Chapter 14 The Fundamental Group

1 The Nature of Algebraic Topology

- In the first eight chapters we have dealt almost exclusively with point-set topology. This chapter introduces the fundamental group which, as the term "group" suggests, is an algebraic concepts.
- The purpose of algebraic topology is to describe the structure of topological spaces by algebraic means, usually groups, rings, or modules.
- Algebraic topology was introduced in a series of papers during the years 1895-1901 by H. Poincaré(1854 $\sim$ 1912).

- Algebraic topology did not develop as an outgrowth of point set topology.

  Although algebraic topology and point-set topology share the common goal of classifying spaces by topological properties,

  the subjects are quite distinct in their historical development, emphasis, and methods.

  (i) Poincaré's first paper preceded Fréchet's work on general metric space by 11 years, and Hausdorff's definition of general topological spaces by 19 years.

  (ii) Poincaré's work on the fundamental group and other aspects of algebraic topology was not influenced by Cantor's theory of sets.

- Algebraic topology developed in response to specific geometric problems in Euclidean spaces, to describe the connectivity or the "hole in the space" by algebraic methods.

- This chapter is restricted to the fundamental group, the first algebraic structure associated by Poincaré with topological spaces, and to its applications.

2 The Fundamental Group

The following two examples in Example 2, which deal with integration on multiply connected domains and with the classification of surfaces, are intended to illustrate the kind of analysis that led to the development of the fundamental group.

For this, we recall the following theorem of calculus.

Theorem 1 (Green's Theorem)**.** *Let C be a simple closed curve in the plane and let D be the simple connected region enclosed by C. If $f(x,y)$ and $g(x,y)$ have their continuous partial derivatives in a closed set containing D, then*

$$\int_C f(x,y)dx + g(x,y)dy = \iint\limits_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)dxdy$$

where C is oriented counterclockwise.

Example 2. (a) Let $F(x,y) = (p(x,y), q(x,y))$ be a continuous vector field defined on an open set containing the annulus of Figure 1 and satisfying the exactness condition: $\dfrac{\partial p}{\partial y} = \dfrac{\partial q}{\partial x}$.

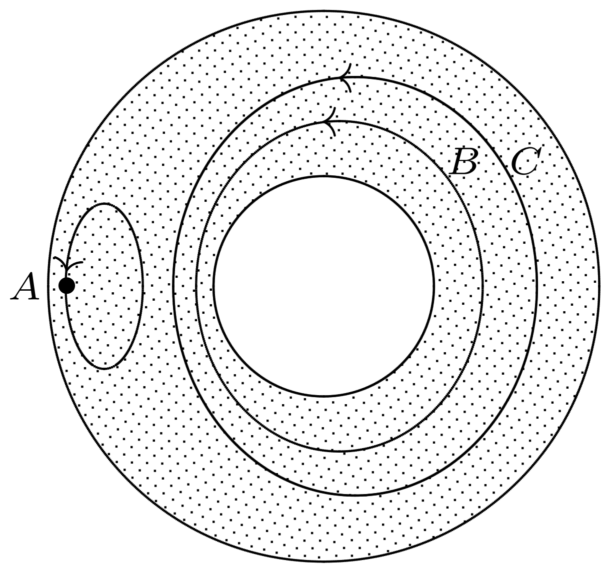


Figure 1: annulus in the plane

i.e., $\exists f(x,y) \cdot\!\ni\!\cdot\ df \equiv f_x dx + f_y dy = pdx + qdy$.

$\Rightarrow \displaystyle\int_A pdx + qdy = 0$ and $\displaystyle\int_B pdx + qdy = \int_C pdx + qdy$

by the Green's Theorem(Theorem 1).

$\Rightarrow$ From the point of view of integrating exact vector fields,

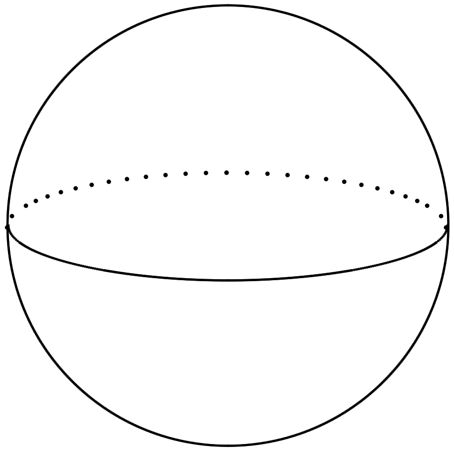
(i) Path A is trivial in the sense $\displaystyle\int_A pdx + qdy = 0$.

(ii) Paths B and C are equivalent.

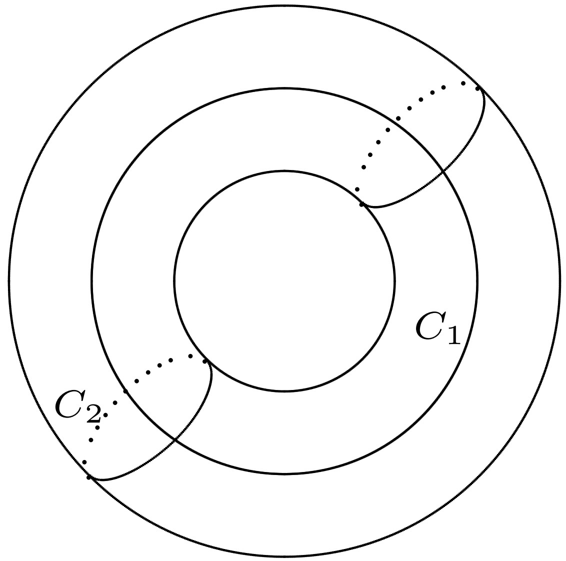
$\Big($ ($\because$) (i) A can be shrunk to a point in the annulus.

(ii) B and C are homotopic paths in the annulus.

(b) The difference between $\mathbf{S}^2$ and $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$.



2-Sphere $\mathbf{S}^2$



Torus $\mathbf{T}^2$

Figure 2: $\mathbf{S}^2$ and $\mathbf{T}^2$

(i) $\mathbf{T}^2$ encloses an inner region and has a "doughnut hole" and $\mathbf{S}^2$ only encloses an inner region.
But the inner region enclosed by $\mathbf{S}^2$ is different from the one enclosed by $\mathbf{T}^2$.
(The difference is difficult to describe at the moment.)

(ii) Every closed path in $\mathbf{S}^2$ is homotopic to a constant path while $\mathbf{T}^2$ has two basic types of paths,
the meridian circle C_1 and the longitudinal circle C_2,
which are not homotopic to constant paths.
(We will see later.)

Paths and path connected spaces are extended to the ideas to describe the concepts of simple and multiple connectedness for general topological spaces in this section. Throughout this chapter, the closed unit interval $[0, 1]$ is denoted by $\mathbf{I}$ and $\partial\mathbf{I} = \{0, 1\}$.

Definition 3. Let X be a topological space.

(1) Let $\alpha, \beta : \mathbf{I} \to X$ be two paths in X. Then α and β are said to be *homotopic* if $\exists$ a continuous function $F : \mathbf{I} \times \mathbf{I} \to X$ such that

$$F(s,0) = \alpha(s) \text{ and } F(s,1) = \beta(s) \text{ for all } s \in \mathbf{I}.$$

The continuous function F is called a *homotopy* or a *free homotopy* between the paths α and β.

(2) Let $\alpha, \beta : \mathbf{I} \to X$ be two paths with $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. Then α and β are said to be *equivalent* or *homotopic modulo end points* or *homotopic relative to* $\{0,1\}$ if $\exists$ a continuous function $F : \mathbf{I} \times \mathbf{I} \to X$ such that

(i) $F(s,0) = \alpha(s)$ and $F(s,1) = \beta(s)$ for all $s \in \mathbf{I}$,

(ii) $F(0,t) = \alpha(0) = \beta(0)$ and $F(1,t) = \alpha(1) = \beta(1) \forall t \in \mathbf{I}$.

The continuous function F is called a *homotopy* (relative to $\{0,1\}$) between the paths α and β. For $t \in \mathbf{I}$, the restriction of F to $\mathbf{I} \times \{t\}$, denoted $F(*,t)$, is called the *t-level* of the homotopy.

(3) A path $\alpha : \mathbf{I} \to X$ with $\alpha(0) = \alpha(1) = x_0 \in X$ is called a *loop at* x_0 *in* X. We write $\Omega_1(X, x_0) = \{\alpha : \mathbf{I} \to X \mid \alpha \text{ is a loop at } x_0\}$.

- Two loops $\alpha, \beta \in \Omega_1(X, x_0)$ are said to be *equivalent* or *homotopic modulo* x_0, denoted $\alpha \simeq_{x_0} \beta$, if there exists a homotopy $F : \mathbf{I} \times \mathbf{I} \to X$ such that

  (i) $F(s, 0) = \alpha(s)$ and $F(s, 1) = \beta(s)$ for all $s \in \mathbf{I}$,

  (ii) $F(0, t) = F(1, t) = x_0$ for all $t \in \mathbf{I}$.

- Such a homotopy F is called a *base point preserving homotopy* or F is said to *stay fixed through the homotopy.*

  The usual practice in studying the loops in a space X is to specify a point x_0 in X to serve as the base point for the loops under consideration.

- This point x_0 is called the *base point* of X.

(4) The loop $c_{x_0} : \mathbf{I} \to X$, defined by $c_{x_0}(s) = x_0$ for all $s \in \mathbf{I}$, is called the *constant loop* at $x_0 \in X$.

- A loop that is equivalent to the constant loop is said to be *null-homotopic*.

Example 4. Consider the annulus X shown in Figure 3 with base point x_0 and closed curves A, B, C which are the images of paths α, β and γ, respectively.

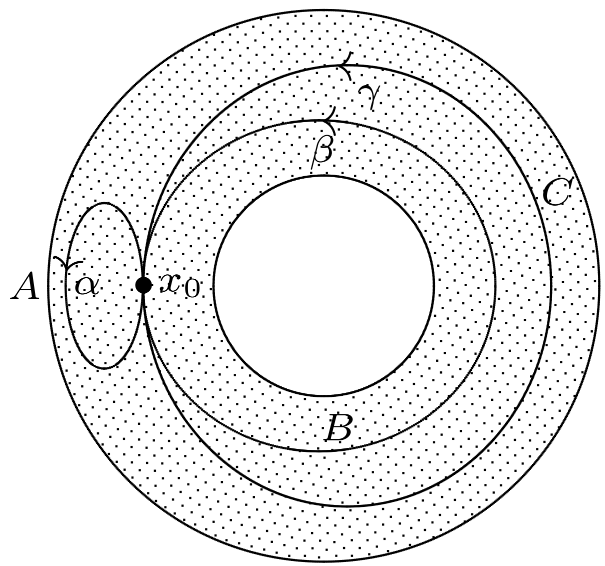


Figure 3: annulus in the plane

Here α, β, and γ are vector-valued functions, and we assume that the parametrizations for β and γ are chosen in such a way that the line segment $t\gamma(s) + (1-t)\beta(s) (0 \leq t \leq 1)$ from $\beta(s)$ to $\gamma(s)$, for each $s \in \mathbf{I}$, lies in the annulus. Then

(1) The loop α is null-homotopic by the homotopy $F : \mathbf{I} \times \mathbf{I} \to X$, defined by $F(s,t) = (1-t)\alpha(s) + tx_0$ for all $(s,t) \in \mathbf{I} \times \mathbf{I}$.

$$\left(\begin{array}{ll} (\because) & \text{Clearly } F : \mathbf{I} \times \mathbf{I} \to X \text{ is well-defined and continuous.} \\ & F(s,0) = \alpha(s) \text{ and } F(s,1) = x_0 = c_{x_0}(s) \text{ for all } s \in \mathbf{I}. \\ & F(0,t) = (1-t)\alpha(0) + tx_0 = (1-t)x_0 + tx_0 = x_0, \\ & F(1,t) = (1-t)\alpha(1) + tx_0 = x_0 \text{ for all } t \in \mathbf{I}. \\ \Rightarrow & F : \alpha \simeq_{x_0} c_{x_0}. \text{ i.e., } \alpha \text{ is null-homotopic by } F. \end{array}\right.$$

(2) The loops β and γ are equivalent by the homotopy $G : \mathbf{I} \times \mathbf{I} \to X$, defined by $G(s,t) = t\gamma(s) + (1-t)\beta(s)$ for all $(s,t) \in \mathbf{I} \times \mathbf{I}$.

$(\because)$ Omit.

Lemma 5 (Pasting Lemma for closed sets)**.** *Let $X = A \cup B$, where A and B are closed in X. If $f : A \to Y$, $g : B \to Y$ are continuous functions $\cdot\!\ni\!\cdot$ $f(x) = g(x)\ \forall x \in A \cap B$, then $h : X \to Y$ defined by*

$$h(x) = \begin{cases} f(x) & \textit{if } x \in A, \\ g(x) & \textit{if } x \in B, \end{cases}$$

is a well-defined continuous function.

Proof. Since $f(x) = g(x)$ for all $x \in A \cap B$, $h : X \to Y$ is well-defined.

To show $h : X \to Y$ is continuous, let C be a closed subset of Y.

$\Rightarrow$ Since $f : A \to Y$ and $g : B \to Y$ are continuous,

$f^{-1}(C)$ and $g^{-1}(C)$ are closed in A and B respectively.

$\Rightarrow$ Since A, B are closed in X, $f^{-1}(C)$ and $g^{-1}(C)$ are closed in X.

$\Rightarrow h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ is closed in X.

$\Rightarrow h : X \to Y$ is continuous. $\square$

Lemma 6 (Pasting Lemma for open sets)**.** *Let* $X = \bigcup_{\alpha \in \mathscr{A}} U_\alpha$*, where* U_α *is open in* X *for each* $\alpha \in \mathscr{A}$*. If* $f_\alpha : U_\alpha \to Y$ *are continuous functions such that* $f_\alpha(x) = f_\beta(x)$ *for all* $x \in U_\alpha \cap U_\beta$*, then* $h : X \to Y$ *defined by* $h(x) = f_\alpha(x)$ *for* $x \in U_\alpha$ *is a well-defined continuous function.*

Proof. It is similar to that of Lemma 5. $\square$

Corollary 7. *Let* X *be a topological space and let* A *and* B *be closed subsets of* X *whose union is* X*. Let* $h : X \to Y$ *be a function. If the restrictions* $h|_A : A \to Y$ *and* $h|_B : B \to Y$ *are continuous, then* $h : X \to Y$ *is continuous.*

Proof. It follows from Lemma 5. $\square$

Corollary 8. *Let X be a topological space and let $U_\alpha, \alpha \in \mathscr{A}$ be open subsets of X whose union is X. Let $h : X \to Y$ be a function. If all the restrictions $h|_{U_\alpha} : U_\alpha \to Y$ are continuous, then $h : X \to Y$ is continuous.*

Proof. It follows from Lemma 6. $\square$

Theorem 9. *Let X be a topological space.*

(1) *The relation of equivalence for paths is an equivalence relation on the set of all paths in X.*

(2) *The relation of equivalence $\simeq_{x_0}$ for loops at $x_0 \in X$ is an equivalence relation on $\Omega_1(X, x_0)$.*

Proof. The proof of (1) is omitted.

(2) Let X be a topological space and fix a base point $x_0 \in X$.

(i) $\simeq_{x_0}$ is reflexive on $\Omega_1(X, x_0)$.

$(\because)$ Let $\alpha \in \Omega_1(X, x_0)$.

$\Rightarrow$ Define $H : \mathbf{I} \times \mathbf{I} \to X$ by $H(s,t) = \alpha(s)$ for all $(s,t) \in \mathbf{I} \times \mathbf{I}$.

$\Rightarrow$ $H(s,0) = \alpha(s)$ and $H(s,1) = \alpha(s)$ for all $s \in \mathbf{I}$.

$H(0,t) = \alpha(0) = x_0$ and $H(1,t) = \alpha(1) = x_0$ for all $t \in \mathbf{I}$.

$\Rightarrow$ $\alpha \simeq_{x_0} \alpha$ by the homotopy H.

(ii) $\simeq_{x_0}$ is symmetric on $\Omega_1(X, x_0)$.

($\because$) Let $\alpha, \beta \in \Omega_1(X, x_0)$ and assume that $\alpha \simeq_{x_0} \beta$.

$\Rightarrow$ There exists a homotopy $H : \mathbf{I} \times \mathbf{I} \to X$ such that

$H(s, 0) = \alpha(s), H(s, 1) = \beta(s)$ for all $s \in \mathbf{I}$ and

$H(0, t) = x_0 = H(1, t)$ for all $t \in \mathbf{I}$.

$\Rightarrow$ Define $G : \mathbf{I} \times \mathbf{I} \to X$ by $G(s, t) = H(s, 1 - t)$

for all $(s, t) \in \mathbf{I} \times \mathbf{I}$.

$\Rightarrow$ G is a well-defined continuous function and

$G(s, 0) = H(s, 1) = \beta(s)$,

$G(s, 1) = H(s, 0) = \alpha(s)$ for all $s \in \mathbf{I}$ and

$G(0, t) = H(0, 1 - t) = x_0$,

$G(1, t) = H(1, 1 - t) = x_0$ for all $t \in \mathbf{I}$.

$\Rightarrow$ $\beta \simeq_{x_0} \alpha$ by the homotopy G.

(iii) $\simeq_{x_0}$ is transitive on $\Omega_1(X, x_0)$.

($\because$) Let $\alpha, \beta, \gamma \in \Omega_1(X, x_0)$ and assume that $\alpha \simeq_{x_0} \beta$, $\beta \simeq_{x_0} \gamma$.

$\Rightarrow$ There exist homotopies $H : \mathbf{I} \times \mathbf{I} \to X, G : \mathbf{I} \times \mathbf{I} \to X \cdot\ni\cdot$

$H(s, 0) = \alpha(s), H(s, 1) = \beta(s)$ for all $s \in \mathbf{I}$ and

$G(s, 0) = \beta(s), G(s, 1) = \gamma(s)$ for all $s \in \mathbf{I}$ and

$H(0, t) = H(1, t) = G(0, t) = G(1, t) = x_0$ for all $t \in \mathbf{I}$.

$\Rightarrow$ Define $K : \mathbf{I} \times \mathbf{I} \to X$ by

$$K(s, t) = \begin{cases} H(s, 2t) & \text{if } 0 \leq s \leq 1, 0 \leq t \leq \dfrac{1}{2} \text{ and} \\ G(s, 2t - 1) & \text{if } 0 \leq s \leq 1, \dfrac{1}{2} \leq t \leq 1. \end{cases}$$

$\Rightarrow$ Since $H(s, 2\frac{1}{2}) = \beta(s) = G(s, 0) = G(s, 2\frac{1}{2} - 1)$ for all $s \in \mathbf{I}$,

$K : \mathbf{I} \times \mathbf{I} \to X$ is well-defined and continuous by Lemma 5,

$K(s, 0) = H(s, 0) = \alpha(s)$ and $K(s, 1) = G(s, 1) = \gamma(s) \forall s \in \mathbf{I}$

$$K(0,t) = \begin{cases} H(0,2t) = x_0 & \text{if } 0 \le t \le \dfrac{1}{2}, \\ G(0,2t-1) = x_0 & \text{if } \dfrac{1}{2} \le t \le 1, \end{cases}$$

$$K(1,t) = \begin{cases} H(1,2t) = x_0 & \text{if } 0 \le t \le \dfrac{1}{2}, \\ G(1,2t-1) = x_0 & \text{if } \dfrac{1}{2} \le t \le 1. \end{cases}$$

i.e., $K(0,t) = K(1,t) = x_0$ for all $t \in \mathbf{I}$.

$\Rightarrow \alpha \simeq_{x_0} \gamma$ by the homotopy K.

$\square$

Definition 10. Let X be a topological space with a base point $x_0 \in X$ and let $\alpha, \beta \in \Omega_1(X, x_0)$.

The *product* of loops α and β is the loop $\alpha * \beta : \mathbf{I} \to X$ defined by

$$\alpha * \beta(s) = \begin{cases} \alpha(2s), & \text{if } 0 \leq s \leq \dfrac{1}{2}, \\ \beta(2s-1), & \text{if } \dfrac{1}{2} \leq s \leq 1. \end{cases}$$

Note that $\alpha(1) = \beta(0)$ and that $\alpha * \beta$ is just the path product in Definition 5.41.

Lemma 11. *Let X be a topological space with a base point $x_0 \in X$ and let $\alpha, \beta \in \Omega_1(X, x_0)$. If $\alpha \simeq_{x_0} \alpha'$ and $\beta \simeq_{x_0} \beta'$, then $\alpha * \beta \simeq_{x_0} \alpha' * \beta'$.*

Proof. Let $\alpha, \alpha', \beta, \beta' \in \Omega_1(X, x_0)$ with $\alpha \simeq_{x_0} \alpha'$ and $\beta \simeq_{x_0} \beta'$.

$\Rightarrow$ There exist base point preserving homotopies $F, G : \mathbf{I} \times \mathbf{I} \to X$

from α to α' and from β to β', respectively.

$\Rightarrow F(s,0) = \alpha(s), F(s,1) = \alpha'(s), G(s,0) = \beta(s), G(s,1) = \beta'(s) \forall s,$

$F(0,t) = F(1,t) = G(0,t) = G(1,t) = x_0$ for all $t \in \mathbf{I}$.

$\Rightarrow$ Define $H : \mathbf{I} \times \mathbf{I} \to X$ by

$$H(s,t) = \begin{cases} F(2s,t), & 0 \leq s \leq 1/2, \quad 0 \leq t \leq 1, \\ G(2s-1,t), & 1/2 \leq s \leq 1, \quad 0 \leq t \leq 1. \end{cases}$$

$\Rightarrow H : \mathbf{I} \times \mathbf{I} \to X$ is a well-defined continuous function by Lemma 5,

$$H(s,0) = \begin{cases} F(2s,0) = \alpha(2s), & 0 \leq s \leq 1/2, \\ G(2s-1,0) = \beta(2s-1), & 1/2 \leq s \leq 1 \end{cases}$$

$$= (\alpha * \beta)(s) \text{ for all } s \in \mathbf{I} \text{ and}$$

$$H(s,1) = \begin{cases} F(2s,1) = \alpha'(2s), & 0 \leq t \leq 1/2, \\ G(2s-1,1) = \beta'(2s-1), & 1/2 \leq s \leq 1 \end{cases}$$

$$= (\alpha' * \beta')(s) \text{ for all } s \in \mathbf{I}$$

$$H(0,t) = F(0,t) = x_0, H(1,t) = G(1,t) = x_0 \forall t \in \mathbf{I}.$$

Pictorially, the base point preserving homotopy H from $\alpha * \beta$ to $\alpha' * \beta'$ can be described as in Figure 4.

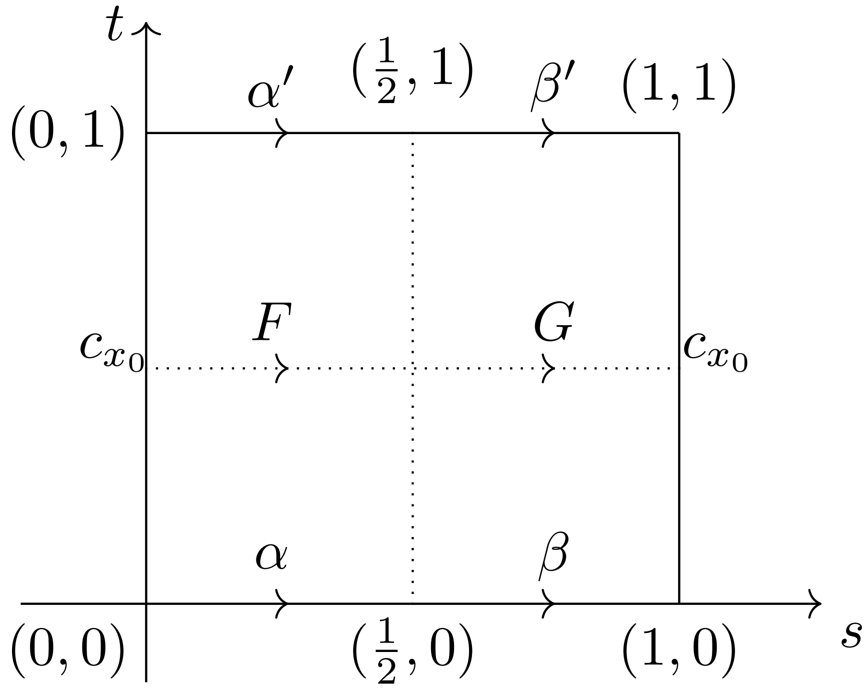


Figure 4: $\alpha * \beta \simeq_{x_0} \alpha' * \beta'$

$\square$

Definition 12. Let X be a topological space with a base point x_0.

(1) For $\alpha \in \Omega_1(X, x_0)$, $[\alpha] = \{\beta \in \Omega_1(X, x_0) \mid \alpha \simeq_{x_0} \beta\}$ is called the *homotopy class* or *equivalence class* of α.

(2) $\pi_1(X, x_0) = \Omega_1(X, x_0)/_{\simeq_{x_0}} = \{[\alpha] \mid \alpha \in \Omega_1(X, x_0)\}$ is called the *fundamental group*, the *Poincaré group* or the *first homotopy group* of X at x_0, where the group operation $\cdot$ of $\pi_1(X, x_0)$, called the *product*, is defined by

$$[\alpha] \cdot [\beta] = [\alpha * \beta] \text{ for all } [\alpha], [\beta] \in \pi_1(X, x_0).$$

Lemma 13. *The product operation $\cdot$ of $\pi_1(X, x_0)$ is well-defined.*

Proof. Let $[\alpha], [\alpha'], [\beta], [\beta'] \in \pi(X, x_0)$ with $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$.

$\Rightarrow \alpha \simeq_{x_0} \alpha'$ and $\beta \simeq_{x_0} \beta'$. Thus by Lemma 11, $\alpha * \beta \simeq_{x_0} \alpha' * \beta'$.

$\Rightarrow [\alpha] \cdot [\beta] = [\alpha * \beta] = [\alpha' * \beta'] = [\alpha'] \cdot [\beta']$.

$\Rightarrow$ The product operation $\cdot$ of $\pi_1(X, x_0)$ is well-defined. $\square$

Lemma 14. *The product operation $\cdot$ of $\pi_1(X, x_0)$ is associative.*

Proof. For three loops α, β, γ in X at x_0, define $F : \mathbf{I} \times \mathbf{I} \to X$ by

$$F(s,t) = \begin{cases} \alpha(\frac{4s}{t+1}), & 0 \leq s \leq \frac{1}{4}(t+1), 0 \leq t \leq 1, \\ \beta(4s - t - 1), & \frac{1}{4}(t+1) \leq s \leq \frac{1}{4}(t+2), 0 \leq t \leq 1, \\ \gamma(\frac{4s-t-2}{2-t}), & \frac{1}{4}(t+2) \leq s \leq 1, 0 \leq t \leq 1. \end{cases}$$

$\Rightarrow$ $F : \mathbf{I} \times \mathbf{I} \to X$ is a well-defined continuous function by Lemma 5,

$F(s, 0) = ((\alpha * \beta) * \gamma)(s)$ and $F(s, 1) = (\alpha * (\beta * \gamma))(s)$ for all $s \in \mathbf{I}$,

$F(0, t) = \alpha(0) = x_0$ and $F(1, t) = \gamma(1) = x_0$ for all $t \in \mathbf{I}$.

$\Rightarrow$ $(\alpha * \beta) * \gamma \simeq_{x_0} \alpha * (\beta * \gamma)$ by the homotopy F.

$\Rightarrow$ $([\alpha] \cdot [\beta]) \cdot [\gamma] = [(\alpha * \beta) * \gamma] = [\alpha * (\beta * \gamma)] = [\alpha] \cdot ([\beta] \cdot [\gamma])$.

Pictorially, the homotopy $F : (\alpha * \beta) * \gamma \simeq_{x_0} \alpha * (\beta * \gamma)$, can be described as in Figure 5.

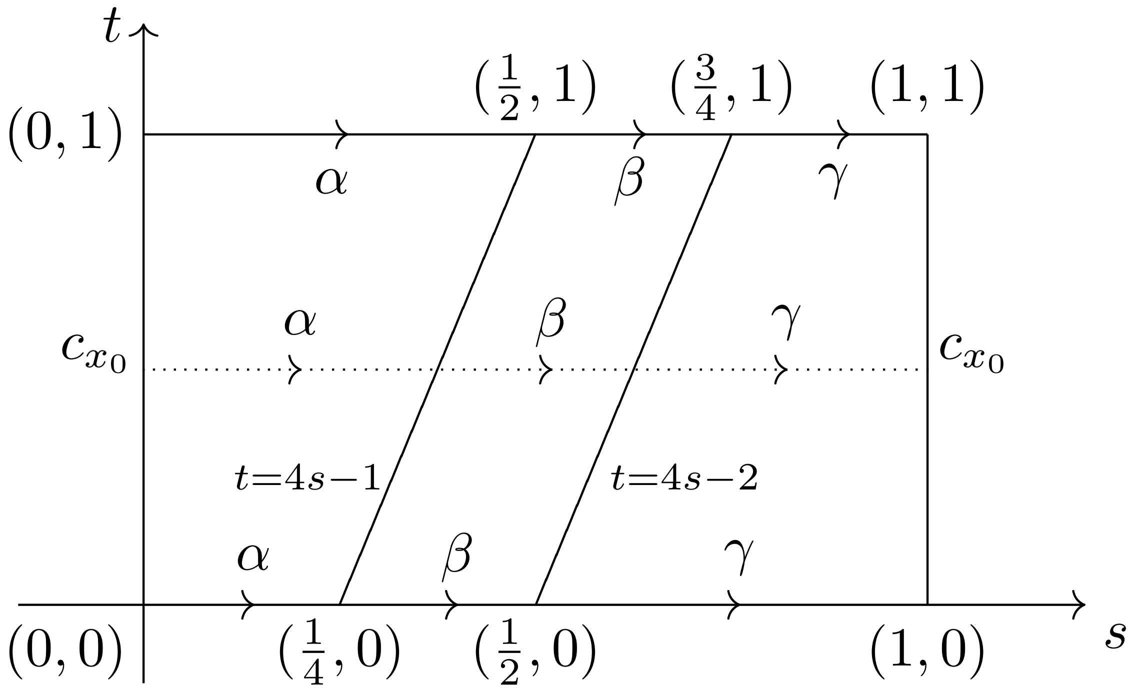


Figure 5: $(\alpha * \beta) * \gamma \simeq_{x_0} \alpha * (\beta * \gamma)$

$\square$

Lemma 15. *The class $[c_{x_0}]$ of the constant loop at x_0 is the identity element of $\pi_1(X, x_0)$ under the product $\cdot$, where c_{x_0} denotes the constant loop at the point x_0.*

i.e., $\forall [\alpha] \in \pi_1(X, x_0)$, $[\alpha] \cdot [c_{x_0}] = [\alpha] = [c_{x_0}] \cdot [\alpha]$. *or,*

$$\alpha * c_{x_0} \simeq_{x_0} \alpha \simeq_{x_0} c_{x_0} * \alpha.$$

Proof. Define $F : \mathbf{I} \times \mathbf{I} \to X$ by

$$F(s,t) = \begin{cases} \alpha(\frac{2s}{t+1}), & 0 \le s \le \frac{t+1}{2}, 0 \le t \le 1, \\ x_0, & \frac{t+1}{2} \le s \le 1, 0 \le t \le 1. \end{cases}$$

$\Rightarrow F : \mathbf{I} \times \mathbf{I} \to X$ is a well-defined continuous function by Lemma 5,

$F(s, 0) = (\alpha * c_{x_0})(s)$ and $F(s, 1) = \alpha(s)$ for all $s \in \mathbf{I}$,

$F(0, t) = \alpha(0) = x_0$ and $F(1, t) = x_0$ for all $t \in \mathbf{I}$.

$\Rightarrow \alpha * c_{x_0} \simeq_{x_0} \alpha$ by the homotopy F.

The homotopy $F : \alpha * c_{x_0} \simeq_{x_0} \alpha$, can be described as in Figure 6.

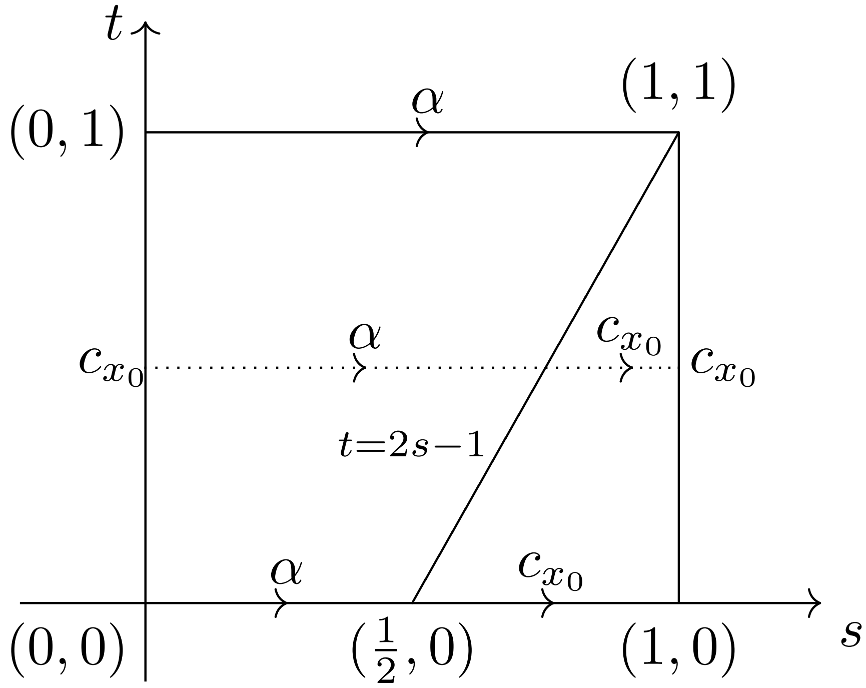


Figure 6: $\alpha * c_{x_0} \simeq_{x_0} \alpha$

$\Rightarrow [\alpha] \cdot [c_{x_0}] = [\alpha * c_{x_0}] = [\alpha]$.

Similarly we can show that $[\alpha] = [c_{x_0}] \cdot [\alpha]$ and it is left for the readers. $\square$

Lemma 16. *For each* $[\alpha] \in \pi_1(X, x_0)$, $[\overline{\alpha}]$ *is the inverse of* $[\alpha]$.
i.e., $[\alpha] \cdot [\overline{\alpha}] = [c_{x_0}] = [\overline{\alpha}] \cdot [\alpha]$ *or,* $\alpha * \overline{\alpha} \simeq_{x_0} c_{x_0} \simeq_{x_0} \overline{\alpha} * \alpha$.

Proof. Let $[\alpha] \in \pi_1(X, x_0)$, represented by a loop $\alpha : \mathbf{I} \to X$ at x_0.

Define $F : \mathbf{I} \times \mathbf{I} \to X$ by

$$F(s,t) = \begin{cases} \alpha(2s), & 0 \leq 2s \leq t, 0 \leq t \leq 1, \\ \alpha(t) = \overline{\alpha}(1-t), & t \leq 2s \leq 2-t, 0 \leq t \leq 1, \\ \overline{\alpha}(2s-1), & 2-t \leq 2s \leq 2, 0 \leq t \leq 1. \end{cases}$$

$\Rightarrow F : \mathbf{I} \times \mathbf{I} \to X$ is a well-defined continuous function by Lemma 5,

$F(s,0) = \alpha(0) = x_0 = c_{x_0}(s)$ and $F(s,1) = (\alpha * \overline{\alpha})(s)$ for all $s \in \mathbf{I}$,

$F(0,t) = \alpha(0) = x_0$ and $F(1,t) = \overline{\alpha}(1) = x_0$ for all $t \in \mathbf{I}$.

$\Rightarrow c_{x_0} \simeq_{x_0} \alpha * \overline{\alpha}$ by the homotopy F.

$\Rightarrow [\alpha] \cdot [\overline{\alpha}] = [c_{x_0}]$.

Pictorially, the homotopy $F : c_{x_0} \simeq_{x_0} \alpha * \overline{\alpha}$, can be described as in Figure 7.

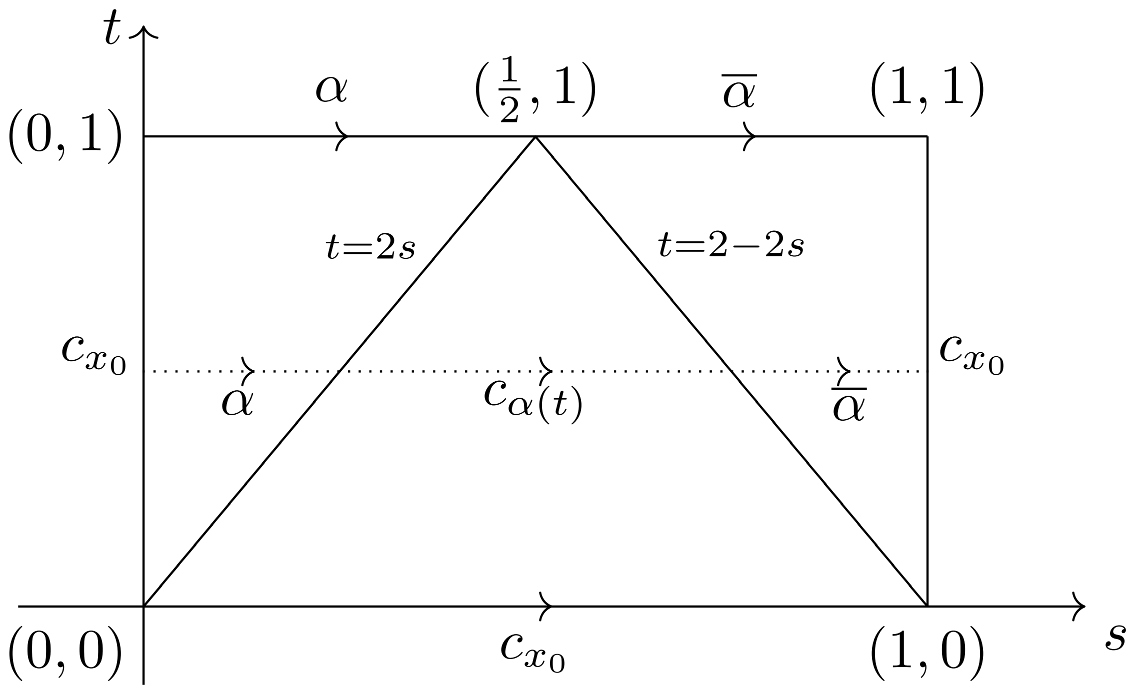


Figure 7: $c_{x_0} \simeq_{x_0} \alpha * \overline{\alpha}$

Similarly we can show that $[c_{x_0}] = [\overline{\alpha}] \cdot [\alpha]$ and it is left for the readers. $\square$

Theorem 17. *Let X be a topological space with a base point x_0. Then $\pi_1(X, x_0)$ together with the multiplication $\cdot$ is a group, called the fundamental group of X based at x_0, in which the identity element is the class of the constant loop at x_0.*

Proof. It follows from Lemmas 13, 14, 15, and 16. $\square$

Theorem 18. *Let X be a path connected space and let $x_0, x_1 \in X$. Then there exists a group isomorphism of $\pi_1(X, x_0)$ onto $\pi_1(X, x_1)$. In this case, we often write simply $\pi_1(X)$ for $\pi_1(X, x_0)$ and call it the fundamental group of X.*

Proof. Let X be a path connected space and let $x_0, x_1 \in X$.

Choose a path α in X from x_0 to x_1.

$\Rightarrow$ Define a function $\alpha_\sharp : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by setting

$\alpha_\sharp([\sigma]) = [\overline{\alpha} * \sigma * \alpha]$ for $[\sigma] \in \pi_1(X, x_0)$.

$\Rightarrow$ (i) $\alpha_\sharp : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is well-defined by Lemma 11.

(ii) $\alpha_\sharp : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is a homomorphism.

($\because$) Let $[\sigma], [\tau] \in \pi_1(X, x_0)$ represented by loops $\sigma, \tau \in \Omega_1(X, x_0)$.

$$\begin{aligned}
\Rightarrow \alpha_\sharp([\sigma] \cdot [\tau]) = \alpha_\sharp([\sigma * \tau]) &= [\overline{\alpha} * (\sigma * \tau) * \alpha] \\
&= [\overline{\alpha} * (\sigma * (\alpha * \overline{\alpha}) * \tau) * \alpha] \\
&= [(\overline{\alpha} * \sigma * \alpha) * (\overline{\alpha} * \tau * \alpha)] = \alpha_\sharp([\sigma]) \cdot \alpha_\sharp([\tau]).
\end{aligned}$$

(iii) $(\alpha_\sharp)^{-1} = \overline{\alpha}_\sharp$. i.e., $\alpha_\sharp \circ \overline{\alpha}_\sharp = 1_{\pi_1(X, x_1)}$ and $\overline{\alpha}_\sharp \circ \alpha_\sharp = 1_{\pi_1(X, x_0)}$.

($\because$) Let $[\rho] \in \pi_1(X, x_1)$ represented by a loop $\rho \in \Omega(X, x_1)$.

$$\begin{aligned}
\Rightarrow (\alpha_\sharp \circ \overline{\alpha}_\sharp)([\rho]) = \alpha_\sharp([\overline{\overline{\alpha}} * \rho * \overline{\alpha}]) &= [\overline{\alpha} * \overline{\overline{\alpha}} * (\rho * \overline{\alpha}) * \alpha] \\
&= [(\overline{\alpha} * \alpha) * \rho * (\overline{\alpha} * \alpha)] = [\rho] = 1_{\pi_1(X, x_1)}([\rho]).
\end{aligned}$$

$\Rightarrow \alpha_\sharp \circ \overline{\alpha}_\sharp = 1_{\pi_1(X, x_1)}$.

Similarly we can show that $\overline{\alpha}_\sharp \circ \alpha_\sharp = 1_{\pi_1(X, x_0)}$.

$\Rightarrow \alpha_\sharp : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is an isomorphism by (i), (ii), (iii). $\square$

Remark 19. (1) By Theorem 18, it is common practice to omit mention of the base point for the fundamental group of a path connected space X. Thus we shall sometimes refer $\pi_1(X)$, the fundamental group of X, since the fundamental group $\pi_1(X, x_0)$ does not depend on the choice of base point $x_0 \in X$.

(2) Note that Theorem 18 does not guarantee that the isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ is unique. Different paths can produce different isomorphisms.

(3) In some application, specification of the base point is important. For example, when comparing fundamental groups of spaces X and Y on the basis of a continuous function $f : X \to Y$, it is usually necessary to specify base points x_0 in X and y_0 in Y and to assume that $f(x_0) = y_0$.

Definition 20. The isomorphism $\alpha_\sharp : \pi_1(X, x_0) \to \pi_1(X, x_1)$ in the proof of Theorem 18 is called the *isomorphism induced by the path* α.

Definition 21. Let X be a topological space.

(1) X is said to be *simply connected* if it is path connected and $\pi_1(X)$ is the trivial group consisting of the identity element only.

(2) X is said to be *contractible to a point* x_0 in X provided that there is a continuous function $F : X \times \mathbf{I} \to X$ such that $F(x, 0) = x, F(x, 1) = x_0, F(x_0, t) = x_0$ for all $x \in X$ for all $t \in \mathbf{I}$. The function F is called a *contraction* of X to the point x_0.

(3) X is said to be *contractible* if there exists a point x_0 in X for which X is contractible to x_0.

Remark 22. In general, for a fixed point x_0 of a space X, we define

$$\Omega_n(X, x_0) = \{\alpha : \mathbf{S}^n \to X \mid \alpha \text{ is continuous and } \alpha(*) = x_0)\},$$

where $* = (1, 0, \cdots, 0) \in \mathbf{S}^n$ and define $\pi_n(X, x_0) = \Omega_n(X, x_0)/_{\cong_{x_0}}$.

Then

- $\pi_0(X, x_0)$ can be identified with the number of path components of a space X and
- $\pi_1(X, x_0)$ in the above notion can be identified with the fundamental group $\pi_1(X, x_0)$ in Definition 12.
- It is known that $\pi_n(X, x_0)$ admits an abelian group structure for each $n \geq 2$ and
- $\pi_n(X, x_0)$ is called the *n-th homotopy group* for $n \geq 1$.

Furthermore, we have the following;

(1) X is path connected if and only if $\pi_0(X, x_0) = 1$. Sometimes a path connected space is called a 0-*connected space*.

(2) X is simply connected if and only if $\pi_0(X, x_0) = 1$ and $\pi_1(X) = 1$, where the second 1 denotes the trivial group. Sometimes a simply connected space is called a 1-*connected space*.

(3) A space X is said to be n-*connected* if $\pi_k(X, x_0) = 1$ for all $k \leq n$.

Example 23. A closed interval $[a, b]$ on $\mathbb{R}$ is contractible to a.

Proof. Define $F : [a, b] \times \mathbf{I} \to [a, b]$ by

$$F(x, t) = ta + (1 - t)x \text{ for all } (x, t) \in [a, b] \times \mathbf{I}.$$

$\Rightarrow F : [a, b] \times \mathbf{I} \to [a, b]$ is a well-defined continuous function and

$$F(x, 0) = x, F(x, 1) = a, F(a, t) = a \text{ for all } x \in [a, b] \text{ and } \forall t \in \mathbf{I}.$$

$\Rightarrow [a, b]$ is contractible to a. In fact, an analogous argument shows that $[a, b]$ is contractible to each of its point. □

Theorem 24. *A convex subset A of $\mathbb{R}^n$ is contractible to each point x_0 of A.*

Proof. Let A be a convex subset of $\mathbb{R}^n$ and let $x_0 \in A$.

$\Rightarrow$ Define $H : A \times \mathbf{I} \to A$ as follows:

Let $a \in A$ and choose the straight line $\alpha_a : \mathbf{I} \to A$ in A connecting a to x_0, i.e., $\alpha_a(t) = (1-t)a + tx_0$.

Define $H(a,t) = \alpha_a(t) = (1-t)a + tx_0 \forall (a,t) \in A \times \mathbf{I}$.

$\Rightarrow$ $H : A \times \mathbf{I} \to A$ is a well-defined continuous function and $H(a,0) = a, H(a,1) = x_0$ for all $a \in A$ and $H(x_0,t) = x_0 \forall t \in \mathbf{I}$.

$\Rightarrow$ A is contractible to the point x_0. $\square$

Theorem 25. *Every contractible space is simply connected.*

Proof. Let X be a contractible space.

$\Rightarrow$ There exists a point $x_0 \in X$ such that X is contractible to x_0.

$\Rightarrow$ There exists a continuous function $F : X \times \mathbf{I} \to X$ such that

$F(x, 0) = x, F(x, 1) = x_0, F(x_0, t) = x_0$ for all $x \in X$ for all $t \in \mathbf{I}$.

(i) X is path connected.

$$\left(\begin{array}{ll} (\because) & \forall x \in X,\ \text{Define } f_x : \mathbf{I} \to X \text{ by } f_x(t) = F(x, t) \forall t \in \mathbf{I}. \\ \Rightarrow & \forall x \in X, f_x \text{ is a path from } x \text{ to } x_0. \\ \Rightarrow & \forall x, y \in X, f_x * \overline{f_y} \text{ is a path from } x \text{ to } y. \end{array}\right.$$

(ii) $\pi_1(X, x_0) = \{[c_{x_0}]\}$.

$$\left(\begin{array}{ll} (\because) & \text{Let } [\alpha] \in \pi_1(X, x_0). \\ \Rightarrow & \text{Define } H : \mathbf{I} \times \mathbf{I} \to X \text{ by} \\ & H(s,t) = F(\alpha(s), t) \text{ for all } (s,t) \in \mathbf{I} \times \mathbf{I}. \\ \Rightarrow & H : \mathbf{I} \times \mathbf{I} \to X \text{ is continuous,} \\ & H(s,0) = F(\alpha(s), 0) = \alpha(s), H(s,1) = F(\alpha(s), 1) = x_0, \\ & H(0,t) = F(\alpha(0), t) = F(x_0, t) = x_0 \text{ for all } t \in \mathbf{I}, \\ & H(1,t) = F(\alpha(1), t) = F(x_0, t) = x_0 \text{ for all } t \in \mathbf{I}. \\ \Rightarrow & H : \alpha \simeq_{x_0} c_{x_0}, \text{ i.e., } [\alpha] = [c_{x_0}]. \\ \Rightarrow & \pi_1(X, x_0) = \{[c_{x_0}]\}. \end{array}\right.$$

$\Rightarrow$ X is simply connected by (i) and (ii) above. $\square$

Remark 26. From Theorems 24 and 25, we conclude that

a single point, an interval, the real line $\mathbb{R}^1$, a disk, a rectangle, Euclidean n-space $\mathbb{R}^n$, and all other convex subspaces of $\mathbb{R}^n$ have trivial fundamental group.

Our first nontrivial example of a fundamental group occurs for the unit circle $\mathbf{S}^1$ in the next section.

Exercises 2

1. Prove that the relation of free homotopic paths in a topological space X is an equivalence relation on the set of all paths in X.

2. Prove that the equivalence of paths in a topological space X is an equivalence relation on the set of all paths in X with a fixed initial point $x_0 \in X$ and a fixed terminal point $x_1 \in X$.

3. Let γ be a loop in X with a base point x_0. Show that $\gamma_\sharp = 1_{\pi_1(X,x_0)}$ if and only if $[\gamma]$ belongs to the center of $\pi_1(X, x_0)$.

4. Let α and β be equivalent paths in a space X from x_0 to x_1. Show that $\alpha_\sharp = \beta_\sharp$.

5. (1) Let α and β be paths in a space X having common initial point x_0 and common terminal point x_1. Prove that α and β are equivalent if and only if the product $\alpha * \overline{\beta}$ is equivalent to the constant loop c_{x_0} at x_0.

(2) Let X be a path connected space.

Prove that X is simply connected if and only if each pair of paths in X having the same initial point and same terminal point are equivalent.

6. Prove that the following spaces are contractible.

(1) The n-dimensional upper hemisphere U^n of $\mathbf{S}^n$:

$$U^n = \{x = (x_1, \cdots, x_{n+1}) \in \mathbf{S}^n \mid x_{n+1} \geq 0\}.$$

(2) The "punctured n-sphere" $\mathbf{S}^n \setminus \{p\}$, where p is any point in $\mathbf{S}^n$.

(3) The topologist's comb.

7. **Definition.** A space X is said to be *weakly contractible* provided that there is a point x_0 in X and a continuous function $F : X \times \mathbf{I} \to X$ such that $F(x, 0) = x, F(x, 1) = x_0$ for all $x \in X$.

The function F is called a *weak contraction*.

Thus a difference between a contraction on X and a weak contraction on X is in the fact that a weak contraction is not required to leave the base point x_0 fixed.

(1) Give an example of a weakly contractible space that is not contractible.

(2) Prove that each weakly contractible space is simply connected.

3 The Fundamental Group of $\mathbf{S}^1$

Definition 27. (1) The function $p : \mathbb{R} \to \mathbf{S}^1$ defined by

$$p(s) = (\cos 2\pi s, \sin 2\pi s) \in \mathbf{S}^1 \subset \mathbb{R}^2 \text{ for all } s \in \mathbb{R}$$

is called the *covering projection of* $\mathbb{R}$ *over* $\mathbf{S}^1$.

Note that $p(k) = (1, 0) \in \mathbf{S}^1$ for all $k \in \mathbb{Z} \subset \mathbb{R}$ and that p functions $[k, k+1]$ around $\mathbf{S}^1$ exactly once in the counterclockwise direction.

Also note that $p(s+t) = (\cos 2\pi(s+t), \sin 2\pi(s+t)) = (\cos 2\pi s, \sin 2\pi s)(\cos 2\pi t, \sin 2\pi t) = p(s)p(t)$ for all $s, t \in \mathbb{R}$, where $(a, b)(c, d) = (ac - bd, ad + bc)$ $\forall (a, b), (c, d) \in \mathbf{S}^1 \subset \mathbb{R}^2$.

$1 \equiv (1, 0) \in \mathbf{S}^1 (\subset \mathbb{R}^2)$ will be designated as a base point of $\mathbf{S}^1$.

(2) Let X be a topological space and let $f : X \to \mathbf{S}^1$ be a continuous function. A continuous function $\widetilde{f} : X \to \mathbb{R}$ is called a *lifting of f to $\mathbb{R}$* or *covering function of f* if $p \circ \widetilde{f} = f$,

i.e., the following diagram commutes.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & \mathbf{S}^1. \end{tikzcd}$$

In particular when $X = \mathbf{I}$, the lifting $\widetilde{\alpha} : \mathbf{I} \to \mathbb{R}$ of a path $\alpha : \mathbf{I} \to \mathbf{S}^1$ is usually called a *covering path* of α.

Remark 28. We shall be particularly interested in lifting a path $\alpha : \mathbf{I} \to \mathbf{S}^1$ to a covering path $\widetilde{\alpha} : \mathbf{I} \to \mathbb{R}$ and a homotopy $F : \mathbf{I} \times \mathbf{I} \to \mathbf{S}^1$ to a covering homotopy $\widetilde{F} : \mathbf{I} \times \mathbf{I} \to \mathbb{R}$.

Lemma 29. *There is a pair U_1, U_2 of subsets of $\mathbf{S}^1$ satisfying the following conditions:*

(1) *U_1, U_2 are path connected and open in $\mathbf{S}^1$.*

(2) *$\mathbf{S}^1 = U_1 \cup U_2$.*

(3) *$p : \mathbb{R} \to \mathbf{S}^1$ functions each path component of $p^{-1}(U_i)$ homeomorphically onto U_i, $\forall i = 1, 2$.*

Proof. There are many possible ways to choose U_1 and U_2, one of which is the following as depicted in as in Figure 8.

$U_1 =$ the open arc on $\mathbf{S}^1$ beginning at the point $(-1, 0)$ and extending counterclockwise to the point $(0, 1)$,

$U_2 =$ the corresponding open arc beginning at the point $(1, 0)$ and extending counterclockwise to the point $(0, -1)$.

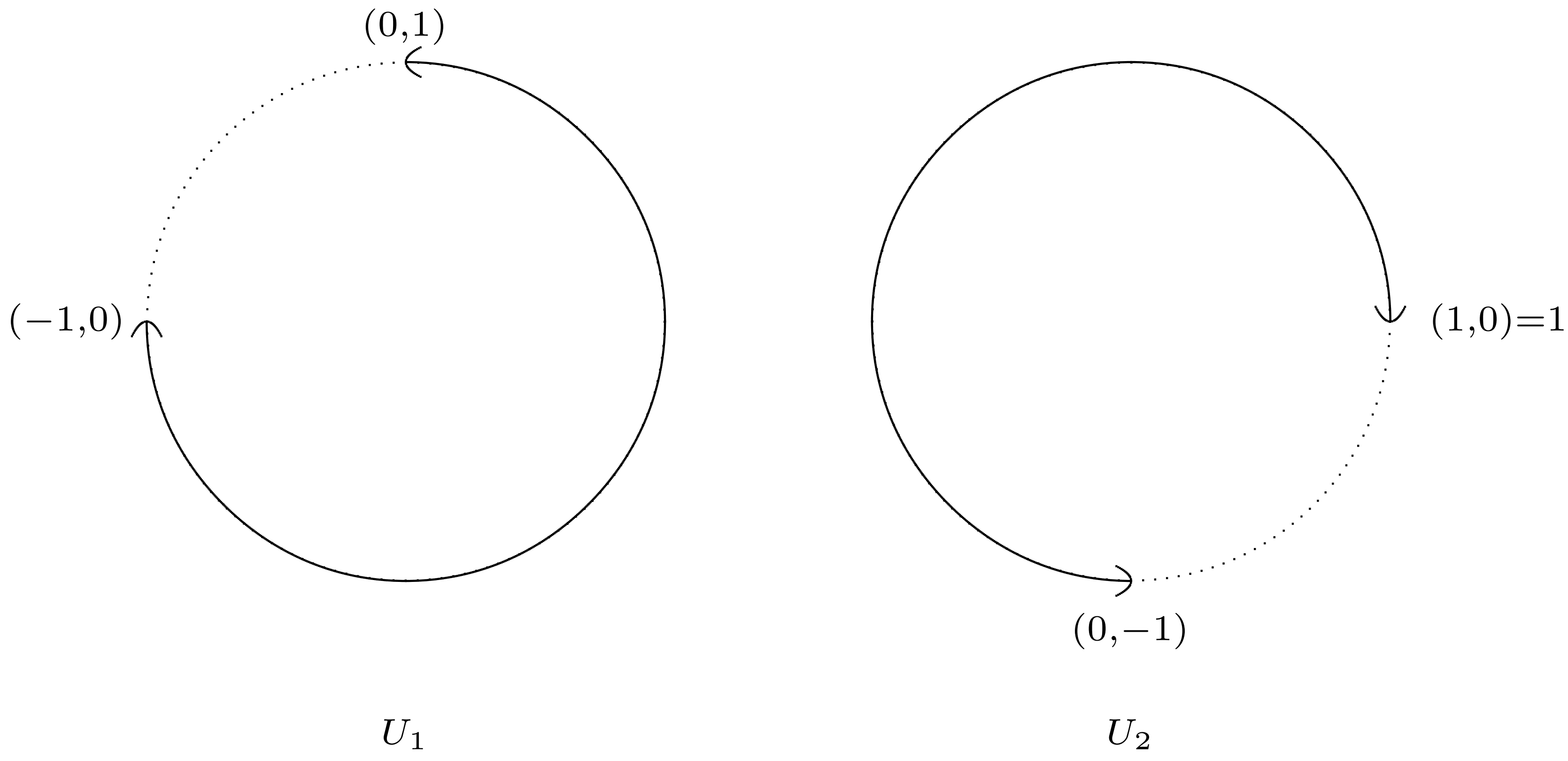


Figure 8: U_1 and U_2

$\Rightarrow$ (1) U_1 and U_2 are clearly path connected open subsets of $\mathbf{S}^1$.

(2) Clearly $U_1 \cup U_2 = \mathbf{S}^1$.

By the definition of the covering projection p, we have

$$p^{-1}(U_1) = \bigcup_{k=-\infty}^{\infty} (k - \frac{1}{2}, k + \frac{1}{4}), \quad p^{-1}(U_2) = \bigcup_{k=-\infty}^{\infty} (k, k + \frac{3}{4}).$$

(3) Note that the path components of $p^{-1}(U_1)$ are the open intervals $(k - \frac{1}{2}, k + \frac{1}{4}), k \in \mathbb{Z}$ and p sends each interval $(k - \frac{1}{2}, k + \frac{1}{4})$ homeomorphically onto U_1. Similarly the path components of $p^{-1}(U_2)$ are the open intervals $(k, k + \frac{3}{4}), k \in \mathbb{Z}$ and p sends each interval $(k, k + \frac{3}{4})$ homeomorphically onto U_2. $\square$

For an open cover $\mathscr{O}$ of a metric space X, recall that a *Lebesgue number* for $\mathscr{O}$ is a positive number $\epsilon > 0$ such that every subset A of X with $\operatorname{diam}(A) < \epsilon$ is contained in some member of $\mathscr{O}$.

Lebesgue Number Theorem. Let X be a compact metric space. Then every open cover of X has a Lebesgue number.

Theorem 30 (Covering Path Property)**.** *Let $\alpha : \mathbf{I} \to \mathbf{S}^1$ be a path with $\alpha(0) = 1 \in \mathbf{S}^1$. Then there exists a unique lifting $\widetilde{\alpha} : \mathbf{I} \to \mathbb{R}$ of α with $\widetilde{\alpha}(0) = 0$.*

i.e., there exists a unique continuous function $\widetilde{\alpha} : \mathbf{I} \to \mathbb{R}$ such that $\widetilde{\alpha}(0) = 0$ and the following diagram commutes.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ \mathbf{I} \arrow[ur, "\widetilde{\alpha}"] \arrow[r, "\alpha"'] & \mathbf{S}^1. \end{tikzcd}$$

Proof. Let $\alpha : \mathbf{I} \to \mathbf{S}^1$ be a path with $\alpha(0) = 1 \in \mathbf{S}^1$. Choose the path connected open subsets U_1 and U_2 of the Figure 8.

Note that $p^{-1}(U_1) = \dot{\bigcup\limits_{k\in\mathbb{Z}}}(k - \frac{1}{2}, k + \frac{1}{4})$, $\quad p^{-1}(U_2) = \dot{\bigcup\limits_{k\in\mathbb{Z}}}(k, k + \frac{3}{4})$,

and that $p|_{(k-\frac{1}{2},k+\frac{1}{4})} : (k - \frac{1}{2}, k + \frac{1}{4}) \to p((k - \frac{1}{2}, k + \frac{1}{4}))$ and

$p|_{(k,k+\frac{3}{4})} : (k - \frac{1}{2}, k + \frac{1}{4}) \to p((k, k + \frac{3}{4}))$ are homeomorphisms,

where $\dot{\bigcup}$ means a disjoint union.

$\Rightarrow \{\alpha^{-1}(U_1), \alpha^{-1}(U_2)\}$ is an open cover of the compact metric space $\mathbf{I}$.

$\Rightarrow$ By the Lebesgue Number Theorem, there exists a Lebesgue number $\epsilon > 0$ for the open covering $\{\alpha^{-1}(U_1), \alpha^{-1}(U_2)\}$ of $\mathbf{I}$.

i.e., $\forall\ A \subset \mathbf{I}$ with $\mathrm{diam}(A) < \epsilon$, $A \subset \alpha^{-1}(U_1)$ or $A \subset \alpha^{-1}(U_2)$.

$\Rightarrow$ Subdivide the interval $\mathbf{I}$ into subintervals

$0 = t_0 < t_1 < \cdots < t_n = 1$ such that $t_i - t_{i-1} < \epsilon \forall i = 1, 2, \cdots, n$.

$\Rightarrow$ Since $\alpha(t_0) = \alpha(0) = 1 \in U_1 \setminus U_2,\ \alpha([t_0, t_1] \subset U_1$.

$\Rightarrow$ Define $\widetilde{\alpha}_1 : [t_0, t_1] \to \mathbb{R}$ by $\widetilde{\alpha}_1(t) = (p|_{(-\frac{1}{2}, \frac{1}{4})})^{-1} \circ \alpha(t) \forall t \in [t_0, t_1]$.

i.e. the following diagrams commute.

$$\begin{tikzcd}
 & (-\frac{1}{2}, \frac{1}{4}) \arrow[r, hook] \arrow[d, "p|"] & \mathbb{R} \arrow[d, "p"] \\
{[t_0, t_1]} \arrow[ur, "\widetilde{\alpha_1}"] \arrow[r, "\alpha|_{[t_0, t_1]}"'] & U_1 \arrow[r, hook] & \mathbf{S}^1
\end{tikzcd}$$

$\Rightarrow$ $\widetilde{\alpha}_1 : [t_0, t_1] \to \mathbb{R}$ is a continuous function,

$\widetilde{\alpha}_1(0) = p|_{(-\frac{1}{2}, \frac{1}{4})}^{-1}(\alpha(0)) = 0$ and $p \circ \widetilde{\alpha}_1(t) = \alpha(t)$ for all $t \in [t_0, t_1]$.

Note that $\widetilde{\alpha}_1 : [t_0, t_1] \to \mathbb{R}$ is uniquely determined by the choice of the unique path component $(-\frac{1}{2}, \frac{1}{4})$ of $p^{-1}(U_1)$ containing 0.

$\Rightarrow$ $\widetilde{\alpha}_1 : [t_0, t_1] \to \mathbb{R}$ is a unique continuous function such that $\widetilde{\alpha}_1(0) = 0$ and $p \circ \widetilde{\alpha}_1(t) = \alpha(t)$ for all $t \in [t_0, t_1]$.

To proceed by induction, assume that for $i \geq 1$, we have defined a unique continuous function $\widetilde{\alpha}_i : [t_0, t_i] \to \mathbb{R}$ such that $\widetilde{\alpha}_i(0) = 0$ and $p \circ \widetilde{\alpha}_i(t) = \alpha(t)$ for all $t \in [t_0, t_i]$.

Note that $\alpha([t_i, t_{i+1}]) \subset U_1$ or $\alpha([t_i, t_{i+1}]) \subset U_2$.

Let $U = U_1$ or U_2, such that $\alpha([t_i, t_{i+1}]) \subset U$.

$\Rightarrow$ Let S_{i+1} be the path component of $p^{-1}(U)$, containing $\widetilde{\alpha}_i(t_i)$ so that $p|_{S_{i+1}} : S_{i+1} \to U$ is a homeomorphism.

$\Rightarrow$ Define $\widetilde{\alpha}_{i+1} : [t_0, t_{i+1}] \to \mathbb{R}$ by setting

$$\widetilde{\alpha}_{i+1}(t) = \begin{cases} \widetilde{\alpha}_i(t) & \text{if } t \in [t_0, t_i] \text{ and} \\ (p|_{S_{i+1}})^{-1} \circ \alpha(t) & \text{if } t \in [t_i, t_{i+1}]. \end{cases}$$

$\Rightarrow \widetilde{\alpha}_{i+1} : [t_0, t_{i+1}] \to \mathbb{R}$ is continuous by the pasting lemma,

$\widetilde{\alpha}_{i+1}(0) = \widetilde{\alpha}_i(0) = 0$ and $p \circ \widetilde{\alpha}_{i+1}(t) = \alpha(t)$ for all $t \in [t_0, t_{i+1}]$.

Note that $\widetilde{\alpha}_{i+1}$ is uniquely determined by $\widetilde{\alpha}_{i+1}(0) = \widetilde{\alpha}_1(0) = 0$.

$\Rightarrow$ By induction argument, we have a unique continuous function

$\widetilde{\alpha} \equiv \widetilde{\alpha}_n : [0,1] \to \mathbb{R}$ such that $\widetilde{\alpha}(0) = 0$ and $p \circ \widetilde{\alpha} = \alpha$. $\square$

Theorem 31 (Covering Homotopy Property)**.** *Let $H : \mathbf{I} \times \mathbf{I} \to \mathbf{S}^1$ be a homotopy such that $H(0,0) = 1 \in \mathbf{S}^1$. Then there exists a unique lifting $\widetilde{H} : \mathbf{I} \times \mathbf{I} \to \mathbb{R}$ such that $p \circ \widetilde{H} = H$ and $\widetilde{H}(0,0) = 0$.*

i.e., there exists a unique continuous function $\widetilde{H} : \mathbf{I} \to \mathbb{R}$ such that $\widetilde{H}(0,0) = 0$ and the following diagram commutes.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ \mathbf{I} \times \mathbf{I} \arrow[ur, "\widetilde{H}"] \arrow[r, "H"'] & \mathbf{S}^1. \end{tikzcd}$$

Proof. Let $H : \mathbf{I} \times \mathbf{I} \to \mathbf{S}^1$ be a homotopy with $H(0,0) = 1 \in \mathbf{S}^1$.

Choose the path connected open subsets U_1 and U_2 of the Figure 8 as in Theorem 30.

$\Rightarrow \{H^{-1}(U_1), H^{-1}(U_2)\}$ is an open cover of the compact metric space

$\mathbf{I} \times \mathbf{I}$.

$\Rightarrow$ By the Lebesgue Number Theorem, there exists a Lebesgue number $\epsilon > 0$ for the open covering $\{H^{-1}(U_1), H^{-1}(U_2)\}$ of $\mathbf{I} \times \mathbf{I}$.

$\Rightarrow$ Subdivide the interval $\mathbf{I}$ into subintervals

$0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1 \quad \cdot\ni\cdot$

$\text{diam}([s_i, s_{i+1}] \times [t_j, t_{j+1}]) < \epsilon (0 \leq i \leq m-1,\ 0 \leq j \leq n-1)$,

$\Rightarrow$ $[s_i, s_{i+1}] \times [t_j, t_{j+1}] \subset H^{-1}(U_1)$ or $[s_i, s_{i+1}] \times [t_j, t_{j+1}] \subset H^{-1}(U_2)$ for each pair (i, j) with $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$.

$\Rightarrow$ $H([s_i, s_{i+1}] \times [t_j, t_{j+1}]) \subset U_1$ or $H([s_i, s_{i+1}] \times [t_j, t_{j+1}]) \subset U_2$ for each pair (i, j) with $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$.

$\Rightarrow$ Since $H(0,0) = H(s_0, t_0) = 1 \in U_1 \setminus U_2$, $H([s_0, s_1] \times [t_0, t_1]) \subset U_1$.

$\Rightarrow$ Define $\widetilde{H}_{(1,1)} : [s_0, s_1] \times [t_0, t_1] \to \mathbb{R}$ by

$$\widetilde{H}_{(1,1)}(s,t) = (p|_{(-\frac{1}{2},\frac{1}{4})})^{-1} \circ H(s,t) \forall (s,t) \in [s_0, s_1] \times [t_0, t_1].$$

i.e. the following diagrams commute.

$$\begin{tikzcd}
 & & (-\frac{1}{2}, \frac{1}{4}) \arrow[r, hook] \arrow[d, "p|"] & \mathbb{R} \arrow[d, "p"] \\
{[s_0, s_1] \times [t_0, t_1]} \arrow[urr, "\widetilde{H}_{(1,1)}"] \arrow[rr, "H|_{[s_0,s_1]\times[t_0,t_1]}"'] & & U_1 \arrow[r, hook] & \mathbf{S}^1
\end{tikzcd}$$

$\Rightarrow \widetilde{H}_{(1,1)} : [s_0, s_1] \times [t_0, t_1] \to \mathbb{R}$ is a continuous function,

$\widetilde{H}_{(1,1)}(0, 0) = (p|_{(-\frac{1}{2},\frac{1}{4})})^{-1}(H(0, 0)) = (p|_{(-\frac{1}{2},\frac{1}{4})})^{-1}(1) = 0$ and

$p \circ \widetilde{H}_{(1,1)}(s, t) = H(s, t)$ for all $(s, t) \in [s_0, s_1] \times [t_0, t_1]$.

$\Rightarrow \widetilde{H}_{(1,1)} : [s_0, s_1] \times [t_0, t_1] \to \mathbb{R}$ is a continuous function such that

$\widetilde{H}_{(1,1)}(0, 0) = 0$ and $p \circ \widetilde{H}_{(1,1)}(s, t) = H(s, t)$

for all $(s, t) \in [s_0, s_1] \times [t_0, t_1]$.

Note that $\widetilde{H}_{(1,1)} : [s_0, s_1] \times [t_0, t_1] \to \mathbb{R}$ is uniquely determined by the initial point $\widetilde{H}_{(1,1)}(0, 0) = 0$.

Proceeding inductively as in the proof of Theorem 30,

we have a unique continuous function $\widetilde{H} \equiv \widetilde{H}_{(m,n)} : \mathbf{I} \times \mathbf{I} \to \mathbb{R}$ such that $p \circ \widetilde{H} = H$ and $\widetilde{H}(0, 0) = 0$.

The details are left for the readers. $\square$

Definition 32. Let $\alpha : \mathbf{I} \to \mathbf{S}^1$ be a loop in $\mathbf{S}^1$, based at $1 \in \mathbf{S}^1$. Then by the Covering Path Property(Theorem 30), there exists a unique lifting $\widetilde{\alpha} : \mathbf{I} \to \mathbb{R}$ of α such that $\widetilde{\alpha}(0) = 0$.

Since $p(\widetilde{\alpha}(1)) = \alpha(1) = 1$, $\widetilde{\alpha}(1) \in \mathbb{Z}$.

The integer $\widetilde{\alpha}(1)$ is called the *degree of the loop* α and is denoted by $\deg(\alpha)$.

Theorem 33. *Let* $\alpha, \beta : \mathbf{I} \to \mathbf{S}^1$ *be two loops in* $\mathbf{S}^1$*, based at* $1 \in \mathbf{S}^1$*. Then* $[\alpha] = [\beta]$ *in* $\pi_1(\mathbf{S}^1, 1)$ *if and only if* $\deg(\alpha) = \deg(\beta)$.

Proof. Let $\alpha, \beta : \mathbf{I} \to \mathbf{S}^1$ be two loops in $\mathbf{S}^1$, based at $1 \in \mathbf{S}^1$.

$\Rightarrow$ There exist liftings $\widetilde{\alpha}, \widetilde{\beta} : \mathbf{I} \to \mathbb{R}$ of $\alpha, \beta : \mathbf{I} \to \mathbf{S}^1$, respectively such that $\widetilde{\alpha}(0) = \widetilde{\beta}(0) = 0$ by the Covering Path Property.

(only if) Assume that $[\alpha] = [\beta]$ in $\pi_1(\mathbf{S}^1, 1)$.

$\Rightarrow$ There exists a homotopy $H : \mathbf{I} \times \mathbf{I} \to \mathbf{S}^1$ such that $H(s, 0) = \alpha(s), H(s, 1) = \beta(s), H(0, t) = H(1, t) = 1$ for all $s, t \in \mathbf{I}$.

$\Rightarrow$ By the Covering Homotopy Property(Theorem 31), there exists a unique lifting $\widetilde{H} : \mathbf{I} \times \mathbf{I} \to \mathbb{R}$ of H $\cdot\!\ni\!\cdot$ $\widetilde{H}(0, 0) = 0$.

$\Rightarrow$ For all $t \in \mathbf{I}$, since $p(\widetilde{H}(0, t)) = H(0, t) = 1$, $\widetilde{H}(0, t) \in \mathbb{Z}$.

$\Rightarrow$ $\widetilde{H}|_{\{0\} \times \mathbf{I}} : \{0\} \times \mathbf{I} \to \mathbb{Z}$ is a well-defined continuous function.

$\Rightarrow$ Since $\{0\} \times \mathbf{I}$ is connected, $\widetilde{H}(\{0\} \times \mathbf{I})$ is a connected subset of $\mathbb{Z}$.

$\Rightarrow$ Since $\mathbb{Z}$ is discrete, $\widetilde{H}(\{0\} \times \mathbf{I})$ is a single point.

$\Rightarrow$ $\widetilde{H}(0,t) = \widetilde{H}(0,0) = 0, \forall t \in \mathbf{I}$. Similarly, $\widetilde{H}(1,t) = \widetilde{H}(1,0) \forall t \in \mathbf{I}$.

$\Rightarrow$ Since $p \circ \widetilde{H}(s,0) = H(s,0) = \alpha(s)$ and $p \circ \widetilde{H}(s,1) = H(s,1) = \beta(s)$,

$\widetilde{H}(s,0) = \widetilde{\alpha}(s)$ and $\widetilde{H}(s,1) = \widetilde{\beta}(s)$ for all $s \in \mathbf{I}$

by the uniqueness of the path liftings starting at 0.

$\Rightarrow$ $\deg(\alpha) = \widetilde{\alpha}(1) = \widetilde{H}(1,0) = \widetilde{H}(1,1) = \widetilde{\beta}(1) = \deg(\beta)$.

(if) Assume that $\deg(\alpha) = \deg(\beta)$.

$\Rightarrow$ $\widetilde{\alpha}(1) = \widetilde{\beta}(1)$. Define $G : \mathbf{I} \times \mathbf{I} \to \mathbb{R}$ by

$G(s,t) = (1-t)\widetilde{\alpha}(s) + t\widetilde{\beta}(s)$ for all $(s,t) \in \mathbf{I} \times \mathbf{I}$.

$\Rightarrow$ G is a well-defined continuous function,

$G(s,0) = \widetilde{\alpha}(s)$ and $G(s,1) = \widetilde{\beta}(s)$ for all $s \in \mathbf{I}$,

$G(0,t) = (1-t)\widetilde{\alpha}(0) + t\widetilde{\beta}(0) = 0$ for all $t \in \mathbf{I}$ and

$G(1,t) = (1-t)\widetilde{\alpha}(1) + t\widetilde{\beta}(1) = \widetilde{\alpha}(1) = \widetilde{\beta}(1)$ for all $t \in \mathbf{I}$.

$\Rightarrow p \circ G : \mathbf{I} \times \mathbf{I} \to \mathbf{S}^1$ is a continuous function and

$p \circ G(s, 0) = p(\widetilde{\alpha}(s)) = \alpha(s)$ and

$p \circ G(s, 1) = p(\widetilde{\beta}(s)) = \beta(s)$ for all $s \in \mathbf{I}$,

$p \circ G(0, t) = p(0) = 1$ and $p \circ G(1, t) = p(\widetilde{\alpha}(1)) = \alpha(1) = 1 \forall t \in \mathbf{I}$.

$\Rightarrow p \circ G : \alpha \simeq_1 \beta$. i.e., $[\alpha] = [\beta]$ in $\pi_1(\mathbf{S}^1, 1)$. $\square$

Theorem 33 shows how to associate each homotopy class of loops in $\pi_1(\mathbf{S}^1, 1)$ with an integer.

The next theorem demonstrates that this correspondence is an isomorphism.

Theorem 34. *The fundamental group* $\pi_1(\mathbf{S}^1, 1)$ *is isomorphic to the group of integers. i.e.,* $\pi_1(\mathbf{S}^1, 1) \cong \mathbb{Z}$.

Proof. Define $d : \pi_1(\mathbf{S}^1, 1) \to \mathbb{Z}$ by $d([\alpha]) = \deg(\alpha) \forall [\alpha] \in \pi_1(\mathbf{S}^1, 1)$.

$\Rightarrow$ (1) d is well-defined and injective by Theorem 33.

(2) d is surjective.

$$\left(\begin{array}{ll} (\because) & \text{Let } k \in \mathbb{Z}. \\ \Rightarrow & \text{Define } \widetilde{\alpha}_k : \mathbf{I} \to \mathbb{R} \text{ and } \alpha_k : \mathbf{I} \to \mathbf{S}^1 \text{ by} \\ & \widetilde{\alpha}_k(s) = ks \text{ and } \alpha_k(s) = (p \circ \widetilde{\alpha}_k)(s) = p(ks) \forall s \in \mathbf{I}, \text{ resp.} \\ \Rightarrow & [\alpha_k] \in \pi_1(\mathbf{S}^1, 1) \text{ and } \widetilde{\alpha}_k \text{ is the lifting of } \alpha_k \\ & \text{starting at } \widetilde{\alpha}_k(0) = 0. \\ \Rightarrow & d([\alpha_k]) = \deg(\alpha_k) = \widetilde{\alpha}_k(1) = k. \end{array} \right.$$

(3) d is a homomorphism.

$(\because)$ Let $[\sigma], [\tau] \in \pi_1(\mathbf{S}^1, 1)$ and let $\widetilde{\sigma}, \widetilde{\tau},$ denote the unique liftings of σ, τ starting at 0, respectively.

$\Rightarrow$ Define a path $\alpha : \mathbf{I} \to \mathbb{R}$ by

$$\alpha(s) = \begin{cases} \widetilde{\sigma}(2s) & \text{if } 0 \leq s \leq \dfrac{1}{2}, \\ \widetilde{\sigma}(1) + \widetilde{\tau}(2s-1) & \text{if } \dfrac{1}{2} \leq s \leq 1. \end{cases}$$

$\Rightarrow$ $\alpha : \mathbf{I} \to \mathbb{R}$ is continuous by the pasting lemma and $p \circ \alpha(s) = (\sigma * \tau)(s)$ for all $s \in \mathbf{I}, \alpha(0) = \widetilde{\sigma}(0) = 0$.

$\Rightarrow$ $\alpha : \mathbf{I} \to \mathbb{R}$ is a lifting of $\sigma * \tau$ starting at 0.

$\Rightarrow$ $\deg(\sigma * \tau) = \alpha(1) = \widetilde{\sigma}(1) + \widetilde{\tau}(1) = \deg(\sigma) + \deg(\tau)$.

$\Rightarrow$ $d([\sigma] \cdot [\tau]) = d([\sigma * \tau]) = \deg(\sigma * \tau)$

$= \deg(\sigma) + \deg(\tau) = d([\sigma]) + d([\tau])$.

$\square$

Exercises 3

1. Complete the inductive definition of the covering homotopy in the proof of the Covering Homotopy Property(Theorem 31).

2. Consider $\mathbf{S}^1$ as the set $z = x + iy$ of complex numbers having modulus 1. Then the covering projection $p : \mathbb{R} \to \mathbf{S}^1$ is, by definition of the exponential function for complex variables,
$p(s) = \cos 2\pi s + i \sin 2\pi s = e^{2\pi i s}$ for all $s \in \mathbb{R}$.

   Use this representation to prove the unique assertion of the Covering Path Property(Theorem 30) by showing the following:

   (1) For a given loop α in $\mathbf{S}^1$ with base point 1, let $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ be covering paths of α with initial point 0. Show that the composition of p with $\widetilde{\alpha}_1 - \widetilde{\alpha}_2$ is a constant path.

   (2) Conclude from part (1) that $\widetilde{\alpha}_1 - \widetilde{\alpha}_2$ has only integral values. Use the connectedness of $\mathbf{I}$ to show that $\widetilde{\alpha}_1 - \widetilde{\alpha}_2$ has only the value 0.

4 The Induced Homomorphisms and Homotopies

Theorem 35. *Let X and Y be topological spaces with base points x_0 and y_0, respectively and let $f : X \to Y$ be a continuous function such that $f(x_0) = y_0$. Define $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $f_*([\alpha]) = [f \circ \alpha]$ for each $[\alpha] \in \pi_1(X, x_0)$, represented by a loop $\alpha : \mathbf{I} \to X$ at x_0. Then $f_* : \pi_1(X, x_0) \to \pi(Y, y_0)$ is a group homomorphism.*

Proof. (1) $f_* : \pi_1(X, x_0) \to \pi(Y, y_0)$ is well-defined.

$$
\left(
\begin{array}{ll}
(\because) & \quad \text{Let } [\sigma], [\tau] \in \pi_1(X, x_0) \text{ and assume that } \sigma \simeq_{x_0} \tau. \\
\Rightarrow & \text{There exists a homotopy } H : \mathbf{I} \times \mathbf{I} \to X \text{ such that} \\
& H(s,0) = \sigma(s), H(s,1) = \tau(s) \text{ for all } s \in \mathbf{I}, \\
& H(0,t) = H(1,t) = x_0 \text{ for all } t \in \mathbf{I}. \\
\Rightarrow & f \circ H : \mathbf{I} \times \mathbf{I} \to Y \text{ is a continuous function and} \\
& f \circ H(s,0) = f(H(s,0)) = f(\sigma(s)) = f \circ \sigma(s), \\
& f \circ H(s,1) = f(H(s,1)) = f(\tau(s)) = f \circ \tau(s), \\
& f \circ H(0,t) = f(x_0) = y_0 = f \circ H(1,t) \text{ for all } t \in \mathbf{I}. \\
\Rightarrow & f \circ \sigma \simeq_{y_0} f \circ \tau \text{ by homotopy } f \circ H. \\
\Rightarrow & [f \circ \sigma] = [f \circ \tau] \in \pi_1(Y, y_0).
\end{array}
\right.
$$

(2) $f_* : \pi_1(X, x_0) \to \pi(Y, y_0)$ is a homomorphism.

$$\left(\begin{array}{ll} (\because) & \text{Let } [\sigma], [\tau] \in \pi_1(X, x_0). \\ \Rightarrow & f_*([\sigma] \cdot [\tau]) = f_*([\sigma * \tau]) = [f \circ (\sigma * \tau)] \\ & = [(f \circ \sigma) * (f \circ \tau)] = [f \circ \sigma] \cdot [f \circ \tau] \\ & = f_*([\sigma]) \cdot f_*([\tau]). \end{array}\right.$$

$\square$

Definition 36. Let $f : X \to Y$ be a continuous function such that $f(x_0) = y_0$. The homomorphism $f_* : \pi_1(X, x_0) \to \pi(Y, y_0)$ of Theorem 35, defined by

$$f_*([\alpha]) = [f \circ \alpha] \text{ for each } [\alpha] \in \pi_1(X, x_0),$$

is called the *induced homomorphism of* $f : X \to Y$.

Theorem 37. *We have the following properties of the induced homomorphisms.*

(1) $(1_{(X,x_0)})_* = 1_{\pi_1(X,x_0)}$ *for any space* X *with a base* x_0.

(2) *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two continuous functions such that* $f(x_0) = y_0$ *and* $g(y_0) = z_0$.

$\Rightarrow (g \circ f)_* = g_* \circ f_* : \pi_1(X, x_0) \to \pi_1(Z, z_0)$.

i.e., the following triangle is commutative;

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[rr, "f_*"] \arrow[dr, "(g\circ f)_*"'] & & \pi_1(Y, y_0) \arrow[dl, "g_*"] \\
& \pi_1(Z, z_0). &
\end{tikzcd}$$

Proof. It follows directly from the definition. □

Definition 38. Let X and Y be topological spaces and let $f, g : X \to Y$ be two continuous functions.

(1) $f, g : X \to Y$ are said to be *homotopic*, written $f \simeq g$, if there exists a continuous function $H : X \times \mathbf{I} \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

Note that $\simeq$ is an equivalence relation on the set of all continuous functions from X to Y.

Such a continuous function $H : X \times \mathbf{I} \to Y$ is called a *homotopy from f to g*, written $H : f \simeq g$. In this case, we say that *f is deformed to g via a homotopy H.*

The equivalence class of a continuous function $f : X \to Y$ under $\simeq$, called the *homotopy class* of f, will be denoted by $[f]$.

(2) Let A be a subspace of X and assume that $f(a) = g(a)$ for all $a \in A$. Then $f, g : X \to Y$ are said to be *homotopic relative to* A, written $f \simeq_A g$, if there exists a continuous function $H : X \times \mathbf{I} \to Y$ such that

(i) $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

(ii) $H(a, t) = f(a) = g(a)$ for all $a \in A$ and for all $t \in \mathbf{I}$.

Note that $\simeq_A$ is an equivalence relation on the set of all continuous functions from X to Y.

Such a continuous function $H : X \times \mathbf{I} \to Y$ is called a *homotopy relative to* A *from* f *to* g, written $H : f \simeq_A g$.

The equivalence classes under $\simeq_A$, are called the *relative homotopy class*.

The set of relative homotopy classes will be denoted by $[X, Y]_A$.

If $A = \emptyset$, we omit the phrase "relative to $\emptyset$" and note that $\simeq_\emptyset \; = \; \simeq$.

(3) Assume that $f, g : X \to Y$ are homeomorphisms. We say that $f, g : X \to Y$ are *isotopic*, written $f \cong g$, if there exists a continuous function $H : X \times \mathbf{I} \to Y$, called an *isotopy from f to g*, written $H : f \cong g$, such that

(i) $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$,

(ii) $H_t : X \to Y$, defined by $H_t(x) = H(x, t)$ for all $x \in X$, is a homeomorphism for each $t \in \mathbf{I}$.

In this case, we say that *f is isotoped to g via an isotopy H*.

Note that $\cong$ is an equivalence relation on the set of all homeomorphisms from X to Y.

The equivalence class of f under $\cong$ is called the *isotopy class* of f.

(4) Assume that $f, g : X \to Y$ are homeomorphisms. Let A be a subspace of X and assume that $f(a) = g(a)$ for all $a \in A$. We say that $f, g : X \to Y$ are *isotopic relative to* A, written $f \cong_A g$ if there exists a continuous function $H : X \times \mathbf{I} \to Y$, called an *isotopy relative to* A *from* f *to* g written $H : f \cong_A g$, such that

(i) $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$,

(ii) $H(a, t) = f(a) = g(a)$ for all $(a, t) \in A \times \mathbf{I}$ and

(iii) $H_t : X \to Y$, defined by $H_t(x) = H(x, t)$ for all $x \in X$, is a homeomorphism for each $t \in \mathbf{I}$.

In this case, we say that f *is isotoped to* g *via* H *relative to* A.

Note that $\cong_A$ is an equivalence relation on the set $\text{Homeo}(X, Y)$ of all homeomorphisms from X to Y. The equivalence class of f under $\cong_A$ is called the *isotopy class relative to* A of f. Again if $A = \emptyset$, we omit the phrase "relative to $\emptyset$" and note that $\cong_\emptyset = \cong$.

Remark 39. For two loops $\sigma, \tau : \mathbf{I} \to X$ with
$$\sigma(0) = \tau(0) = \sigma(1) = \tau(1) = x_0,$$
the notation $\sigma \simeq_{\{0,1\}} \tau$ is the same as the previous notation $\sigma \simeq_{x_0} \tau$. We will use both notations.

Definition 40. (1) A continuous function $f : X \to Y$ is called a *homotopy equivalence* if there is a continuous function $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. If such an f exists, we say that X and Y are *of the same homotopy type* or that X is *homotopically equivalent* to Y and write $X \simeq Y$.

(2) A continuous function $f : X \to Y$ is said to be *null-homotopic* if it is homotopic to a constant function.

(3) A topological space X is said to be *weakly contractible* if X is of the same homotopy type as a single point $x_0 \in X$.

In other words, X is weakly contractible if there exist continuous functions $H : X \times \mathbf{I} \to X$ and $G : \{x_0\} \times \mathbf{I} \to \{x_0\}$ such that $H : i_{\{x_0\}} \circ c_{\{x_0\}} \simeq 1_X$ and $G : c_{\{x_0\}} \circ i_{\{x_0\}} \simeq 1_{\{x_0\}}$(automatic), where $i_{\{x_0\}} : \{x_0\} \to X$ denotes the inclusion function and $c_{\{x_0\}} : X \to \{x_0\}$ denotes the constant function. Thus X is weakly contractible if $\exists$ a point $x_0 \in X$ and a continuous function $H : X \times \mathbf{I} \to X$ such that $H(x, 0) = x_0$ and $H(x, 1) = x$ for all $x \in X$.

Recall that a topological space X is *contractible to a point* $x_0 \in X$

if there exist a continuous function $H : X \times \mathbf{I} \to X$ such that $H(x, 0) = x_0$ and $H(x, 1) = x$ for all $x \in X$ and $H(x_0, t) = x_0$ for all $t \in \mathbf{I}$.

Thus every contractible space is weakly contractible.

A difference between a contractible space and a weakly contractible space is in the fact that the homotopy for a weakly contractible space is not required to leave the base point x_0 fixed.

Definition 41. Let A be a subspace of X and let $i : A \to X$ denote the inclusion function of a subspace A into X. Then

(1) A is called a *retract* of X if there exists a continuous function $r : X \to A$ such that $r \circ i = 1_A$.

In this case, $r : X \to A$ is called a *retract function* of X into A.

(2) A is called a *deformation retract* of X if there exists a continuous function $r : X \to A$ such that $r \circ i = 1_A$ and $i \circ r \simeq 1_X$.

In this case, $r : X \to A$ is called a *deformation retract function* of X into A.

Or equivalently, A is a deformation retract of X if there there exists a continuous function $H : X \times \mathbf{I} \to X$ such that $H(x, 0) \in A$ and $H(x, 1) = 1_X(x) = x$ for all $x \in X$.

The homotopy $H : X \times \mathbf{I} \to X$ is called a *deformation retraction* of X onto A. In this situation, note that $r : X \to A$ and $H : X \times \mathbf{I} \to X$ are related by $r(x) = H(x, 0)$ for all $x \in X$.

(3) A is called a *strong deformation retract* of X if there exists a continuous function $r : X \to A$ such that $r \circ i = 1_A$ and $i \circ r \simeq_A 1_X$.

In this case, $r : X \to A$ is called a *strong deformation retract function* of X into A.

Or equivalently, A is a strong deformation retract of X if there there exists a continuous function $H : X \times \mathbf{I} \to X$ such that

(i) $H(x, 0) \in A$ and $H(x, 1) = 1_X(x) = x$ for all $x \in X$,

(ii) $H(a, t) = a$ for all $(a, t) \in A \times \mathbf{I}$.

The homotopy H is called a *strong deformation retraction* of X onto A.

Since for each a in A, $H(a, t) = a$ for each $t \in \mathbf{I}$, it is sometimes said that the points of A *stay fixed* throughout the deformation retraction. In this situation, note that $r : X \to A$ and $H : X \times \mathbf{I} \to X$ are related by $r(x) = H(x, 0)$ for all $x \in X$.

Remark 42. Let A be a subspace of X.

(1) There exists a continuous function $r : X \to A$ such that $r \circ i = 1_A$ and $i \circ r \simeq 1_X$ if and only if there exists a continuous function $H : X \times \mathbf{I} \to X$ such that
 (i) $H(x, 0) \in A$ and $H(x, 1) = x$ for all $x \in X$,
 (ii) $H(a, 0) = a$ for all $a \in A$.

(2) There exists a continuous function $r : X \to A$ such that $r \circ i = 1_A$ and $i \circ r \simeq_A 1_X$ if and only if there exists a continuous function $H : X \times \mathbf{I} \to X$ such that
 (i) $H(x, 0) \in A$ and $H(x, 1) = x$ for all $x \in X$,
 (ii) $H(a, t) = a$ for all $(a, t) \in A \times \mathbf{I}$.

Example 43. (1) In an annulus, both the inner and outer circles are retracts.

(2) A closed subinterval $[c, d]$ of a given interval $[a, b]$ is a retract of $[a, b]$.

(3) The set of endpoints $A = \{a, b\}$ is not a retract of a closed interval $[a, b]$, where $a < b$, for the following reason:

Since $[a, b]$ is connected and A is not, there is no continuous function from $[a, b]$ onto A.

(4) The closed n-dimensional unit ball D^n is a strong deformation retract of $\mathbb{R}^n$.

(5) The n-dimensional unit sphere $\mathbf{S}^n$ is a strong deformation retract of $\mathbb{R}^{n+1} \setminus \{0\}$ for each $n \in \mathbb{N}$.

Theorem 44. *Let X be a space and let $x_0 \in X$. Let $c_{x_0} : X \to \{x_0\}$ denote the constant function and $i_{x_0} : \{x_0\} \to X$ denote the inclusion function. Then the following are equivalent.*

(1) *X is weakly contractible to x_0.*

(2) $1_X \simeq i_{x_0} \circ c_{x_0}$.

(3) *$c_{x_0} : X \to \{x_0\}$ is a homotopy equivalence.*

(4) *$i_{x_0} : \{x_0\} \to X$ is a homotopy equivalence.*

(5) *$\{x_0\}$ is a deformation retract of X.*

(6) *X and $\{x_0\}$ are of the same homotopy type.*

Proof. ((1) $\Rightarrow$ (2), (3), (4), (5), (6))

Assume that X is weakly contractible to x_0.

$\Rightarrow$ There exists a continuous function $H : X \times \mathbf{I} \to X$ such that $H(x, 0) = x$ and $H(x, 1) = x_0$ for all $x \in X$.

$\Rightarrow H(x,0) = 1_X(x), H(x,1) = i_{x_0} \circ c_{x_0}(x)$ for all $x \in X$.

$\Rightarrow$ (2) $1_X \simeq i_{x_0} \circ c_{x_0}$.

Since $c_{x_0} \circ i_{x_0} = 1_{\{x_0\}}$ automatically,

$1_X \simeq i_{x_0} \circ c_{x_0}$ implies (3), (4), (5) and (6).

The proofs of other implications are left for the readers. $\square$

Example 45. Since the function $H : \mathbb{R}^n \times \mathbf{I} \to \mathbb{R}^n$ defined by $H(x,t) = tx$ for all $(x,t) \in \mathbb{R}^n \times \mathbf{I}$ is continuous, $H(x,0) = \mathbf{0} = (0, 0, \cdots, 0) \in \mathbb{R}^n$ and $H(x,1) = x$ for all $x \in \mathbb{R}^n$,

$\mathbb{R}^n$ is weakly contractible to the origin $\mathbf{0} = (0, 0, \cdots, 0) \in \mathbb{R}^n$ and

hence $\mathbb{R}^n$ and $\{\mathbf{0}\}$ are of the same homotopy type by Theorem 44.

The following Theorem 46 is simply the restatement of (1) $\Leftrightarrow$ (2) in Theorem 44.

Theorem 46. *A space X is weakly contractible if and only if the identity function $1_X : X \to X$ is homotopic to a constant function.*

Proof. It follows from (1) $\Leftrightarrow$ (2) of Theorem 44.

(only if) Suppose that X is weakly contractible.

$\Rightarrow$ There exists a continuous function $H : X \times \mathbf{I} \to X$ such that $H(x, 0) = x$ and $H(x, 1) = x_0$ for all $x \in X$.

$\Rightarrow$ Since $H(x, 0) = x = 1_X(x)$ and $H(x, 1) = x_0 = c_{x_0}(x)$ for all $x \in X$, where $c_{x_0} : X \to X$ is the constant function from X to X defined by $c_{x_0}(x) = x_0$ for all $x \in X$.

$\Rightarrow$ The identity function $1_X : X \to X$ is homotopic to the constant function $c_{x_0} : X \to X$.

(if) Suppose that $1_X \simeq c_{x_0}$ where $c_{x_0} : X \to X$ is a constant function defined by $c_{x_0}(x) = x_0$ for all $x \in X$.

$\Rightarrow$ There exists a homotopy $H : X \times \mathbf{I} \to X$ such that

$$H(x,0) = 1_X(x) = x \text{ and } H(x,1) = c_{x_0}(x) = x_0 \text{ for all } x \in X.$$

$\Rightarrow$ X is weakly contractible to the point x_0. $\square$

Example 47. Let $X = \mathbb{R}^n$. The function $H : X \times \mathbf{I} \to X$ defined by $H(x,t) = tx$, for all $(x,t) \in X \times \mathbf{I}$, is a homotopy from the zero function c_0 to $1_{\mathbb{R}^n}$. Hence, $\mathbb{R}^n$ is weakly contractible. Furthermore, any convex subset of $\mathbb{R}^n$ is weakly contractible.

Recall that $\mathbb{R}^n$ and its convex subsets are contractible and thus they are weakly contractible.

Theorem 48. *Let $f, g : X \to Y$ be two homotopic functions by means of a homotopy $H : X \times \mathbf{I} \to Y$, and let $x_0 \in X$.*

Then we have a commutative triangle;

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[dr, "g_*"'] & \pi_1(Y, f(x_0)) \arrow[d, "\alpha_\sharp"] \\
& \pi_1(Y, g(x_0))
\end{tikzcd}$$

where $\alpha_\sharp$ is the isomorphism induced by the path α in Y from $f(x_0)$ to $g(x_0)$ given by $\alpha(t) = H(x_0, t)$ for all $t \in \mathbf{I}$.

Proof. Assume that $f, g : X \to Y$ are homotopic functions

by means of a homotopy $H : X \times \mathbf{I} \to Y$ and let $[\sigma] \in \pi_1(X, x_0)$.

We will construct a homotopy $F : \overline{\alpha} * (f \circ \sigma) * \alpha \simeq_{g(x_0)} g \circ \sigma$

relative to $\{0, 1\}$.

For this, consider the continuous function $G : \mathbf{I} \times \mathbf{I} \to Y$ defined by

$G(s, t) = H(\sigma(s), t)$ for all $s,\ t \in \mathbf{I}$.

$\Rightarrow G(s, 0) = H(\sigma(s), 0) = f(\sigma(s)) = (f \circ \sigma)(s)$,

$G(s, 1) = g(\sigma(s)) = (g \circ \sigma)(s)$ for all $s \in \mathbf{I}$,

$G(0, t) = H(\sigma(0), t) = H(x_0, t) = \alpha(t)$ and

$G(1, t) = H(\sigma(1), t) = H(x_0, t) = \alpha(t)$ for all $t \in \mathbf{I}$.

$\Rightarrow G : f \circ \sigma \simeq g \circ \sigma$ and $G(0, t) = G(1, t) = \alpha(t)$.

Pictorially the homotopy G can be described as in Figure 9.

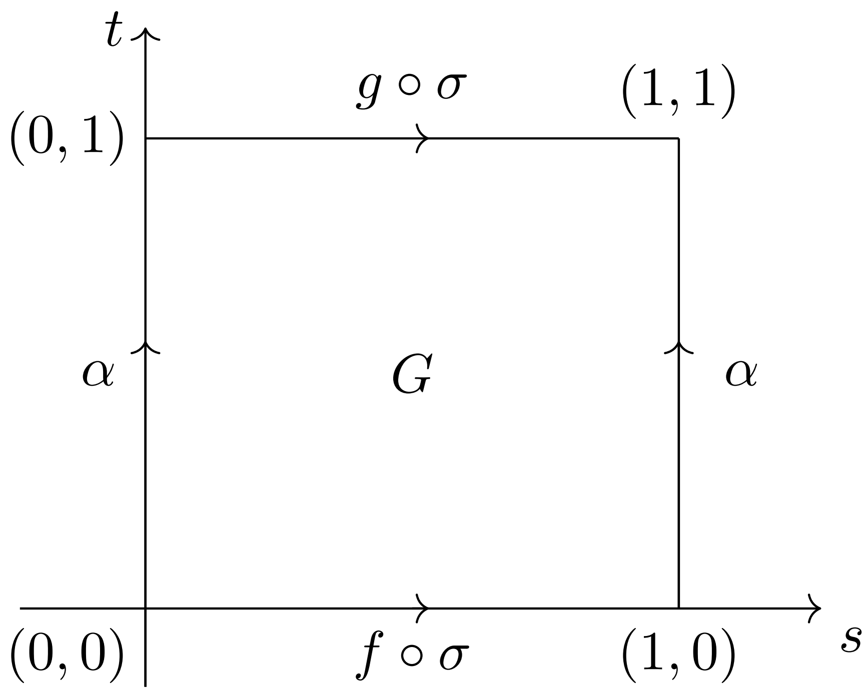


Figure 9: $G : f \circ \sigma \simeq g \circ \sigma$

$\Rightarrow$ Define $F : \mathbf{I} \times \mathbf{I} \to Y$ by

$$F(s,t) = \begin{cases} \overline{\alpha}(2s), & 0 \leq s \leq \dfrac{1-t}{2}, \\ G(\dfrac{4s+2t-2}{3t+1}, t), & \dfrac{1-t}{2} \leq s \leq \dfrac{t+3}{4}, \\ \alpha(4s-3), & \dfrac{t+3}{4} \leq s \leq 1. \end{cases}$$

$\Rightarrow$ $F : \mathbf{I} \times \mathbf{I} \to Y$ is a well-defined continuous function by Lemma 5,

$F(s,0) = (\overline{\alpha} * ((f \circ \sigma) * \alpha))(s)$ and

$F(s,1) = G(s,1) = (g \circ \sigma)(s)$ for all $s \in \mathbf{I}$,

$F(0,t) = \overline{\alpha}(0) = \alpha(1) = g(x_0)$ and

$F(1,t) = \alpha(1) = g(x_0)$ for all $t \in \mathbf{I}$.

The homotopy F from $g \circ \sigma$ to $\overline{\alpha} * (f \circ \sigma) * \alpha$, obtained from G pictorially can be described as in Figure 10.

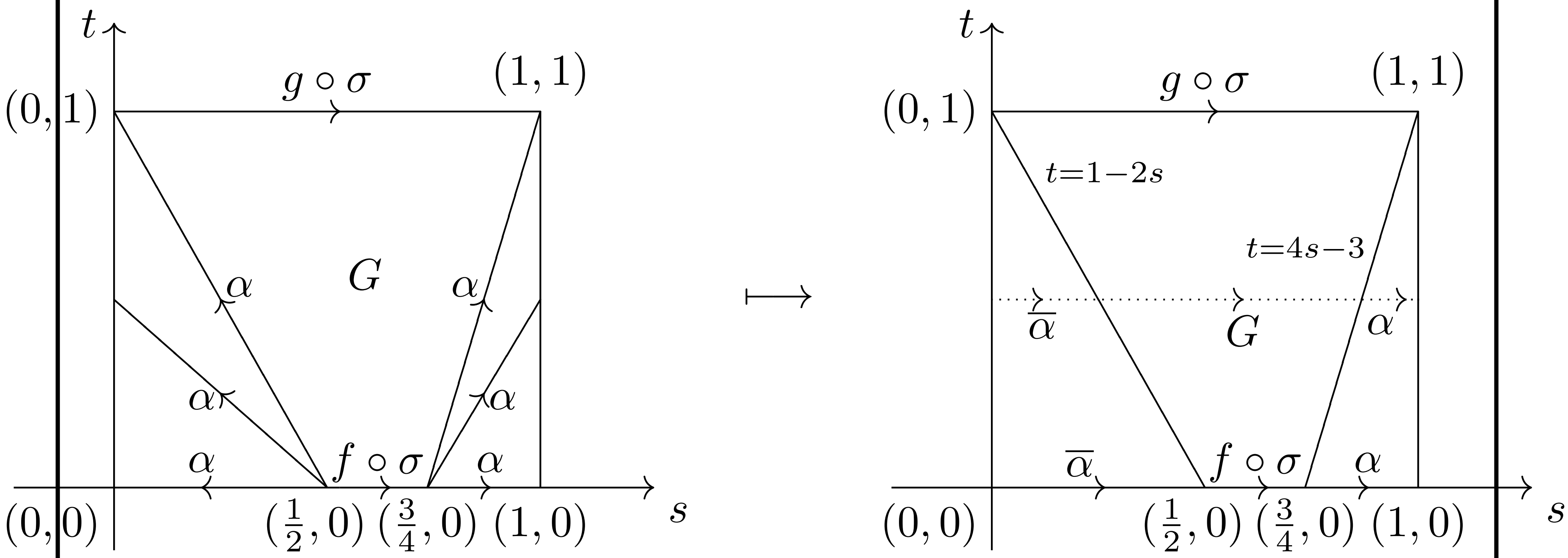


Figure 10: $F : \overline{\alpha} * (f \circ \sigma) * \alpha \simeq_{g(x_0)} g \circ \sigma$ from G

$\Rightarrow F : \overline{\alpha} * (f \circ \sigma) * \alpha \simeq_{g(x_0)} g \circ \sigma.$

$\Rightarrow \alpha_\sharp \circ f_*([\sigma]) = [\overline{\alpha} * (f \circ \sigma) * \alpha] = [g \circ \sigma] = g_*([\sigma]) \forall [\sigma] \in \pi_1(X, x_0).$

$\Rightarrow \alpha_\sharp \circ f_* = g_*.$ $\square$

Corollary 49. *If X and Y are path connected spaces of the same homotopy type, then their fundamental groups are isomorphic.*

Proof. Assume that X and Y are path connected spaces of the same homotopy type.

$\Rightarrow$ There exist continuous functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$.

Choose $x_0 \in X$ and $y_0 \in Y$ and let $g \circ f(x_0) = x_1$, $f \circ g(y_0) = y_1$.

$\Rightarrow$ Apply Theorem 48 to get the following commutative triangles;

$$
\begin{tikzcd}
\pi_1(X, x_0) \arrow[rr, "(g\circ f)_*"] \arrow[dr, "\cong", "(1_X)_*"'] & & \pi_1(X, x_1) \arrow[dl, "\cong"', "\alpha_\sharp"] \\
& \pi_1(X, x_0) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_1(Y, y_0) \arrow[rr, "(f\circ g)_*"] \arrow[dr, "\cong", "(1_Y)_*"'] & & \pi_1(Y, y_1) \arrow[dl, "\cong"', "\beta_\sharp"] \\
& \pi_1(Y, y_0) &
\end{tikzcd}
$$

where α and β are paths induced from the homotopies.

$\Rightarrow$ By Theorem 37, we have the following:

$$g_* \circ f_* = (g \circ f)_* = (\alpha_\sharp)^{-1} \circ (1_X)_* = (\alpha_\sharp)^{-1} \circ 1_{\pi_1(X,x_0)} = (\alpha_\sharp)^{-1},$$
$$f_* \circ g_* = (f \circ g)_* = (\beta_\sharp)^{-1} \circ (1_Y)_* = (\beta_\sharp)^{-1} \circ 1_{\pi_1(Y,y_0)} = (\beta_\sharp)^{-1}.$$

$\Rightarrow$ Since $\alpha_\sharp$ and $\beta_\sharp$ are isomorphisms,

$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ and

$g_* : \pi_1(Y, f(x_0)) \to \pi_1(X, x_1)$ are isomorphisms.

$\Rightarrow$ Since X and Y are path connected,

$\pi_1(X)$ is isomorphic to $\pi_1(Y)$. □

Corollary 50. *Let X and Y be path connected spaces and let $x_0 \in X$. If $f : X \to Y$ is a homeomorphism, then $f_* : \pi_1(X, x_0) \to \pi_1(X, f(x_0))$ is an isomorphism.*

Proof. It is left as an exercise for the reader. □

Exercises 4

1. Let X and Y be path connected spaces and let $x_0 \in X$. Let $f : X \to Y$ be a homeomorphism. Show that $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.

2. Let X and Y be path connected spaces. Assume that $\pi_1(X)$ and $\pi_1(Y)$ are not isomorphic. Show that X and Y are not homeomorphic.

3. Let A be a path connected subspace of a path connected space X. Let $i : A \to X$ denote the inclusion function and let $r : X \to A$ be a continuous function. Then we have the following.
   (1) If $r : X \to A$ is a retract, then $r_* : \pi_1(X, a_0) \to \pi_1(A, r(a_0))$ is an epimorphism and $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ is a monomorphism for any $a_0 \in A$.
   (2) If $r : X \to A$ is a deformation retract, then $r_* : \pi_1(X, a_0) \to \pi_1(A, r(a_0))$ and $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ are isomorphisms for any $a_0 \in A$.

(3) If $r : X \to A$ is a strong deformation retract, then $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ and $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ are isomorphisms for any $a_0 \in A$.

4. Complete the proof of Theorem 44.

Theorem 44. Let X be a space and let $x_0 \in X$. Let $c_{x_0} : X \to \{x_0\}$ denote the constant function and $i_{x_0} : \{x_0\} \to X$ denote the inclusion function. Then the following are equivalent.

(1) X is weakly contractible to x_0.

(2) $1_X \simeq i_{x_0} \circ c_{x_0}$.

(3) $c_{x_0} : X \to \{x_0\}$ is a homotopy equivalence.

(4) $i_{x_0} : \{x_0\} \to X$ is a homotopy equivalence.

(5) $\{x_0\}$ is a deformation retract of X.

(6) X and $\{x_0\}$ are of the same homotopy type.

5 Additional Examples of Fundamental Groups

Our work in this chapter has revealed that the fundamental group of $\mathbf{S}^1$ is the additive group of integers and that the fundamental group of a weakly contractible space is trivial. It should be clear by now that the fundamental group is difficult to determine rigorously. This section presents several theorems that are useful in determining fundamental groups and some additional examples.

For a subspace A of a topological space X, we first recall Remark 42:

(1) A is a deformation retraction of X if and only if there exists a continuous function $H : X \times \mathbf{I} \to X$ such that

(i) $H(x, 0) \in A$ and $H(x, 1) = x$ for all $x \in X$,

(ii) $H(a, 0) = a$ for all $a \in A$.

(2) A is a strong deformation retraction of X if and only if there exists a continuous function $H : X \times \mathbf{I} \to X$ such that

(i) $H(x, 0) \in A$ and $H(x, 1) = x$ for all $x \in X$,

(ii) $H(a, t) = a$ for all $(a, t) \in A \times \mathbf{I}$.

Example 51. Consider the annuli A, B, the unit circle $\mathbf{S}^1$, and the circles C_2, C_4, defined as follows;

$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid 1 \leq x_1^2 + x_2^2 \leq 4\}$,

$\mathbf{S}^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$ and

$C_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 4\}$ as shown in Figure 11(a),

$B = \{(x_1, x_2) \in \mathbb{R}^2 \mid r_1^2 \leq x_1^2 + x_2^2 \leq r_2^2\}$ for any $0 < r_1 < r_2$ and

$C_r = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = r^2\}$ for any $r_1 < r < r_2$ as shown in Figure 11(b). Then $\mathbf{S}^1$ and C_2 are strong deformation retract of A and C_r is a strong deformation retract of D for any $r_1 < r < r_2$ in general.

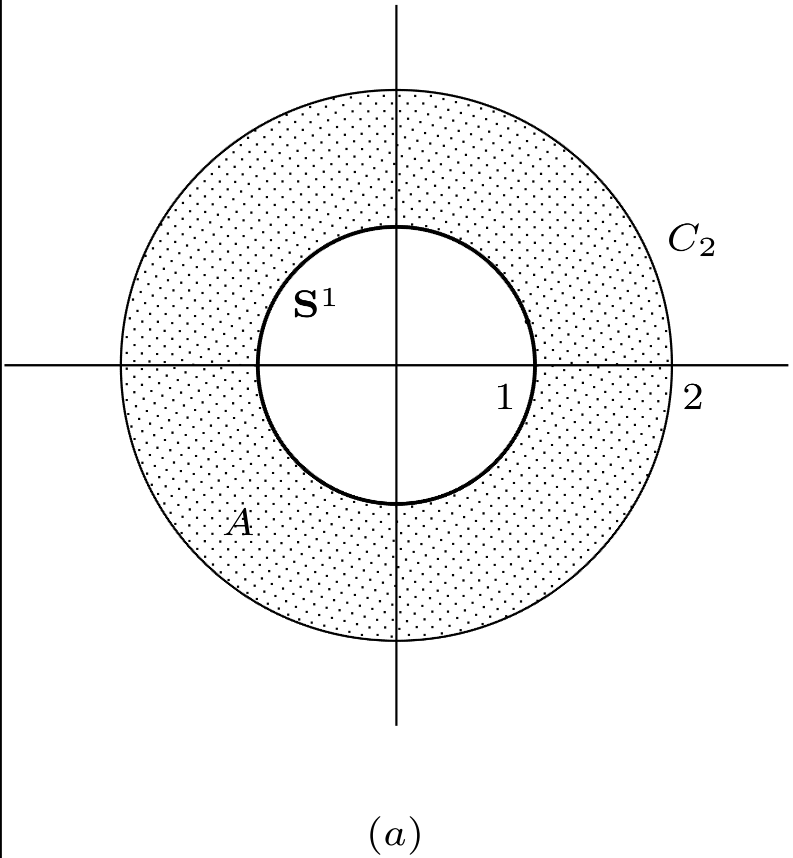


Figure 11: annulus in the plane

Proof. (1) To show that $\mathbf{S}^1$ in Figure 11(a) is a strong deformation retract of A, define $H : A \times \mathbf{I} \to A$ by

$$H(x,t) = tx + (1-t)\frac{x}{||x||} \text{ for all } (x,t) \in A \times \mathbf{I}.$$

$\Rightarrow H : A \times \mathbf{I} \to A$ is a strong deformation retraction of A onto its inner circle $\mathbf{S}^1$.

$$\left(\because \right) \quad \begin{aligned} &H(x,0) = \frac{x}{||x||} \in \mathbf{S}^1, H(x,1) = x \text{ for all } x \in A, \\ &H(y,t) = y \text{ for all } y \in \mathbf{S}^1. \end{aligned}$$

(2) To show that C_2 in Figure 11(a) is a strong deformation retract of A, define $G : A \times \mathbf{I} \to A$ by

$$G(x,t) = t(x_1,x_2) + (1-t)\left(\frac{2x_1}{\sqrt{x_1^2+x_2^2}}, \frac{2x_2}{\sqrt{x_1^2+x_2^2}}\right)$$

for all $x = (x_1, x_2) \in A$ and $t \in \mathbf{I}$.

$\Rightarrow G : A \times \mathbf{I} \to A$ is a deformation retraction of A onto its outer circle C_2.

$$\left(\begin{array}{l} (\because) \quad G(x,0) = \left(\dfrac{2x_1}{\sqrt{x_1^2+x_2^2}}, \dfrac{2x_2}{\sqrt{x_1^2+x_2^2}}\right) \in C_2 \\ \qquad\qquad \text{for all } x = (x_1,x_2) \in A, \\ \qquad\quad G(x,1) = x \text{ for all } x \in A, \text{ and } G(y,t) = y \text{ for all } y \in C_2. \end{array} \right.$$

(3) To show that C_r in Figure 11(b) is a strong deformation retract of B, define $H : B \times \mathbf{I} \to B$ by

$$H(x,t) = t(x_1, x_2) + (1-t)\left(\frac{rx_1}{\sqrt{x_1^2+x_2^2}}, \frac{rx_2}{\sqrt{x_1^2+x_2^2}}\right)$$

for all $x = (x_1, x_2) \in B$ and $t \in \mathbf{I}$.

$\Rightarrow$ $H : B \times \mathbf{I} \to B$ is a deformation retraction of B onto its inside circle C_r.

$$\left(\begin{array}{l} (\because) \quad H(x,0) = \left(\dfrac{rx_1}{\sqrt{x_1^2+x_2^2}}, \dfrac{rx_2}{\sqrt{x_1^2+x_2^2}}\right) \in C_r \\ \qquad\qquad \text{for all } x = (x_1, x_2) \in B, \\ \qquad\quad H(x,1) = x \text{ for all } x \in B, \text{ and } H(y,t) = y \text{ for all } y \in C_r. \end{array} \right.$$

$\square$

Example 52.

(1) Any point of $\mathbb{R}^n$ is a strong deformation retract of $\mathbb{R}^n$.

(2) Any point of the n-ball $\mathbf{B}^n$ is a strong deformation retract of $\mathbf{B}^n$.

(3) Any point of the n-cube $\mathbf{I}^n$ is a strong deformation retract of $\mathbf{I}^n$.

Proof. (1) For any point $p \in \mathbb{R}^n$, consider $F : \mathbb{R}^n \times \mathbf{I} \to \mathbb{R}^n$ defined by

$$F(x,t) = tx + (1-t)p \text{ for all } (x,t) \in \mathbb{R}^n \times \mathbf{I}.$$

(2) For any point $p \in \mathbf{B}^n$, consider $G : \mathbf{B}^n \times \mathbf{I} \to \mathbf{B}^n$ defined by

$$G(x,t) = tx + (1-t)p \text{ for all } (x,t) \in \mathbf{B}^n \times \mathbf{I}.$$

(3) For any point $p \in \mathbf{I}^n$, consider $H : \mathbf{I}^n \times \mathbf{I} \to \mathbf{I}^n$ defined by

$$H(x,t) = tx + (1-t)p \text{ for all } (x,t) \in \mathbf{I}^n \times \mathbf{I}.$$

$\square$

Theorem 53. *Let D be a deformation retract of a path connected space X and let $x_0 \in D$. Then $\pi_1(X, x_0)$ and $\pi_1(D, x_0)$ are isomorphic.*

Proof. Let D be a deformation retract of a path connected space X

and let $x_0 \in D$. Let $i : D \to X$ denote the inclusion function.

$\Rightarrow$ There exists a continuous function $r : X \to D$ such that

$r \circ i = 1_D$ and $i \circ r \simeq 1_X$.

$\Rightarrow$ D and X are of the same homotopy type.

$\Rightarrow$ $i_* : \pi_1(D, x_0) \to \pi_1(X, x_0)$ is an isomorphism by Corollary 49. $\square$

Example 54. Let A be an annulus and let $\mathbb{R}^2 \setminus \{p\}$ be the punctured plane.

$\Rightarrow$ Both inner and outer circles of A are deformation retracts and a circle containing p in its inner region is a deformation retract of $\mathbb{R}^2 \setminus \{p\}$.

$\Rightarrow$ By Theorem 53, $\pi_1(A) \cong \mathbb{Z}$ and $\pi_1(\mathbb{R}^2 \setminus \{p\}) \cong \mathbb{Z}$.

Theorem 55. *Let (X, x_0) and (Y, y_0) be topological spaces with base points. Then $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$.*

Proof. Let $p : X \times Y \to X,\ q : X \times Y \to Y$ denote the natural projection functions. Define

a function $h : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$ by

$$h([\alpha]) = (p_*[\alpha], q_*[\alpha]) = ([p \circ \alpha], [q \circ \alpha]) \forall [\alpha] \in \pi_1(X \times Y, (x_0, y_0)).$$

$\Rightarrow$ (1) h is a homomorphism.

$$\left(\begin{array}{ll} (\because) & \text{Let } [\alpha], [\beta] \in \pi_1(X \times Y, (x_0, y_0)). \\ \Rightarrow & h([\alpha] \cdot [\beta]) = h([\alpha * \beta]) = (p_*([\alpha * \beta]), q_*([\alpha * \beta])) \\ & \qquad = ([p \circ (\alpha * \beta)], [q \circ (\alpha * \beta)]) \\ & \qquad = ([(p \circ \alpha) * (p \circ \beta)], [(q \circ \alpha) * (q \circ \beta)]) \\ & \qquad = ([p \circ \alpha] \cdot [p \circ \beta], [q \circ \alpha] \cdot [q \circ \beta]) \\ & \qquad = ([p \circ \alpha], [q \circ \alpha])([p \circ \beta], [q \circ \beta]) = h([\alpha])h([\beta]). \end{array}\right.$$

(2) $h : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$ is surjective.

$$\left(\begin{array}{ll} (\because) & \text{Suppose that } ([\alpha_1], [\alpha_2]) \in \pi_1(X, x_0) \oplus \pi_1(Y, y_0). \\ \Rightarrow & \text{Define a loop } \alpha : \mathbf{I} \to X \times Y \\ & \text{by } \alpha(t) = (\alpha_1(t), \alpha_2(t)), \forall t \in \mathbf{I}. \\ \Rightarrow & [\alpha] \in \pi_1(X \times Y, (x_0, y_0)) \text{ and } h([\alpha]) = ([\alpha_1], [\alpha_2]). \\ \Rightarrow & h : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \oplus \pi_1(Y, y_0) \text{ is surjective.} \end{array} \right.$$

(3) $h : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$ is injective.

$$\left(\begin{array}{ll}
(\because) & \text{Let } [\alpha], [\beta] \in \pi_1(X \times Y, (x_0, y_0)) \text{ and suppose } h([\alpha]) = h([\beta]). \\
\Rightarrow & ([p \circ \alpha], [q \circ \alpha]) = ([p \circ \beta], [q \circ \beta]). \\
\Rightarrow & [p \circ \alpha] = [p \circ \beta] \text{ and } [q \circ \alpha] = [q \circ \beta]. \\
\Rightarrow & p \circ \alpha \simeq_{x_0} p \circ \beta \text{ and } q \circ \alpha \simeq_{y_0} q \circ \beta. \\
\Rightarrow & \exists \text{ homotopies } F_1 : \mathbf{I} \times \mathbf{I} \to X \text{ and } F_2 : \mathbf{I} \times \mathbf{I} \to Y \text{ such that} \\
 & F_1(s, 0) = p \circ \alpha(s) \text{ and } F_1(s, 1) = p \circ \beta(s) \text{ for all } s \in \mathbf{I}, \\
 & F_1(0, t) = F_1(1, t) = x_0 \text{ for all } t \in \mathbf{I} \text{ and} \\
 & F_2(s, 0) = q \circ \alpha(s) \text{ and } F_2(s, 1) = q \circ \beta(s) \text{ for all } s \in \mathbf{I}, \\
 & F_2(0, t) = F_2(1, t) = y_0 \text{ for all } t \in \mathbf{I}. \\
\Rightarrow & \text{Define the homotopy } F : \mathbf{I} \times \mathbf{I} \to X \times Y \text{ by} \\
 & F(s, t) = (F_1(s, t), F_2(s, t)) \text{ for all } (s, t) \in \mathbf{I} \times \mathbf{I}. \\
\Rightarrow & F : \alpha \simeq_{(x_0, y_0)} \beta \text{ and hence } [\alpha] = [\beta]. \\
\Rightarrow & h : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \oplus \pi_1(Y, y_0) \text{ is injective.}
\end{array}\right.$$

$\Rightarrow h : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$ is an isomorphism. $\square$

Example 56. (1) Since the torus $\mathbf{T}^2$ is homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^1$, $\pi_1(\mathbf{T}^2) \cong \pi_1(\mathbf{S}^1) \bigoplus \pi_1(\mathbf{S}^1) \cong \mathbb{Z} \bigoplus \mathbb{Z}$.

(2) An n-dimensional torus $\mathbf{T}^n$ is the product of n factors of $\mathbf{S}^1$. $\pi_1(\mathbf{T}^n)$ is isomorphic to the direct sum of n copies of $\mathbb{Z}$.

(3) Since a closed cylinder C is the product of $\mathbf{S}^1$ and $[a, b]$, $\pi_1(C) \cong \pi_1(\mathbf{S}^1) \bigoplus \pi_1([a, b]) \cong \mathbb{Z} \bigoplus \{0\} \cong \mathbb{Z}$.

Theorem 57. *For $n \geq 2$, the n-sphere $\mathbf{S}^n$ is simply connected.*

Proof. Clearly $\mathbf{S}^n$ is path connected.

Let $[\alpha] \in \pi_1(\mathbf{S}^n, a)$, represented by a path $\alpha : \mathbf{I} \to \mathbf{S}^n$.

Let $V_1 = \{x = (x_1, \cdots, x_{n+1}) \in \mathbf{S}^n \mid x_{n+1} < \frac{1}{2}\}$ and

$$V_2 = \{x = (x_1, \cdots, x_{n+1}) \in \mathbf{S}^n \mid x_{n+1} > -\frac{1}{2}\}.$$

$\Rightarrow$ V_1 and V_2 are simply connected open subsets of $\mathbf{S}^n$,

$V_1 \cup V_2 = \mathbf{S}^n$ and $V_1 \cap V_2$ is nonempty and path connected.

Let $a \in V_1 \cap V_2 \subset \mathbf{S}^n$ be the base point of $\mathbf{S}^n$ and

let c_a denote the constant loop at a in V_1 or V_2.

$\Rightarrow$ Since $\alpha : \mathbf{I} \to \mathbf{S}^n$ is continuous, $\{\alpha^{-1}(V_1), \alpha^{-1}(V_2)\}$ is an open covering of the compact metric space $\mathbf{I}$.

$\Rightarrow$ By the Lebesgue Number Theorem, there exists a Lebesgue number $\epsilon > 0$ for the open covering $\{\alpha^{-1}(V_1), \alpha^{-1}(V_2)\}$ of $\mathbf{I}$.

$\Rightarrow$ We can subdivide $\mathbf{I}$ into $0 = s_0 < s_1 < \cdots < s_l = 1$ such that $\alpha([s_{i-1}, s_i]) \subset V_1$ or $\alpha([s_{i-1}, s_i]) \subset V_2$ for each $i = 1, 2, \cdots, l$.

$\Rightarrow$ We can subdivide $\mathbf{I}$ into $0 = s_0 < s_1 < \cdots < s_m = 1$ such that $\alpha([s_{i-1}, s_i]) \subset V_1$ or $\alpha([s_{i-1}, s_i]) \subset V_2$ for each $i = 1, 2, \cdots, m$ and $\alpha(s_1), \cdots, \alpha(s_{m-1}) \in V_1 \cap V_2$ $(\alpha(s_0) = \alpha(s_m) = a \in V_1 \cap V_2)$ by removing the point s_i in the subdivision $s_0 < s_1 < \cdots < s_l$ and renumbering if $\alpha([s_{i-1}, s_{i+1}]) \subset V_1$ or $\alpha([s_{i-1}, s_{i+1}]) \subset V_2$.

$\Rightarrow$ For each $i = 1, 2, \cdots, m$, define a path $\alpha_i : \mathbf{I} \to \mathbf{S}^n$ by

$$\alpha_i(t) = \alpha((1-t)s_{i-1} + ts_i) \text{ for all } t \in \mathbf{I} \text{ and}$$

choose a path γ_i in $V_1 \cap V_2$ such that $\gamma_i(0) = a$ and $\gamma_i(1) = \alpha(s_i)$.

$\Rightarrow$ $\alpha_i : \mathbf{I} \to \mathbf{S}^n$ is continuous for each $i = 1, 2, \cdots, m$,

$\alpha \simeq_* \alpha_1 * \alpha_2 * \cdots * \alpha_m$ in $\mathbf{S}^n$(See Figure 12),

$\overline{\gamma}_i * \gamma_i$ is equivalent to the constant loop at a in V_1 or V_2, and

$\alpha_1 * \overline{\gamma}_1, \gamma_i * \alpha_{i+1} * \overline{\gamma}_{i+1}(1 \leq i \leq m-2)$ and $\gamma_{m-1} * \alpha_m$ are loops

in V_1 or V_2, which are equivalent to the constant loop at a

in V_1 or V_2, respectively, because V_1 and V_2 are simply connected,

i.e., $\alpha_1 * \overline{\gamma}_1 \simeq_a c_a, \gamma_i * \alpha_{i+1} * \overline{\gamma}_{i+1} \simeq_a c_a(1 \leq i \leq m-2)$ and

$\gamma_{m-1} * \alpha_m \simeq_a c_a$ in V_1 or V_2.

$$
\begin{aligned}
\Rightarrow [\alpha] &= [\alpha_1 * \alpha_2 * \cdots * \alpha_m] \\
&= [\alpha_1 * (\overline{\gamma}_1 * \gamma_1) * \alpha_2 * (\overline{\gamma}_2 * \gamma_2) * \cdots * (\overline{\gamma}_{m-1} * \gamma_{m-1}) * \alpha_m] \\
&= [(\alpha_1 * \overline{\gamma}_1) * (\gamma_1 * \alpha_2 * \overline{\gamma}_2) * \cdots * (\gamma_{m-2} * \alpha_{m-1} * \overline{\gamma}_{m-1}) \\
&\quad *(\gamma_{m-1} * \alpha_m)] \\
&= [\alpha_1 * \overline{\gamma}_1] \cdot [\gamma_1 * \alpha_2 * \overline{\gamma}_2] \cdot \cdots \cdot [\gamma_{m-2} * \alpha_{m-1} * \overline{\gamma}_{m-1}] \cdot [\gamma_{m-1} * \alpha_m] \\
&= [c_a] \cdot [c_a] \cdot \cdots \cdot [c_a] \cdot [c_a] = [c_a].
\end{aligned}
$$

$\Rightarrow \pi_1(\mathbf{S}^n, a) = \{[c_a]\} = 1$, the trivial group.

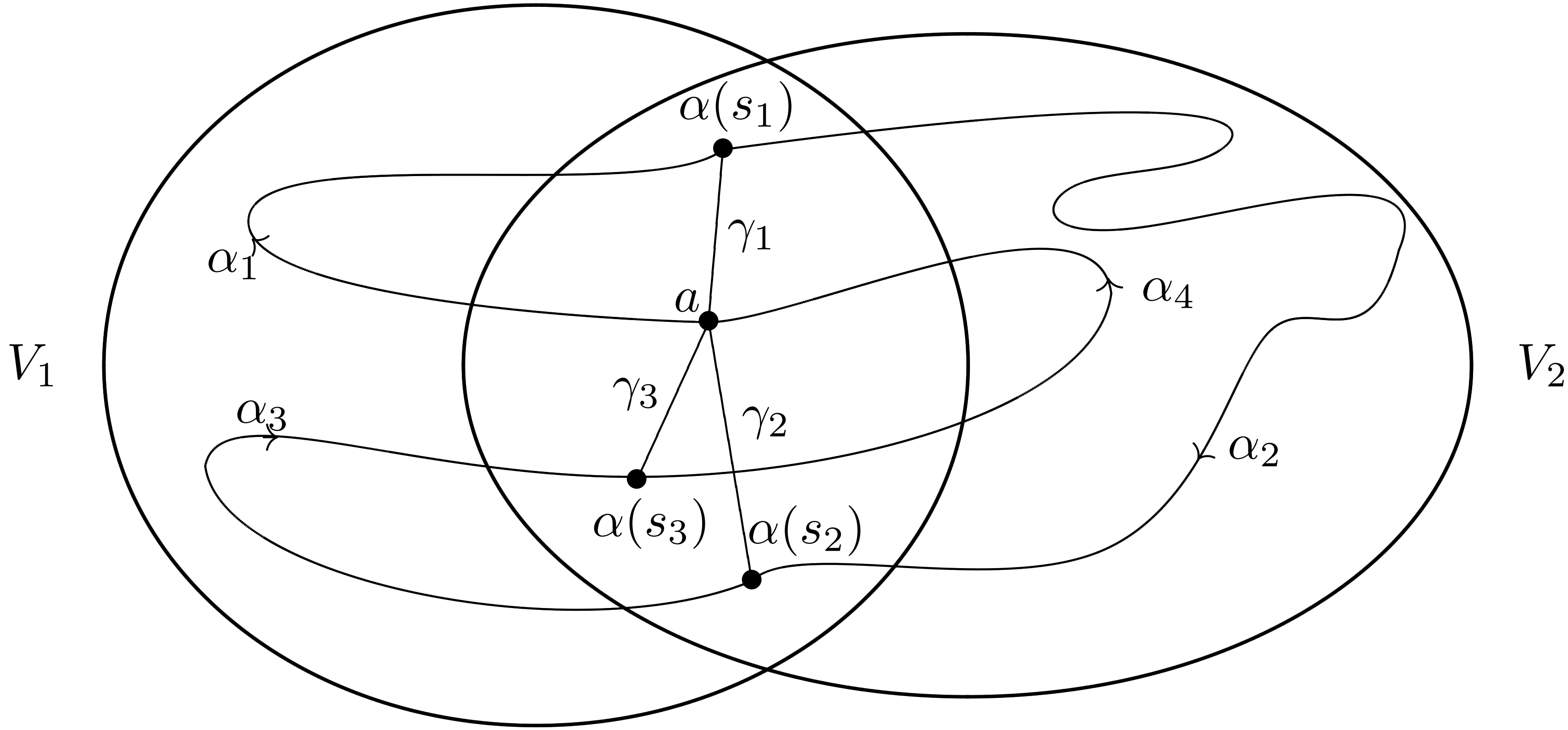


Figure 12: $\alpha \simeq_* (\alpha_1 * \overline{\gamma}_1) * (\gamma_1 * \alpha_2 * \overline{\gamma}_2) * (\gamma_2 * \alpha_3 * \overline{\gamma}_3) * (\gamma_3 * \alpha_4)$

$\square$

Question 58. Where in the preceding proof was the assumption $n \geq 2$ used?

The examples of fundamental groups given in this chapter are all abelian.

There are relatively simple topological spaces which have nonabelian fundamental groups;

• the doubly punctured plane (plane with two points removed) and

• the subspace of the plane consisting of two tangent circles (a figure eight) are two examples.

Showing that the fundamental groups of these spaces are nonabelian would require a considerable departure from the mainstream of this chapter, so these demonstrations will be discussed in the next chapter.

Exercises 5

1. Show that $\mathbf{S}^1$ is a strong deformation retract of the cylinder $\mathbf{S}^1 \times \mathbf{I}$. Use this to prove that the fundamental group of a cylinder is isomorphic to $\mathbb{Z}$.

2. Explain in detail where the assumption $n \geq 2$ was used in the proof of Theorem 57.

3. Generalize the proof of Theorem 57 to prove the following.

   Theorem. Suppose X is a space with an open cover $\{V_a \mid a \in \mathscr{A}\}$ such that

   (1) $\bigcap_{a \in \mathscr{A}} V_a \neq \emptyset$

   (2) V_a is simply connected for each $a \in \mathscr{A}$,

   (3) $V_a \bigcap V_b$ is path connected for $a \neq b$ in $\mathscr{A}$.

   Then X is simply connected.

4. Determine the fundamental group of the Möbius strip.

5. (1) Prove that $\mathbf{S}^{n-1}$ is a strong deformation retract of $\mathbb{R}^n \setminus \{0\}$.
   (2) Use part (1) to prove that the punctured n-space $\mathbb{R}^n \setminus \{p\} (p \in \mathbb{R}^n)$ is simply connected for $n \geq 3$.
6. Let X be a space consisting of two spheres $\mathbf{S}^m$ and $\mathbf{S}^n$ joined at a point, where $m, n \geq 2$. Prove that X is simply connected.
7. Give an example of a simply connected space that is not contractible.
8. Let $X = \mathbb{R}^3 \times \mathbb{R}^3 \setminus \Delta$ be a subspace of $\mathbb{R}^6$, where $\Delta = \{(x, x) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x \in \mathbb{R}^3\}$. Show the following.
   (1) X and the 2-sphere $\mathbf{S}^2$ are of the same homotopy type.
   (2) X is simply connected.
   (3) X is not contractible.
9. Compute $\pi_1(\mathbf{S}^1 \times \mathbf{S}^1 \times \cdots \times \mathbf{S}^1)$.
10. Prove that $\mathbf{S}^1$ and $\mathbf{S}^n$ do not have the same homotopy type for $n \geq 2$. Conclude that $\mathbb{R}^1$ and $\mathbb{R}^n$ are not homeomorphic for $n \geq 2$.

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