$A \bowtie^f J = \big\{\big(a, f(a) + j\big) \,\big|\, a \in A \text{ and } j \in J\big\}.$$

We call the ring $A \bowtie^f J$ the *amalgamation of A with B along J with respect to f*.

In fact, $A \bowtie^f J$ is the pullback $\widehat{f} \times_{B/J} \pi$ of $\widehat{f}$ and π, where $\pi : B \to B/J$ is the canonical projection and $\widehat{f} = \pi \circ f$:

$$\begin{tikzcd} A \bowtie^f J = \widehat{f} \times_{B/J} \pi \arrow[r, "p_A"] \arrow[d, "p_B"'] & A \arrow[d, "\widehat{f}"] \\ B \arrow[r, "\pi"] & B/J. \end{tikzcd}$$

Let $\iota : A \to A \bowtie^f J$ be the natural embedding defined by $a \longmapsto (a, f(a))$ for all $a \in A$. For a multiplicative subset S of A, put $S' := \{(s, f(s)) \mid s \in S\}$. Clearly, S' and $f(S)$ are multiplicative subsets of $A \bowtie^f J$ and B, respectively.

Theorem 3.2. *Let $f : A \to B$ be a ring homomorphism, J an ideal of B and S a multiplicative subset of A. Then the following statements are equivalent.*

(1) *$A \bowtie^f J$ is an S'-Noetherian ring.*
(2) *A is an S-Noetherian ring and J is an S'-Noetherian $A \bowtie^f J$-module (with the $A \bowtie^f J$-module structure naturally induced by p_B, where $p_B : A \bowtie^f J \to B$ defined by $(a, f(a) + j) \longmapsto f(a) + j$).*
(3) *A is an S-Noetherian ring and $f(A) + J$ is an $f(S)$-Noetherian ring.*

Proof. (1) $\Leftrightarrow$ (2) Proposition 2.3.

(1) $\Rightarrow$ (3) By Proposition 2.3, A is S-Noetherian. Note that $\operatorname{Ker}(p_B) = f^{-1}(J) \times \{0\}$. Hence we have

$$\frac{A \bowtie^f J}{f^{-1}(J) \times \{0\}} \simeq f(A) + J. \tag{1}$$

Since $p_B(S') = f(S)$, we conclude that $f(A) + J$ is an $f(S)$-Noetherian ring [17, Lemma 2.2].

(3) $\Rightarrow$ (2) Note that every $A \bowtie^f J$-submodule of J is an ideal of $f(A) + J$. Let J_0 be an $A \bowtie^f J$-submodule of J. Then J_0 is an ideal of $f(A) + J$. Since $f(A) + J$ is $f(S)$-Noetherian, there exist an element $s \in S$ and $j_1, \ldots, j_k \in J_0$ such that $f(s)J_0 \subseteq (j_1, \ldots, j_k)(f(A) + J) \subseteq J_0$. Hence we obtain

$$\big(s, f(s)\big) \cdot J_0 \subseteq \big(A \bowtie^f J\big) \cdot j_1 + \cdots + \big(A \bowtie^f J\big) \cdot j_k \subseteq J_0,$$

which says that J_0 is an S'-finite $A \bowtie^f J$-module. Thus J is an S'-Noetherian $A \bowtie^f J$-module. $\square$

When S is a subset of units of A, we regain

Corollary 3.3. *(See [4, Proposition 5.6].) Let $f : A \to B$ be a ring homomorphism and J an ideal of B. Then $A \bowtie^f J$ is Noetherian if and only if A and $f(A) + J$ are Noetherian.*

By a similar observation as in the paragraph before [4, Proposition 5.7], Theorem 3.2 seems to have a moderate interest. In order to obtain more useful criteria for the S-Noetherian property of $A \bowtie^f J$, we specialize Theorem 3.2 in some relevant cases. To do this, we first recall the shape of prime ideals of $A \bowtie^f J$. Let $f : A \to B$ be a ring homomorphism and J an ideal of B. Note that $f(A) + J$ is a subring of B. For prime ideals P and Q of A and B, respectively, we set

$$P \bowtie^f J := \big\{\big(p, f(p) + j\big) \bigm| p \in P \text{ and } j \in J\big\}; \quad \text{and}$$
$$\overline{Q}^f := \big\{\big(a, f(a) + j\big) \bigm| a \in A,\ j \in J \text{ and } f(a) + j \in Q\big\}.$$

Then the prime ideals of $A \bowtie^f J$ are exactly of the type $P \bowtie^f J$ or $\overline{Q}^f$ for some prime ideals P of A and some prime ideals Q of B which do not contain J. (See [5, Proposition 2.6(3)] or [8, Theorem 1.4].)

Theorem 3.4. *Let $f : A \to B$ be a ring homomorphism, J an ideal of B, S a multiplicative subset of A and $S' := \{(s, f(s)) \mid s \in S\}$. If A is an S-Noetherian ring and B is an S-finite A-module (with the A-module structure induced by f), then $A \bowtie^f J$ is an S'-Noetherian ring, and hence $f(A) + J$ is an $f(S)$-Noetherian ring.*

Proof. Let $\mathcal{P}$ be a prime ideal of $A \bowtie^f J$ such that $\mathcal{P} \cap S' = \emptyset$.

Case I. $\mathcal{P} := P \bowtie^f J$ for a prime ideal P of A which is disjoint from S.
Since A is an S-Noetherian ring, there exist an element $s_1 \in S$ and $a_1, \ldots, a_n \in P$ such that $s_1 P \subseteq (a_1, \ldots, a_n) \subseteq P$. Since B is an S-finite A-module, there exists an element $s_2 \in S$ such that $s_2 \cdot J \subseteq A \cdot j_1 + \cdots + A \cdot j_m \subseteq J$ (or $f(s_2)J \subseteq f(A)j_1 + \cdots + f(A)j_m \subseteq J$) for some $j_1, \ldots, j_m \in J$. Put $s = s_1 s_2$. Then for each $p \in P$, there exist $r_1, \ldots, r_n \in A$ such that $sp = \sum_{i=1}^n r_i a_i$, and for each $j \in J$, we can find $r'_1, \ldots, r'_m \in A$ such that $s \cdot j = \sum_{k=1}^m r'_k \cdot j_k$ (or $f(s)j = \sum_{k=1}^m f(r'_k) j_k$). Note that $(r, f(r)) \in A \bowtie^f J$ for all $r \in A$ and $(a, f(a)), (0, j) \in P \bowtie^f J$ for all $a \in P$ and $j \in J$. Hence for each $(p, f(p) + j) \in P \bowtie^f J$, we have

$$\begin{aligned}
\big(s, f(s)\big)\big(p, f(p) + j\big) &= \big(sp, f(s)f(p) + f(s)j\big) \\
&= \left(\sum_{i=1}^n r_i a_i, \sum_{i=1}^n f(r_i) f(a_i) + \sum_{k=1}^m f\big(r'_k\big) j_k\right) \\
&= \sum_{i=1}^n \big(r_i, f(r_i)\big)\big(a_i, f(a_i)\big) + \sum_{k=1}^m \big(r'_k, f\big(r'_k\big)\big)(0, j_k) \\
&\in \big(\big\{\big(a_i, f(a_i)\big), (0, j_k) \bigm| 1 \leqslant i \leqslant n \text{ and } 1 \leqslant k \leqslant m\big\}\big).
\end{aligned}$$

Therefore we obtain

$$\big(s, f(s)\big)\big(P \bowtie^f J\big) \subseteq \big(\big\{\big(a_i, f(a_i)\big), (0, j_k) \bigm| 1 \leqslant i \leqslant n \text{ and } 1 \leqslant k \leqslant m\big\}\big) \subseteq P \bowtie^f J,$$

which implies that $P \bowtie^f J$ is S'-finite.

Case II. $\mathcal{P} := \overline{Q}^f$ for a prime ideal Q of B which does not contain J.
Note that $p_B(\overline{Q}^f) = \{f(a) + j \mid (a, f(a) + j) \in \overline{Q}^f\}$ is an A-submodule of B, where p_B is the projection as in Theorem 3.2(2). Since B is an S-finite A-module and A is an S-Noetherian ring, there exists an element $s_1 \in S$ such that

$$s_1 \cdot p_B\big(\overline{Q}^f\big) = f(s_1) p_B\big(\overline{Q}^f\big) \subseteq f(A)\big(f(a_1) + j_1\big) + \cdots + f(A)\big(f(a_n) + j_n\big) \subseteq p_B\big(\overline{Q}^f\big) \tag{2}$$

for some $f(a_1) + j_1, \ldots, f(a_n) + j_n \in p_B(\overline{Q}^f)$ by Lemma 2.1. Note that $f^{-1}(J)$ is an ideal of A, because J is an ideal of B. Since A is S-Noetherian, there exist an element $s_2 \in S$ and $z_1, \ldots, z_m \in A$ such that

$$s_2 f^{-1}(J) \subseteq (z_1, \ldots, z_m) \subseteq f^{-1}(J). \tag{3}$$

Note that $(z_i, 0) = (z_i, f(z_i) - f(z_i))$; so $(z_i, 0) \in \overline{Q}^f$ for each $i = 1, \ldots, m$. Let $(a, f(a) + j) \in \overline{Q}^f$. Then from Formula (2), we have

$$\big(s_1, f(s_1)\big)\big(a, f(a)+j\big) = \big(s_1 a, f(s_1)\big(f(a)+j\big)\big) = \left(s_1 a, \sum_{k=1}^{n} f(r_k)\big(f(a_k)+j_k\big)\right)$$

for some $r_1, \ldots, r_n \in A$. Note that $f(s_1)f(a) - \sum_{k=1}^{n} f(r_k)f(a_k) \in J$; so $s_1 a - \sum_{k=1}^{n} r_k a_k \in f^{-1}(J)$. Hence from Formula (3), we get

$$s_2\left(s_1 a - \sum_{k=1}^{n} r_k a_k\right) = r'_1 z_1 + \cdots + r'_m z_m$$

for some $r'_1, \ldots, r'_m \in A$. Note that $(r, f(r)) \in A \bowtie^f J$ for all $r \in A$ and $(z_i, 0), (a_k, f(a_k)+j_k) \in \overline{Q}^f$ for all $i = 1, \ldots, m$ and $k = 1, \ldots, n$. By putting $s = s_1 s_2$, we have

$$\begin{aligned}
\big(s, f(s)\big)\big(a, f(a)+j\big) &= \big(s_2 s_1 a, f(s_2 s_1)\big(f(a)+j\big)\big) \\
&= \left(s_2 \sum_{k=1}^{n} r_k a_k + \sum_{i=1}^{m} r'_i z_i, f(s_2)\sum_{k=1}^{n} f(r_k)\big(f(a_k)+j_k\big)\right) \\
&= \sum_{i=1}^{m} \big(r'_i, f(r'_i)\big)(z_i, 0) + \sum_{k=1}^{n} \big(s_2 r_k, f(s_2) f(r_k)\big)\big(a_k, f(a_k)+j_k\big) \\
&\in \big(\big\{(z_i, 0), \big(a_k, f(a_k)+j_k\big) \,\big|\, 1 \leqslant i \leqslant m \text{ and } 1 \leqslant k \leqslant n\big\}\big).
\end{aligned}$$

Therefore we obtain

$$\big(s, f(s)\big)\overline{Q}^f \subseteq \big(\big\{(z_i, 0), \big(a_k, f(a_k)+j_k\big) \,\big|\, 1 \leqslant i \leqslant m \text{ and } 1 \leqslant k \leqslant n\big\}\big) \subseteq \overline{Q}^f;$$

so $\overline{Q}^f$ is S'-finite.

From Cases I and II, $\mathcal{P}$ is S'-finite, which indicates that $A \bowtie^f J$ is an S'-Noetherian ring [1, Corollary 5]. □

Let $D \subseteq E$ be an extension of commutative rings, $\{X_1, \ldots, X_n\}$ a set of indeterminates over E, $D + (X_1, \ldots, X_n)E[X_1, \ldots, X_n] := \{f \in E[X_1, \ldots, X_n] \mid$ the constant term of f belongs to $D\}$ and $D + (X_1, \ldots, X_n)E[\![X_1, \ldots, X_n]\!] := \{f \in E[\![X_1, \ldots, X_n]\!] \mid$ the constant term of f is contained in $D\}$. Then the composite ring extensions $D + (X_1, \ldots, X_n)E[X_1, \ldots, X_n]$ and $D + (X_1, \ldots, X_n)E[\![X_1, \ldots, X_n]\!]$ are special examples of the amalgamated algebra. (For the details, see [4, Example 2.5].) As simple consequences of Theorem 3.4, we give sufficient conditions for the rings $D + (X_1, \ldots, X_n)E[X_1, \ldots, X_n]$ and $D + (X_1, \ldots, X_n)E[\![X_1, \ldots, X_n]\!]$ to be S-Noetherian when S is an anti-Archimedean subset of D. (Recall that a multiplicative subset S of a commutative ring R is *anti-Archimedean* if $\bigcap_{n \in \mathbb{N}} s^n R \cap S \neq \emptyset$ for every $s \in S$.)

Corollary 3.5. *(Cf. [15, Theorem 3.6].) Let $D \subseteq E$ be an extension of commutative rings, $\{X_1, \ldots, X_n\}$ a set of indeterminates over E, J an ideal of $E[X_1, \ldots, X_n]$ and S an anti-Archimedean subset of D. If D is an S-Noetherian ring and E is an S-finite D-module, then $D[X_1, \ldots, X_n] + J$ is an S-Noetherian ring. In particular, $D + (X_1, \ldots, X_n)E[X_1, \ldots, X_n]$ is an S-Noetherian ring.*

Proof. Note that if S is an anti-Archimedean subset of D and D is an S-Noetherian ring, then $D[X_1, \ldots, X_n]$ is also an S-Noetherian ring [1, Proposition 9]; and that if E is an S-finite D-module, then $E[X_1, \ldots, X_n]$ is an S-finite $D[X_1, \ldots, X_n]$-module. Thus the result follows directly from Theorem 3.4. The second assertion is the case when $J = (X_1, \ldots, X_n)E[X_1, \ldots, X_n]$. □

Corollary 3.6. *(Cf. [15, Theorem 4.4].) Let $D \subseteq E$ be an extension of commutative rings, $\{X_1, \ldots, X_n\}$ a set of indeterminates over E, J an ideal of $E[\![X_1, \ldots, X_n]\!]$ and S an anti-Archimedean subset of D consisting of nonzerodivisors. If D is an S-Noetherian ring and E is an S-finite D-module, then $D[\![X_1, \ldots, X_n]\!] + J$ is an S-Noetherian ring. In particular, $D + (X_1, \ldots, X_n)E[\![X_1, \ldots, X_n]\!]$ is an S-Noetherian ring.*

Proof. Note that if S is an anti-Archimedean subset of D consisting of nonzerodivisors and D is S-Noetherian, then $D[\![X_1, \ldots, X_n]\!]$ is S-Noetherian [1, Proposition 10]; and that if E is an S-finite D-module, then $E[\![X_1, \ldots, X_n]\!]$ is an S-finite $D[\![X_1, \ldots, X_n]\!]$-module. Thus the result is an immediate consequence of Theorem 3.4. The second assertion is the case when $J = (X_1, \ldots, X_n)E[\![X_1, \ldots, X_n]\!]$. □

By Corollaries 3.5 and 3.6, we have

Corollary 3.7. *(See [10, Proposition 2.1] (or [13, Corollary 2.2] and [14, Corollary 1]).) Let $D \subseteq E$ be an extension of commutative rings and $\{X_1, \ldots, X_n\}$ a set of indeterminates over E. If D is a Noetherian ring and E is a finitely generated D-module, then $D + (X_1, \ldots, X_n)E[X_1, \ldots, X_n]$ and $D + (X_1, \ldots, X_n)E[\![X_1, \ldots, X_n]\!]$ are Noetherian rings.*

Finally, we study how the S-Noetherian property of Nagata's idealization (as a particular case of the amalgamation) $R(+)M$ related to those of R and M. To do this, we recall some facts on the idealization. Let R be a commutative ring with identity and M a unitary R-module. The *idealization* of M in R (or *trivial extension* of R by M) is a commutative ring

$$R(+)M := \{(r, m) \mid r \in R \text{ and } m \in M\}$$

under the usual addition and the multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$ for all $(r_1, m_1), (r_2, m_2) \in R(+)M$. (For more on the relation between the amalgamation and the idealization, see [4, Remark 2.8].) It was shown that if S is a multiplicative subset of R, then $S(+)M$ is a multiplicative subset of $R(+)M$ [12, Lemma 25.4] (or [3, Theorem 3.8]); and the prime ideals of $R(+)M$ have the form $P(+)M$, where P is a prime ideal of R [12, Theorem 25.1] (or [3, Theorem 3.2]).

Theorem 3.8. *Let R be a commutative ring with identity, M a unitary R-module and S a multiplicative subset of R. Then $R(+)M$ is an $S(+)M$-Noetherian ring if and only if R is an S-Noetherian ring and M is S-finite.*

Proof. ($\Rightarrow$) Assume that $R(+)M$ is an $S(+)M$-Noetherian ring. Note that a map $\varphi : R(+)M \to R$ defined by $\varphi(r, m) = r$ for all $(r, m) \in R(+)M$ is an epimorphism and $\varphi(S(+)M) = S$; so R is S-Noetherian [17, Lemma 2.2]. Since $(0)(+)M$ is an ideal of $R(+)M$, there exist an $(s, m) \in S(+)M$ and $m_1, \ldots, m_n \in M$ such that $(s, m)((0)(+)M) \subseteq ((0, m_1), \ldots, (0, m_n))(R(+)M)$. Hence $sM \subseteq m_1 R + \cdots + m_n R$, and thus M is S-finite.

($\Leftarrow$) Let $P(+)M$ be a prime ideal of $R(+)M$ that is disjoint from $S(+)M$. Then P is a prime ideal of R which is disjoint from S. Since R is S-Noetherian, there exist an $s \in S$ and a finitely generated ideal J of R such that $sP \subseteq J \subseteq P$. Also, since M is S-finite, we can find a suitable $t \in S$ and a finitely generated R-submodule F of M such that $tM \subseteq F$. Therefore $(st, 0)(P(+)M) \subseteq tJ(+)F \subseteq P(+)M$. Note that $tJ(+)F$ is an ideal of $R(+)M$, because $tJM \subseteq F$. Since J and F are finitely generated, $tJ(+)F$ is also finitely generated. Hence $P(+)M$ is S-finite, and thus $R(+)M$ is an S-Noetherian ring [1, Corollary 5]. $\square$

It was shown that (r, m) is a unit in $R(+)M$ if and only if r is a unit in R [12, Theorem 25.1(6)] (or [3, Theorem 3.7]). Applying Theorem 3.8 to the case when $S = \{1\}$, we can recover the following well-known result.

Corollary 3.9. *(See [3, Theorem 4.8].) Let R be a commutative ring with identity and M a unitary R-module. Then $R(+)M$ is Noetherian if and only if R is Noetherian and M is finitely generated.*

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