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Moduli spaces of framed symplectic and orthogonal bundles on $\mathbb{P}^2$ and the K-theoretic Nekrasov partition functions

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Dedicated to the late Professor Kentaro Nagao

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ABSTRACT

Let K be the compact Lie group $\mathrm{USp}(N/2)$ or $\mathrm{SO}(N,\mathbb{R})$. Let $\mathcal{M}_n^K$ be the moduli space of framed K-instantons over S^4 with the instanton number n. By Donaldson (1984), $\mathcal{M}_n^K$ is endowed with a natural scheme structure. It is a Zariski open subset of a GIT quotient of $\mu^{-1}(0)$, where μ is a holomorphic moment map such that $\mu^{-1}(0)$ consists of the ADHM data.

The purpose of the paper is to study the geometric properties of $\mu^{-1}(0)$ and its GIT quotient, such as complete intersection, irreducibility, reducedness and normality. If $K = \mathrm{USp}(N/2)$ then μ is flat and $\mu^{-1}(0)$ is an irreducible normal variety for any n and even N. If $K = \mathrm{SO}(N,\mathbb{R})$ the similar results are proven for low n and N.

As an application one can obtain a mathematical interpretation of the K-theoretic Nekrasov partition function of Nekrasov and Shadchin (2004).

1. Introduction

1.1

The Nekrasov partition function was formulated by Nekrasov [1] in the 4-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory in physics, especially relevant to the Seiberg–Witten prepotential [2]. It is defined as a generating function of the equivariant integration of the trivial cohomology class 1 over Uhlenbeck partial compactification of the (framed) instanton moduli space on $\mathbb{R}^4$ for all the instanton numbers. Its logarithm turned out to contain the Seiberg–Witten prepotential as the coefficient of the lowest degree (with respect to two variables from the torus action on $\mathbb{R}^4$), which is the remarkable result proven by Nakajima–Yoshioka [3], Nekrasov–Okounkov [4] (both for $\mathrm{SU}(N)$) and Braverman–Etingof [5] (for any gauge group) in completely independent methods. The Nekrasov partition function is an equivariant version of Donaldson-type invariants for $\mathbb{R}^4$, where the ordinary Donaldson invariants are integrals over Uhlenbeck compactifications of instanton moduli spaces on compact 4-manifolds. More generally, a Nekrasov partition function for the theory with matters is defined via the equivariant integration of a cohomology class other than the trivial class. For instance, in the works of Göttsche–Nakajima–Yoshioka [6,7], such partition functions are used to express wallcrossing terms of Donaldson invariants and a relation between them and Seiberg–Witten invariants.

Our interest in this paper is the K-theoretic Nekrasov partition function. It arose from the 5-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory proposed by Nekrasov [8]. Mathematically, one change in its definition is made [1]: the

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equivariant integration of elements in the equivariant K-theory of coherent sheaves instead of cohomology classes. There is a technical, but a subtle problem here. We need a scheme structure on the Uhlenbeck space for the definition, but there are several choices (see [5]). Therefore it is not clear what is the correct definition. For type A, one can use framed moduli spaces of coherent sheaves instead, which are smooth. The K-theoretic Nekrasov partition functions also appeared in mathematics literatures as K-theoretic Donaldson invariants studied in [9,10] in which the K-theory version of Nekrasov conjecture and blowup equations are proven for type A.

Nekrasov–Shadchin [11] took a different approach when the gauge group is of classical type. They defined the partition function using the K-theory class of the Koszul complex which defines the ADHM data of the instanton moduli spaces given in [12].

The main purpose of this paper is to show that Nekrasov–Shadchin's definition coincides with a generating function of the coordinate rings of the instanton moduli spaces for the classical gauge groups. The answer for the gauge group $\mathrm{SU}(N)$ has been already known by a general result of Crawley-Boevey on quiver varieties [13] (see Section 1.6 for the precise motivating questions which we pursue). Our study for the gauge groups $\mathrm{USp}(N/2)$ and $\mathrm{SO}(N,\mathbb{R})$ informs of some geometry of the moduli spaces in algebraic geometry, as the instanton moduli space for $\mathrm{USp}(N/2)$ and $\mathrm{SO}(N,\mathbb{R})$ is isomorphic to the moduli space of (framed) vector bundles with symplectic and orthogonal structures on $\mathbb{P}^2$ respectively (see Section 1.4 for explanation on the scheme structures of moduli spaces).

1.2

Let us fix the notation and explain earlier results in order to state our result precisely. Let K be a compact connected simple Lie group. Let P_K be a principal K-bundle over the 4-sphere S^4. Since $\pi_3(K)\cong\mathbb{Z}$, an integer n uniquely determines the topological type of P_K. Let $\mathcal{M}_n^K$ be the quotient of the space of ASD connections (instantons) by the group of the gauge transformations trivial at infinity $\infty\in S^4$ [14, §5.1.1]. By [15, Table 8.1] $\mathcal{M}_n^K$ is a $\mathcal{C}^\infty$-manifold with $\dim\mathcal{M}_n^K = 4nh^\vee$, where $h^\vee$ is the dual Coxeter number of K. In this paper, we will also consider the case $K=\mathrm{SO}(4,\mathbb{R})$. Its universal cover $\mathrm{Spin}(4)$ is the product $\mathrm{SU}(2)\times\mathrm{SU}(2)$, and hence $\mathrm{SO}(4,\mathbb{R})$-bundles over S^4 are classified by pairs of integers (n_L,n_R). We have $\mathcal{M}^K_{(n_L,n_R)}\cong\mathcal{M}^{\mathrm{SU}(2)}_{n_L}\times\mathcal{M}^{\mathrm{SU}(2)}_{n_R}$, and hence $\dim\mathcal{M}^K_{(n_L,n_R)}=8(n_L+n_R)$.

1.3

Let $K=\mathrm{SU}(N)$. Donaldson [16] showed that $\mathcal{M}_n^K$ is naturally isomorphic to the moduli space of framed holomorphic vector bundles E with $c_2(E)=n$ over $\mathbb{P}^2$, where a framing is a trivialization of E over the line l_∞ at infinity. This was proved by using the ADHM description [12] and the relation between moment maps and geometric invariant theory (GIT). Let $K=\mathrm{USp}(N/2)$ or $\mathrm{SO}(N,\mathbb{R})$ and G be its complexification. Since G-bundles can be identified with rank N vector bundles with symplectic or orthogonal structures, we have the following description of $\mathcal{M}_n^K$.

Let $G'_k := \mathrm{O}(k)$ (resp. $\mathrm{Sp}(k/2)$) if $G=\mathrm{Sp}(N/2)$ (resp. $\mathrm{SO}(N)$). Let $\mathfrak{p}(\mathbb{C}^k)$ be the space of symmetric endomorphisms of $\mathbb{C}^k$ (see (2.1) for the precise definition). The ADHM data are elements $x\in\mathbf{N}$ satisfying $\mu(x)=0$, where $\mathbf{N}:=\mathfrak{p}(\mathbb{C}^k)^{\oplus 2}\oplus\mathrm{Hom}(\mathbb{C}^N,\mathbb{C}^k)$ and μ is the holomorphic moment map defined on $\mathbf{N}$. The G'_k-action on $\mathbf{N}$ induces a G'_k-action on $\mu^{-1}(0)$. Now $\mathcal{M}_n^K$ is the image of the regular locus $\mu^{-1}(0)^{\mathrm{reg}}$ of the GIT quotient $\mu^{-1}(0)\to\mu^{-1}(0)/\!\!/G'_k$. Here the second Chern number k of associated vector bundles is determined from n by the argument in [15, §10] as

$$k=\begin{cases} n & \text{if } G=\mathrm{Sp}(N/2),\\ 2n & \text{if } G=\mathrm{SO}(N), \text{ where } N\geq 5,\\ 4n & \text{if } G=\mathrm{SO}(3).\end{cases} \tag{1.1}$$

The GIT quotient $\mu^{-1}(0)/\!\!/G'_k$ endows the Donaldson–Uhlenbeck partial compactification of $\mathcal{M}_n^K$ with a scheme structure except the case $G=\mathrm{SO}(3)$ (cf. Section 2.6). We call $\mu^{-1}(0)/\!\!/G'_k$ the scheme-theoretic Donaldson–Uhlenbeck partial compactification of $\mathcal{M}_n^K$ unless $G=\mathrm{SO}(3)$. As we will see in Section 1.6, the K-theoretic Nekrasov partition function coincides with the generating function of the coordinate rings of algebraic functions on these Donaldson–Uhlenbeck spaces if $G=\mathrm{Sp}(N/2)$.

1.4

The scheme structure of $\mathcal{M}_n^K$ is given by $\mu^{-1}(0)^{\mathrm{reg}}/G'_k$. Since $\mu^{-1}(0)^{\mathrm{reg}}/G'_k$ is a Zariski-open subset of $\mu^{-1}(0)/\!\!/G'_k$, $\mathcal{M}_n^K$ is a quasi-affine scheme. We notice that $\mathcal{M}_n^K$ may have two scheme structures because it is possible that $\mathrm{Lie}(K)=\mathrm{Lie}(K')$ for a different classical group K' (hence $\mathcal{M}_n^K=\mathcal{M}_n^{K'}$) but $\mathcal{M}_n^K$ and $\mathcal{M}_n^{K'}$ have the different ADHM descriptions. Such pairs of (K,K') are $(\mathrm{SU}(2),\mathrm{USp}(1))$, $(\mathrm{SU}(2),\mathrm{SO}(3,\mathbb{R}))$, $(\mathrm{USp}(2),\mathrm{SO}(5,\mathbb{R}))$ and $(\mathrm{SU}(4),\mathrm{SO}(6,\mathbb{R}))$. A scheme-theoretic isomorphism $\mathcal{M}_n^K\overset{\cong}{\to}\mathcal{M}_n^{K'}$ is induced by sending the associated vector bundles E to itself, $\mathrm{ad}E$, $(\Lambda^2E)_0$ and Λ^2E respectively. The notations in the above are as follows: $\mathrm{ad}E$ is the trace-free part of $\mathrm{End}(E)$, and $(\Lambda^2E)_0$ is the kernel of the symplectic form $\Lambda^2E\to\mathcal{O}$.

In the above isomorphism we used the following two assertions: First by Donaldson's theorem [16], $\mathcal{M}_n^K$ is canonically isomorphic to the moduli space of framed rank N vector bundles E with $c_2(E) = k$ and G-structure. Then the latter space is endowed with a (smooth) scheme structure and the canonical isomorphism is scheme-theoretic. See Appendix B.

1.5

There is a canonical $G \times G'_k$-action on $\mathbf{N}$. The canonical G-action on $\mathbf{N}$ induces a G-action on $\mathcal{M}_n^K$. We define a $(\mathbb{C}^*)^2$-action on $\mathbf{N}$ by $(q_1, q_2).(B_1, B_2, i) = (q_1B_1, q_2B_2, \sqrt{q_1q_2}i)$. More precisely this action is well-defined only on a double cover of $\mathbb{C}^* \times \mathbb{C}^*$, but we follow the convention in physics. It commutes with the $G \times G'_k$-action and induces a $(\mathbb{C}^*)^2$-action on $\mathcal{M}_n^K$. Since i always appears together with i^* for generators in $\mathbb{C}[\mathbf{N}]^{G'_k}$ (see [17, Theorems 2.9.A and 6.1.A]), the $(\mathbb{C}^*)^2$-action is well-defined on $\mathcal{M}_n^K$.

Let T_G be a maximal torus of G. We identify $T_G = (\mathbb{C}^*)^l$, where $l = \operatorname{rank}(G)$. Let $T := T_G \times (\mathbb{C}^*)^2 = (\mathbb{C}^*)^{l+2}$. Then T acts on $\mu^{-1}(0)/\!\!/G'_k$ and $\mathcal{M}_n^K$. Nekrasov–Shadchin [11] defined the K-theoretic instanton partition function Z^K for the gauge group K by an explicit integral formula. Their formula can be interpreted as follows.

Fix the instanton number n. Let us consider μ as a section of the trivial vector bundle $\mathbf{E} = \operatorname{Lie} G'_k \times \mathbf{N}$ over $\mathbf{N}$. We endow $\mathbf{E}$ with a $G'_k \times T$-equivariant structure so that μ is a T-equivariant section. Then μ defines a Koszul complex

$$\Lambda^{\operatorname{rank}\mathbf{E}}\mathbf{E}^\vee \to \cdots \to \Lambda^2\mathbf{E}^\vee \to \mathbf{E}^\vee \to \Lambda^0\mathbf{E}^\vee = \mathcal{O}_{\mathbf{N}}. \tag{1.2}$$

The alternating sum $\sum_i (-1)^i \Lambda^i\mathbf{E}^\vee$ of terms gives an element in $K^{T\times G'_k}(\mathbf{N})$, the Grothendieck group of $T \times G'_k$-equivariant vector bundles over $\mathbf{N}$. We then take a pushforward of the class with respect to the obvious map $p\colon \mathbf{N} \to \mathrm{pt}$. This is not a class in the representation ring $R(T \times G'_k)$ because $\mathbf{N}$ is not proper. However, it is a well-defined class in $\operatorname{Frac}(R(T \times G'_k))$, the fractional field of $R(T \times G'_k)$ by *equivariant integration*: Check first that the origin 0 is the unique fixed point in $\mathbf{N}$ with respect to $T \times T'_k$, where T'_k is a maximal torus of G'_k (see Appendix A). Therefore the pushforward homomorphism p_* can be defined as the inverse of i_* thanks to the fixed point theorem of the equivariant K-theory, where $i\colon \{0\} \to \mathbf{N}$ is the inclusion. In practice, we take the Koszul resolution of the skyscraper sheaf at the origin as above, replacing $\mathbf{E}$ by $\mathbf{N}$, and then divide the class by $\sum_i (-1)^i \Lambda^i\mathbf{N}^\vee$.

In our situation, $\mathbf{N}$ has only positive weights with respect to the factor $(\mathbb{C}^*)^2$ of T (see Appendix A). Therefore the rational function can be expanded as an element in a *completed* ring of $T \times G'_k$-characters

$$\hat{R}(T \times G'_k) := R(T_G \times G'_k)[[q_1^{-1}, q_2^{-1}]],$$

where q_m denote the T-characters of the 1-dimensional representations for the second factor $(\mathbb{C}^*)^2$ of $T = T_G \times (\mathbb{C}^*)^2$. The expansion is considered as a formal character, as the sum of dimensions of weight spaces: Each weight space is finite dimensional and its dimension is given by the coefficient of $f \in \hat{R}(T \times G'_k)$ of the monomial corresponding to the weight.

Next we take the G'_k-invariant part. This is given by an integral over the maximal compact subgroup of T'_k by Weyl's integration formula. Finally taking the generating function for all $n \in \mathbb{Z}_{\geq 0}$ with the formal variable $\mathfrak{q}$, we define the instanton partition function

$$Z^K := \sum_{n\geq 0} \mathfrak{q}^n \sum_i (-1)^i p_*(\Lambda^i\mathbf{E}^\vee)^{G'_k} \in \hat{R}(T)[[\mathfrak{q}]],$$

where $\hat{R}(T) := R(T_G)[[q_1^{-1}, q_2^{-1}]]$.

Explicit computation of Z^K has been performed for small instanton numbers in physics literature. See [18,19] for instance.

1.6

Nekrasov–Shadchin's definition of Z^K is mathematically rigorous, but depends on the ADHM description, hence is not intrinsic in the instanton moduli space $\mathcal{M}_n^K$. We can remedy this problem under the following geometric assumptions:

(1) μ is a flat morphism;
(2) $\mu^{-1}(0)/\!\!/G'_k$ is a normal variety and $\mathcal{M}_n^K$ has complement of codimension ≥ 2 in $\mu^{-1}(0)/\!\!/G'_k$.

Under these assumptions, $\sum_i (-1)^i p_*(\Lambda^i\mathbf{E}^\vee)^{G'_k}$ is the formal character of the ring $\mathbb{C}[\mathcal{M}_n^K]$ of regular functions on $\mathcal{M}_n^K$: First, (1) asserts μ gives a regular sequence so that the Koszul complex (1.2) is a resolution of the coordinate ring $\mathbb{C}[\mu^{-1}(0)]$ of $\mu^{-1}(0)$. Secondly (2) asserts $\mathbb{C}[\mathcal{M}_n^K] = \mathbb{C}[\mu^{-1}(0)/\!\!/G'_k] = \mathbb{C}[\mu^{-1}(0)]^{G'_k}$ by extension of regular functions on $\mathcal{M}_n^K = \mu^{-1}(0)^{\mathrm{reg}}/G'_k$ to $\mu^{-1}(0)/\!\!/G'_k$. As explained in Section 1.4, the scheme structure of $\mathcal{M}_n^K$ is independent of the choice of the ADHM description.

When $K = SU(N)$, (1), (2) are proved by Crawley-Boevey [20] in a general context of quiver varieties. Moreover the Gieseker space $\widetilde{\mathcal{M}}_n^N$ of framed rank N torsion free sheaves $\mathcal{E}$ on $\mathbb{P}^2$ with $c_2(\mathcal{E}) = n$ gives a resolution of singularities of $\mu^{-1}(0)/\!\!/G'_k$. The higher direct image sheaves $R^l\widetilde{p}_*\mathcal{O}_{\widetilde{\mathcal{M}}_n^N} = 0$ for $l \geq 1$, where $\widetilde{p}\colon \widetilde{\mathcal{M}}_n^N \to \mathrm{pt}$ is the obvious map. For, $\widetilde{\mathcal{M}}_n^N$ is symplectic and thus the canonical sheaf $\mathcal{K}_{\widetilde{\mathcal{M}}_n^N}$ is isomorphic to $\mathcal{O}_{\widetilde{\mathcal{M}}_n^N}$. By the Grauert–Riemenschneider vanishing theorem, $R^l\pi_*\mathcal{K}_{\widetilde{\mathcal{M}}_n^N} = 0$ for $l \geq 1$, where π denotes the resolution of singularities. Now the vanishing of higher direct image sheaves comes from the E_2-degeneration of the spectral sequence $R^l p_* R^m \pi_* \Rightarrow R^{l+m}\widetilde{p}_*$, where $p: \mu^{-1}(0)/\!\!/G'_k \to \mathrm{pt}$ is the obvious affine map. Therefore $\mathbb{C}[\widetilde{\mathcal{M}}_n^N] \cong \mathbb{C}[\mu^{-1}(0)/\!\!/G'_k]$. The T-action lifts to $\widetilde{\mathcal{M}}_n^N$, where fixed points are parametrized by N-tuples of Young diagrams corresponding to direct sums of monomial ideal sheaves. Therefore the character of $\mathbb{C}[\widetilde{\mathcal{M}}_n^N]$ is given by a sum over N-tuples of Young diagrams, which is the original definition of the instanton partition function in [1]. See [9] for more detail on the latter half of this argument.

1.7

A goal of this paper is to prove (1), (2) for $K = \mathrm{USp}(N/2)$ (see Theorem 3.1.) A key of the proof is a result of Panyushev [21], which gives the flatness of μ for $N = 0$.

We also study $\mu^{-1}(0)$ for $K = \mathrm{SO}(N, \mathbb{R})$ for $(N, k) = (2, k), (N, 2)$ or $(3, 4)$ (see Theorems 4.1–4.3 respectively). The case $N = 2$ is less interesting since there are no $\mathrm{SO}(2, \mathbb{R})$-instantons except for $k = 0$. But $\mu^{-1}(0)$ *does* make sense, hence the study of its properties is a mathematically meaningful question. We show that μ is not flat, i.e., (1) is not true in this case. Similarly there is no instanton for $(N, k) = (3, 2)$. We will see that μ is flat, but $\mu^{-1}(0)/\!\!/G'_k$ with reduced scheme structure is isomorphic to $\mathbb{C}^2$, so (2) is false in this case. In the case $(N, 2)$ with $N \geq 4$, we prove that $\mu^{-1}(0)/\!\!/G'_k$ is isomorphic to the product of $\mathbb{C}^2$ and the closure $\mathbf{P}$ of the minimal nilpotent $\mathrm{O}(N)$-orbit. When $(N, k) = (3, 4)$, we show that μ is flat and $\mu^{-1}(0)/\!\!/G'_k$ is a union of two copies of $\mathbb{C}^2 \times \mathbf{P}$ meeting along $\mathbb{C}^2 \times \{0\}$. Here $\mathbf{P}$ is the minimal (= regular) nilpotent $\mathrm{O}(3)$-orbit closure, which is isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$. In particular, (2) is false. The isomorphism $\mathcal{M}_n^{\mathrm{SU}(2)} \cong \mathcal{M}_n^{\mathrm{SO}(3)}$ implies that both $\mathcal{M}_n^{\mathrm{SU}(2)}$ and $\mathcal{M}_n^{\mathrm{SO}(3)}$ are $\mathbb{C}^2 \times (\mathbf{P}\backslash\{0\})$ for $n = 1$. However their ADHM descriptions are different, as it is $\mathbb{C}^2 \times \mathbf{P}$ for SU(2). This phenomenon happens only for SO(3). See Theorem 2.6. These examples show that the definition of Z^K depends on the ADHM description, and hence must be studied with care.

The author plans to study the case $(N \geq 4, k \geq 4)$ in the near future.

1.8

The paper is organized as follows.

In Section 2 we give the set-up for the entire part of the paper. Specifically we set up linear algebra of vector spaces with nondegenerate forms in Sections 2.1–2.3. In Section 2.4 we identify the closed G'_k-orbits in $\mathbf{N}$. By this identification we see that $\mathcal{M}_n^K$ is the G'_k-orbit space of stable–costable representations in $\mu^{-1}(0)$. In Section 2.6 we stratify $\mu^{-1}(0)/\!\!/G'_k$ with $\mathcal{M}_n^K$ as a stratum. Each stratum is isomorphic to a product of $\mathcal{M}_{n'}^K$ and a symmetric product of $\mathbb{C}^2$ for some $n' \leq n$.

In Sections 3 and 4 we state Theorems 3.1, 4.1–4.3 which describes the geometry of $\mu^{-1}(0)$ when $K = \mathrm{USp}(N/2)$ and $\mathrm{SO}(N, \mathbb{R})$.

In Section 5 we explain Kraft–Procesi's classification theory of nilpotent pairs. In our study of $\mu^{-1}(0)$, their theory is quite useful to see contribution from the factor $\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^k)$ of $\mathbf{N}$. In the case $k = 2$, the geometry of $\mu^{-1}(0)$ is immediately deduced from Kraft–Procesi's theory on $\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^k)$ since $\mathfrak{p}(\mathbb{C}^2)$ consists of the scalars.

In Section 6 and Section 7 we will see how Kraft–Procesi's theory is applied to the cases $k = 2$ and $N = 2$. This will prove Theorem 4.1 and a part of Theorem 4.2.

In Section 8 we prove Theorem 4.3 (the case $(N, k) = (3, 4)$). The proof involves more than Kraft–Procesi's theory since the pairs $(B_1, B_2) \in \mathfrak{p}(\mathbb{C}^4)^{\oplus 2}$ are no more commuting pairs. Since $[B_1, B_2] \neq 0$ in general, the study of $\mu^{-1}(0)$ does not solely come from the factor $\mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^4)$. So we study the commutator map $\mathfrak{p}(\mathbb{C}^4)^{\oplus 2} \to \mathrm{Lie}(\mathrm{Sp}(2)), (B_1, B_2) \mapsto [B_1, B_2]$.

In Section 9 we finish the proof of Theorem 4.2 (the case $(N, k) = (2, 4)$) based on the study of the commutator map in the above.

2. Preliminary

We set up conventions and notation and review basic material for an entire part of the paper.

We are working over $\mathbb{C}$. Vector spaces are all finite dimensional and schemes are of finite type. We say that a morphism between schemes is irreducible (resp. normal and Cohen–Macaulay) if all the nonempty fibres are irreducible (resp. normal and Cohen–Macaulay). If $\mathcal{M}$ is a scheme then $\mathcal{M}^{\mathrm{sm}}$ and $\mathcal{M}^{\mathrm{sing}}$ are the smooth locus and the singular locus of $\mathcal{M}$ respectively.

Let G be an algebraic group and $\mathfrak{g} := \mathrm{Lie}(G)$ the Lie algebra of G. Let $\mathcal{M}$ be a G-scheme. Let $G^x := \{g \in G \,|\, g.x = x\}$ the stabilizer subgroup of $x \in \mathcal{M}$. Let $\mathfrak{g}^x := \mathrm{Lie}(G^x)$.

2.1. The right adjoint

Let V_1 and V_2 be vector spaces with nondegenerate bilinear forms $(\ ,\ )_{V_1}$ and $(\ ,\ )_{V_2}$ respectively. Then for any $i \in \mathrm{Hom}(V_1, V_2)$, we have the right adjoint $i^* \in \mathrm{Hom}(V_2, V_1)$, i.e. $(v, i^*w)_{V_1} = (iv, w)_{V_2}$, where $v \in V_1$ and $w \in V_2$. The map

$$* : \mathrm{Hom}(V_1, V_2) \to \mathrm{Hom}(V_2, V_1), \quad i \mapsto i^*$$

is a $\mathbb{C}$-linear isomorphism. Further if V_3 is a vector space with a nondegenerate bilinear form then for $i \in \mathrm{Hom}(V_1, V_2)$, $j \in \mathrm{Hom}(V_2, V_3)$, we have $(ji)^* = i^*j^*$.

2.2. Anti-symmetric and symmetric forms

Let V be a vector space of dimension k with a nondegenerate form $(\ ,\ )_V$. Let $\varepsilon \in \{-1,+1\}$. Let $(\ ,\ )_\varepsilon$ be a nondegenerate bilinear form $(u,v)_\varepsilon = \varepsilon(v,u)_\varepsilon$ on V. If $\varepsilon = +1$ (resp. -1) then $(\ ,\ )_\varepsilon$ is an orthogonal form (resp. symplectic form). We say V is orthogonal (resp. symplectic) if $\varepsilon = +1$ (resp. $\varepsilon = -1$).

We decompose $\mathfrak{gl}(V) = \mathfrak{k}(V) \oplus \mathfrak{p}(V)$ as a vector space, where

$$\begin{aligned} \mathfrak{k}(V) &:= \{X \in \mathfrak{gl}(V) \mid (Xu,v)_\varepsilon = -(u,Xv)_\varepsilon\} = \{X \in \mathfrak{gl}(V) \mid X^* = -X\} \\ \mathfrak{p}(V) &:= \{X \in \mathfrak{gl}(V) \mid (Xu,v)_\varepsilon = (u,Xv)_\varepsilon\} = \{X \in \mathfrak{gl}(V) \mid X^* = X\}. \end{aligned} \tag{2.1}$$

Let $\mathfrak{k} := \mathfrak{k}(V)$ and $\mathfrak{p} := \mathfrak{p}(V)$ for short. The followings are immediate to check:

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}, \qquad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}. \tag{2.2}$$

Let $G(V) := \{g \in \mathrm{GL}(V) \mid (gv,gv')_\varepsilon = (v,v')_\varepsilon \text{ for all } v,v' \in V\}$. Then $\mathfrak{k} = \mathfrak{g}(V)$, where $\mathfrak{g}(V) := \mathrm{Lie}\, G(V)$. We have $\dim G(V) = \dim \mathfrak{k} = \frac{1}{2}k(k-\varepsilon)$. So $\dim \mathfrak{p} = \frac{1}{2}k(k+\varepsilon)$.

Remark 2.1. If V is a symplectic vector space of dimension 2 then $\mathfrak{p}(V)$ consists of scalars.

If $\varepsilon = -1$ then $G(V)$ is denoted by $\mathrm{Sp}(V)$ (called the symplectic group) with the Lie algebra $\mathfrak{sp}(V)$. If $\varepsilon = +1$ then $G(V)$ is denoted by $\mathrm{O}(V)$ (called the orthogonal group) with the Lie algebra $\mathfrak{o}(V)$. Unless no confusion arises, we shorten the notation as Sp, $\mathfrak{sp}$, O and $\mathfrak{o}$.

2.3

It is direct to check that if $(\ ,\ )_{V_1} = (\ ,\ )_\varepsilon$ and $(\ ,\ )_{V_2} = (\ ,\ )_{-\varepsilon}$ for some $\varepsilon \in \{-1,+1\}$ then $** = -\mathrm{Id}$. If both $(\ ,\ )_{V_1}$ and $(\ ,\ )_{V_2}$ are $(\ ,\ )_\varepsilon$ for the same $\varepsilon \in \{-1,+1\}$ then $** = \mathrm{Id}$. In particular if V is symplectic and W is orthogonal then for $i \in \mathrm{Hom}(W,V)$, $ii^* \in \mathfrak{sp}$ by (2.1) since $(ii^*)^* = i^{**}i^* = -ii^*$. Similarly $i^*i \in \mathfrak{o}$ for $i \in \mathrm{Hom}(W,V)$.

2.4. Orbit-closedness and semisimplicity of quiver representations

Let

$$\mathbf{M} := \mathrm{Hom}(V,V)^{\oplus 2} \oplus \mathrm{Hom}(V,W) \oplus \mathrm{Hom}(V,W)$$

where V and W are vector spaces. An element of $\mathbf{M}$ is called an ADHM quiver representation. Let $W \cong \mathbb{C}^N$ be any isomorphism. Then we have an obvious ($\mathrm{GL}(V)$-equivariant) linear isomorphism

$$c\colon \mathbf{M} \to \mathrm{Hom}(V,V)^{\oplus 2} \oplus \mathrm{Hom}(V,\mathbb{C})^{\oplus N} \oplus \mathrm{Hom}(V,\mathbb{C})^{\oplus N}.$$

We identify the target space of c with the space of representations of the deframed quiver given in Fig. 1 (cf. Crawley-Boevey's trick [13, p. 57]). The number of arrows from the bottom vertex to the top is N, and the number of arrows from the top vertex to the bottom is also N.

Let $\varepsilon \in \{-1,+1\}$. In the rest of paper we fix V and W as a k-dimensional vector space with $(\ ,\ )_\varepsilon$ and an N-dimensional vector space with $(\ ,\ )_{-\varepsilon}$ respectively. Let

$$\mathbf{N} := \left\{(B_1,B_2,i,j) \in \mathbf{M} \mid B_1 = B_1^*, B_2 = B_2^*, j = i^*\right\}.$$

It is clear that $\mathbf{N}$ is isomorphic to the one we used in Section 1.3 by the projection.

We use the notation $\overset{\perp}{\oplus}$ for a direct sum of vector spaces orthogonal with respect to the given nondegenerate forms.

It is well-known that $x \in \mathbf{M}$ has a closed $\mathrm{GL}(V)$-orbit if and only if $c(x)$ is semisimple (cf. [22]). We have the corresponding result for $G(V)$ as below, whose proof is left as an exercise for the readers.

Theorem 2.2. *Let $x := (B_1,B_2,i,j) \in \mathbf{N}$. Then the following properties are equivalent.*

(a) *$c(x)$ is semisimple (i.e. the direct sum of simple quiver representations).*
(b) *there exists a decomposition*

$$V = V^s \overset{\perp}{\oplus} \overset{\perp}{\bigoplus}_a V_a \overset{\perp}{\oplus} \overset{\perp}{\bigoplus}_b (V_b \oplus V_b')$$

such that
(1) *V_b and V_b' are dual isotropic subspaces of V for each index b;*
(2) *$c(B_1|_{V^s}, B_1|_{V^s}, i, j|_{V^s})$, $(B_1|_{V_a}, B_2|_{V_a}, 0, 0)$, $(B_1|_{V_b}, B_2|_{V_b}, 0, 0)$ and $(B_1|_{V_b'}, B_2|_{V_b'}, 0, 0)$ are simple quiver representations.*

(c) *$G(V).x$ is closed in $\mathbf{N}$.*

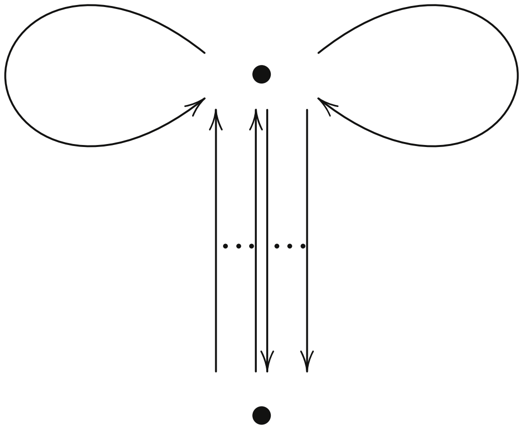

Fig. 1. A deframed quiver of the ADHM quiver.

We remark that $(B_1|_{V_b}, B_2|_{V_b}, 0, 0)$ and $(B_1|_{V'_b}, B_2|_{V'_b}, 0, 0)$ in the above statement are dual to each other.

Let us recall the two GIT stability conditions for the $\mathrm{GL}(V)$-actions on $\mathbf{M}$ (cf. [23, Chap. 2]).

Definition 2.3. $(B_1, B_2, i, j) \in \mathbf{M}$ is *stable* (resp. *costable*) if the following condition holds:

(1) (*stability*) there is no subspace $S \subsetneq V$ such that $B_1(S) \subset S$, $B_2(S) \subset S$ and $\mathrm{Im}\, i \subset S$,
(2) (*costability*) there is no nonzero subspace $T \subset V$ such that $B_1(T) \subset T$, $B_2(T) \subset T$ and $T \subset \mathrm{Ker}\, j$.

Note that $x \in \mathbf{M}$ is stable and costable if and only if $c(x)$ is simple when $W \neq 0$.

Let $*_{\mathbf{M}}: \mathbf{M} \to \mathbf{M}$, $(B_1, B_2, i, j) \mapsto (B_1^*, B_2^*, -j^*, i^*)$. Then $\mathbf{N}$ is the $*_{\mathbf{M}}$-invariant subspace of $\mathbf{M}$. If $x \in \mathbf{M}$ is stable (resp. costable) then $*_{\mathbf{M}}(x)$ is costable (resp. stable). Therefore stability and costability are equivalent on $\mathbf{N}$. In particular, $x \in \mathbf{N}$ is stable if and only if $c(x)$ is simple when $W \neq 0$.

2.5. ADHM description for framed Sp-bundles and SO-bundles

We describe the framed Sp-bundles and SO-bundles in terms of ADHM data (cf. [16], see also [24]). Let $V := \mathbb{C}^k$ with $(\ ,\ )_\varepsilon$ and $W := \mathbb{C}^N$ with $(\ ,\ )_{-\varepsilon}$, where $\varepsilon \in \{-1, +1\}$ and $N \geq 1$. Let $\mu: \mathbf{N} \to \mathfrak{g}(V)$ be the moment map given by $(B_1, B_2, i, j) \mapsto [B_1, B_2] + ij$.

Let $\mu^{-1}(0)^{\mathrm{reg}}$ be the (regular) locus of stable–costable quiver representations in $\mu^{-1}(0)$. We have $\mu^{-1}(0)^{\mathrm{reg}} \subset \mu^{-1}(0)^{\mathrm{sm}}$ since if $x \in \mathbf{N}$ is stable then the differential $d\mu_x$ is surjective. By Theorem 2.2 the image of $\mu^{-1}(0)^{\mathrm{reg}}$ of the GIT quotient map $\mu^{-1}(0) \to \mu^{-1}(0)/\!\!/G(V)$ is a Zariski open subset of $\mu^{-1}(0)/\!\!/G(V)$ and it is a $G(V)$-orbit space $\mu^{-1}(0)^{\mathrm{reg}}/G(V)$. In particular $\mu^{-1}(0)^{\mathrm{reg}}/G(V)$ is a smooth quasi-affine scheme. In fact it is also irreducible by [25, Propositions 2.24 and 2.25].

Remark 2.4. By Section 1.4, the scheme structures of $\mathcal{M}_n^{\mathrm{SO}(N,\mathbb{R})}$ and $\mathcal{M}_n^{\mathrm{USp}(N/2)}$ are defined as the quasi-affine scheme $\mu^{-1}(0)^{\mathrm{reg}}/G(V)$, where the relation between n and k are given there.

Definition 2.5. Let $\varepsilon = -1$ (resp. $\varepsilon = +1$). We say $x = (B_1, B_2, i, j) \in \mu^{-1}(0)$ is a SO-*datum* (resp. Sp-*datum*) if $[B_1, B_2] + ij = 0$.

2.6. Stratification

Let $\mu_k: \mathbf{N} \to \mathfrak{g}(V)$, $(B_1, B_2, i, j) \mapsto [B_1, B_2] + ij$ (the holomorphic moment map). To emphasize $k = \dim V$, we use G_k for $G(V)$. It was denoted by G'_k in Section 1. Let $x := (B_1, B_2, i, j) \in \mu_k^{-1}(0)$. Assume $G_k.x$ is closed in $\mathbf{N}$. By Theorem 2.2, $(B_1|_{V^s}, B_2|_{V^s}, i, j|_{V^s})$ corresponds to a framed vector bundle, and $(B_1|_{V_a}, B_2|_{V_a}, 0, 0)$, $(B_1|_{V_b}, B_2|_{V_b}, 0, 0)$ and $(B_1|_{V'_b}, B_2|_{V'_b}, 0, 0)$ are all commuting pairs in $\mathfrak{p}(V)$. Let us focus on commuting pairs here. We simplify the situation: $B_1, B_2 \in \mathfrak{p}(V)$ with $[B_1, B_2] = 0$, where V is orthogonal or symplectic. By the semisimplicity (Theorem 2.2), B_1 and B_2 are simultaneously diagonalizable. Therefore all V_a, V_b, and V'_b of Theorem 2.2(b) are 1-dimensional. When V is symplectic, V_a does not appear and $B_1|_{V_b \oplus V'_b}$, $B_2|_{V_b \oplus V'_b}$ are scalars, as $\mathfrak{p}(V) = \mathbb{C}$ if $\dim V = 2$. When V is orthogonal, the index set of b can be absorbed in the index set of a. We have the summary as follows.

Theorem 2.6. *Let $S^n\mathbb{A}^2$ be the nth symmetric product of $\mathbb{A}^2$.*

(1) *Suppose V is symplectic. Then there exists a canonical set-theoretical bijection*

$$\mu_k^{-1}(0)/\!\!/G_k = \coprod_{0\le k'\le k} \mu_{k'}^{-1}(0)^{\mathrm{reg}}/G_{k'} \times S^{\frac{k-k'}{2}}\mathbb{A}^2.$$

(2) *Suppose V is orthogonal. Then there exists a canonical set-theoretical bijection*

$$\mu_k^{-1}(0)/\!\!/G_k = \coprod_{0\le k'\le k} \mu_{k'}^{-1}(0)^{\mathrm{reg}}/G_{k'} \times S^{k-k'}\mathbb{A}^2.$$

Note that this stratification is nothing but the one of the Uhlenbeck space except the case $G = \mathrm{SO}(3)$ (cf. [14, p. 158, §4.4.1]). For $G = \mathrm{SO}(3)$, $\mu_k^{-1}(0)/\!\!/G_k$ is different from the Uhlenbeck space, where the symmetric product is $S^{\frac{k-k'}{4}}\mathbb{A}^2$ by the formula (1.1).

Since $\mu_k^{-1}(0)^{\mathrm{reg}}/G_k$ is a free quotient (unless $\mu_k^{-1}(0)^{\mathrm{reg}} = \emptyset$), we have $\dim \mu_k^{-1}(0)^{\mathrm{reg}}/G_k = \dim \mathbf{N} - 2\dim G_k$. Using $\dim\mathfrak{p} - \dim\mathfrak{t} = k$ if V is orthogonal, and $\dim\mathfrak{p} - \dim\mathfrak{t} = -k$ if V is symplectic, we have

$$\dim \mu_k^{-1}(0)^{\mathrm{reg}}/G_k = \begin{cases} k(N-2) & \text{if } V \text{ is symplectic,} \\ k(N+2) & \text{if } V \text{ is orthogonal,}\end{cases} \tag{2.3}$$

whenever $\mu_k^{-1}(0)^{\mathrm{reg}} \neq \emptyset$. Therefore we get the dimension of the strata. Hence by Section 1.3, we obtain the following.

Lemma 2.7. (1) *Assume V is symplectic. Then $\mu_{k'}^{-1}(0)^{\mathrm{reg}}/G_{k'} \times S^{\frac{k-k'}{2}}\mathbb{A}^2$ is nonempty if and only if either $N = 3$ and $k' \in 4\mathbb{Z}_{\ge 0}$, or $N \ge 4$ and $k' \in 2\mathbb{Z}_{\ge 0}$. If it is nonempty, it is of dimension $k'(N-2) + (k-k')$.*

(2) *If V is orthogonal then $\mu_{k'}^{-1}(0)^{\mathrm{reg}}/G_{k'} \times S^{k-k'}\mathbb{A}^2$ is nonempty and of dimension $k'(N+2) + 2(k-k')$.* □

Remark 2.8. (1) Suppose V is symplectic. By $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$, if $N = 3$ and $k \in 4\mathbb{Z}_{\ge 0}$ then $\mu_k^{-1}(0)^{\mathrm{reg}}/G_k$ is irreducible (Section 1.3). Hence, the strata of $\mu_4^{-1}(0)/\!\!/G_4$ are $\mu_4^{-1}(0)^{\mathrm{reg}}/G_4$ and $S^2\mathbb{A}^2$, both of which are irreducible varieties of dimension 4. See an alternative proof in Theorem 4.3(3).

By $\mathrm{Spin}(4) \cong \mathrm{SU}(2)\times\mathrm{SU}(2)$, if $N = 4$ and $k \in 2\mathbb{Z}_{\ge 0}$ then $\mu_k^{-1}(0)^{\mathrm{reg}}/G_k$ has the $(k/2+1)$ irreducible components.

(2) We will prove in Theorem 3.1 that all the strata in Lemma 2.7(2) are irreducible normal.

3. Geometry of the moduli spaces of Sp-data

Let V be an orthogonal vector space of dimension k, and W be a symplectic vector space of dimension $N \ge 2$. Let $G := G(V) = \mathrm{O}(V)$.

The main theorem of this section is the following.

Theorem 3.1. *μ is normal, irreducible, flat, reduced and surjective. Hence $\mu^{-1}(0)$ is a normal irreducible variety of dimension $k(N+2) + k(k-1)/2$. And the Donaldson–Uhlenbeck space $\mu^{-1}(0)/\!\!/G$ is a normal irreducible variety of dimension $k(N+2)$, and $(\mu^{-1}(0)/\!\!/G)\backslash(\mu^{-1}(0)^{\mathrm{reg}}/G)$ is of codimension ≥ 2 in $\mu^{-1}(0)/\!\!/G$.*

The proof will appear in Section 3.2.

Corollary 3.2. *The canonical embedding induces the equality of the ring of regular functions: $\mathbb{C}[\mu^{-1}(0)/\!\!/G] = \mathbb{C}[\mu^{-1}(0)^{\mathrm{reg}}/G]$.*

3.1

Let $m\colon \mathfrak{p}\times\mathfrak{p} \to \mathfrak{t}$ by $(B_1, B_2) \mapsto [B_1, B_2]$.

Theorem 3.3. *For any $X \in \mathfrak{t}$, $m^{-1}(X)$ is an irreducible normal variety and a complete intersection in $\mathfrak{p}\times\mathfrak{p}$ of the $\dim\mathfrak{t}$ hypersurfaces. And the smooth locus of $m^{-1}(X)$ is the locus of $x \in m^{-1}(X)$ such that the differential dm_x is surjective.*

Proof. By [21, (3.5)(1)] and [21, Theorem 3.2], $m^{-1}(X)$ is an irreducible complete intersection. See also [26] for a different proof. The normality is due to [21, Corollary 4.4].

For $x \in m^{-1}(X)$, x is a smooth point of $m^{-1}(X)$ if and only if $\dim T_x m^{-1}(X) = \dim m^{-1}(X)$ ($=2\dim\mathfrak{p} - \dim\mathfrak{t}$) if and only if dm_x is surjective (cf. [21, the proof of Proposition 4.2]). □

For $B \in \mathfrak{gl}$, $\mathfrak{gl}^B = \{A \in \mathfrak{gl} \mid [A, B] = 0\}$. Let $\mathfrak{p}^B := \mathfrak{gl}^B \cap \mathfrak{p}$.

Let $\mathfrak{gl}_l := \{B \in \mathfrak{gl} \mid \dim\mathfrak{gl}^B = l\}$ and $\mathfrak{p}_l := \{B \in \mathfrak{p} \mid \dim\mathfrak{p}^B = l\}$ $(l \in \mathbb{Z}_{\ge 0})$.

Lemma 3.4 ([27, Prop. 5]). *For any $B \in \mathfrak{p}$, $\dim\mathfrak{p}^B - \dim\mathfrak{t}^B = k$.* □

Lemma 3.5. *For any $l \in \mathbb{Z}_{\geq 0}$, $\mathfrak{p}_{\geq l}$ is a closed subvariety of $\mathfrak{p}$. And $\mathfrak{p}_k$ is Zariski open dense in $\mathfrak{p}$.*

Proof. By [28, p. 7], for the conjugation action of $\mathrm{GL}(V)$ on $\mathfrak{gl}$, the map $A \in \mathfrak{gl} \mapsto \dim \mathrm{GL}(V)^A = \dim \mathfrak{gl}^A$ is upper-semicontinuous. Thus $\mathfrak{gl}_l$ is a locally closed subvariety of $\mathfrak{gl}$. By (2.2), for any $B \in \mathfrak{p}$, $\mathfrak{gl}^B = \mathfrak{p}^B \oplus \mathfrak{t}^B$. By Lemma 3.4, for $B \in \mathfrak{p}$, $B \in \mathfrak{p}_l$ if and only if $B \in \mathfrak{gl}_{2l-k}$. Since $\mathfrak{p}$ is a closed subvariety of $\mathfrak{gl}$, $\mathfrak{p}_l = \mathfrak{p} \cap \mathfrak{gl}_{2l-k}$ is a locally closed subvariety of $\mathfrak{p}$.

The second claim comes from the upper-semicontinuity, $\mathfrak{p}_k = \mathfrak{p} \cap \mathfrak{gl}_k \neq \emptyset$ and $\mathfrak{gl}_l = \emptyset$ for $l < k$. □

Let $\pi_i : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ be the ith projection ($i = 1, 2$).

Note that for $X = 0$, the singular locus of $m^{-1}(0)$ is of codimension ≥ 3 by [26, p. 6414, Lemma] (cf. [21, Theorem (4.3)]).

Let M be a smooth variety, and $\phi : M \to \mathfrak{t}$ be a morphism. The fibre-product $(\mathfrak{p} \times \mathfrak{p}) \times_{\mathfrak{t}} M$ is cut out by the $\dim \mathfrak{t}$-equations $[B_1, B_2] - \phi(x) = 0$.

Let $S := \{x \in \mathfrak{p} \times \mathfrak{p} \mid dm_x \text{ is surjective}\}$.

Proposition 3.6. *Let M be a smooth irreducible variety and $\phi : M \to \mathfrak{t}$ be a morphism. Then $(\mathfrak{p} \times \mathfrak{p}) \times_{\mathfrak{t}} M$ is an irreducible normal variety and a complete intersection in $\mathfrak{p} \times \mathfrak{p} \times M$ of $\dim \mathfrak{t}$ hypersurfaces.*

Proof. Let $\tilde{m} : (\mathfrak{p} \times \mathfrak{p}) \times_{\mathfrak{t}} M \to M$ be the projection. Note that $\tilde{m}$ is surjective since so is m by Theorem 3.3. For each $x \in M$, $\tilde{m}^{-1}(x) \cong m^{-1}(\phi(x))$, which means any fibre dimension of $\tilde{m}$ is $2\dim \mathfrak{p} - \dim \mathfrak{t}$. Therefore $\dim(\mathfrak{p} \times \mathfrak{p}) \times_{\mathfrak{t}} M = 2\dim \mathfrak{p} - \dim \mathfrak{t} + \dim M$ which proves $(\mathfrak{p} \times \mathfrak{p}) \times_{\mathfrak{t}} M$ is a complete intersection.

The complete intersection property implies that every irreducible component of $(\mathfrak{p} \times \mathfrak{p}) \times_{\mathfrak{t}} M$ has the same dimension. Suppose $(\mathfrak{p} \times \mathfrak{p}) \times_{\mathfrak{t}} M$ is reducible. There are two distinct irreducible components Z, Z'. The restrictions to Z, Z' of $\tilde{m}$ are dominant due to the equi-dimensionality of the fibres of $\tilde{m}$ and irreducibility of M. For a generic element $x \in M$, $\tilde{m}^{-1}(x) \cap Z$ and $\tilde{m}^{-1}(x) \cap Z'$ are distinct closed subschemes in $\tilde{m}^{-1}(x)$ with the same dimension $2\dim \mathfrak{p} - \dim \mathfrak{t}$. But this contradicts the fact that $m^{-1}(\phi(x))$ ($\cong \tilde{m}^{-1}(x)$) is an irreducible scheme with dimension $2\dim \mathfrak{p} - \dim \mathfrak{t}$ (Theorem 3.3).

By the upper-semicontinuity, S is an open subvariety of $\mathfrak{p} \times \mathfrak{p}$. By Theorem 3.3, $m^{-1}(X) \backslash S$ is of codimension ≥ 2 in $m^{-1}(X)$ for each $X \in \mathfrak{t}$. Thus $(\mathfrak{p} \times \mathfrak{p}) \backslash S$ is of codimension ≥ 2 in $\mathfrak{p} \times \mathfrak{p}$. By the smooth base change ([29, III.10.1 (b)]), $\tilde{m}_{S \times_{\mathfrak{t}} M}$ is a smooth morphism. By [29, III.10.4], $S \times_{\mathfrak{t}} M$ is smooth. By Serre's criterion ([29, Proposition II.8.23]), we get the normality. □

We have further description of the smooth locus of each fibre of m as follows, which is not necessary in the proof of Theorem 3.1.

Proposition 3.7. *Let $X \in \mathfrak{t}$.*

(1) *$m| : m^{-1}(X) \cap (\pi_1^{-1}(\mathfrak{p}_k) \cup \pi_2^{-1}(\mathfrak{p}_k)) \to \mathfrak{t}$ is a smooth morphism.*

(2) *The codimension of $m^{-1}(X) \backslash (\pi_1^{-1}(\mathfrak{p}_k) \cup \pi_2^{-1}(\mathfrak{p}_k))$ in $m^{-1}(X)$ is larger than 1.*

Proof. (1) Let $L_B : \mathfrak{p} \to \mathfrak{t}$ by $A \mapsto [B, A]$, where $B \in \mathfrak{p}$. If $B \in \mathfrak{p}_k$ then L_B is surjective since $\dim \mathfrak{t} + k = \dim \mathfrak{p}$. Since for $(B_1, B_2) \in \pi_1^{-1}(\mathfrak{p}_k) \cup \pi_2^{-1}(\mathfrak{p}_k)$, the differential $dm_{(B_1,B_2)} : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{t}$ maps (A_1, A_2) to $[B_1, A_2] + [A_1, B_2]$. Since $\mathfrak{t} = \mathrm{Im} L_{B_1} + \mathrm{Im} L_{B_2} \subset \mathrm{Im}\, dm_{(B_1,B_2)}$, $dm_{(B_1,B_2)}$ is surjective. This proves (1).

(2) Will be proven in Appendix C. □

Remark 3.8. Let $\tilde{M} := (\mathfrak{p} \times \mathfrak{p}) \times_{\mathfrak{t}} M$, $\tilde{S} := S \times_{\mathfrak{t}} M$ and $\tilde{S}' := (\pi_1^{-1}(\mathfrak{p}_k) \cup \pi_2^{-1}(\mathfrak{p}_k)) \times_{\mathfrak{t}} M$. Then $\tilde{S}' \subset \tilde{S} \subset \tilde{M}$. By Proposition 3.6, $\tilde{S}$ is a smooth variety such that $\tilde{M} \backslash \tilde{S}$ is of codimension ≥ 2 in $\tilde{M}$. In fact, we can strengthen the result further as follows: $\tilde{S}'$ is a smooth variety such that $\tilde{M} \backslash \tilde{S}'$ is of codimension ≥ 2. By Proposition 3.7(1) and the smooth base change, $\tilde{S}'$ is a smooth variety. By Proposition 3.7(2), $(\mathfrak{p} \times \mathfrak{p}) \backslash (\pi_1^{-1}(\mathfrak{p}_k) \cup \pi_2^{-1}(\mathfrak{p}_k))$ is of codimension ≥ 2 in $\mathfrak{p} \times \mathfrak{p}$. Thus $\tilde{M} \backslash \tilde{S}'$ is of codimension ≥ 2 in $\tilde{M}$.

3.2. The proof of Theorem 3.1

Let $M := \mathrm{Hom}(W, V)$ and $\phi : M \to \mathfrak{t}$, $i \mapsto -ii^*$. By Proposition 3.6, $\mu^{-1}(0) = (\mathfrak{p} \times \mathfrak{p}) \times_{\mathfrak{t}} M$ is an irreducible normal variety and a complete intersection in $\mathfrak{p} \times \mathfrak{p} \times M$. By the method of associated cones [30, II.4.2], μ is normal, irreducible, flat and reduced. Since m is surjective, so is μ.

By [28, p. 5], $\mu^{-1}(0) /\!\!/ G$ is an irreducible normal variety. If one can show $\mu^{-1}(0)^{\mathrm{reg}} \neq \emptyset$ then by Lemma 2.7(2), $\mu^{-1}(0)^{\mathrm{reg}}/G$ is of the complement codimension ≥ 2, since $N \geq 2$.

It remains to show $\mu^{-1}(0)^{\mathrm{reg}} \neq \emptyset$. Let $i \in \mathrm{Hom}(W, V) \backslash 0$. Let $X := -ii^*$. Let $B_1 \in \mathfrak{p}_k$. Let a_n be the eigenvalues of B_1 ($n = 1, 2, \ldots, k$). Let V_{a_n} be the a_n-eigenspace. Let $p_n : V \to V_{a_n}$ be the projection. We may take B_1 so that $p_n(\mathrm{Im}(i)) \neq 0$ for all n. By Theorem 3.3, for any $B_1 \in \mathfrak{p}_k$, there exists $B_2 \in \mathfrak{p}$ such that $[B_1, B_2] = X$. Now, $(B_1, B_2, i, i^*) \in \mu^{-1}(0)^{\mathrm{reg}}$, since $V = \sum_P P(B_1) i(W)$, where P runs over one-variable polynomials. □

4. Geometry of the moduli spaces of SO-data: statements

We assume that V is symplectic and W is orthogonal in the rest of the paper, except Section 5.1 where preliminaries from linear algebra will be given. Let $k = \dim V$ and $N = \dim W$.

If $k = 0$ then we have $\mathbf{N} = 0$. If $N = 1$ and $k \geq 2$, $\mu^{-1}(0)^{\mathrm{reg}} = \emptyset$ as the right hand side of (2.3) is negative. We will study the next simplest cases, $k = 2$ (Theorem 4.1), $N = 2$ (Theorem 4.2) and $(k, N) = (4, 3)$ (Theorem 4.3). μ is not flat nor irreducible in general unlike the case of the moduli spaces of Sp-data.

In the following theorems, $\mathbf{P}$ denotes the minimal nilpotent $\mathrm{O}(W)$-orbit closure in $\mathfrak{o}(W)$. If $N = 3$ or $N \geq 5$ then $\mathbf{P}$ is irreducible and normal. When $N = 4$, $\mathbf{P}$ has two irreducible components, which are isomorphic by the action of an element in $\mathrm{O}(W)\backslash\mathrm{SO}(W)$. Each irreducible component is the closure of a $\mathrm{SO}(W)$-orbit. See [31, Theorems 5.1.4 and 5.1.6].

Theorem 4.1. *Let $k = 2$. Then the following properties are verified.*

(1) *$\mu^{-1}(0)$ is isomorphic to $\mathbb{C}^2 \times \{i \in \mathrm{Hom}(W, V) \mid ii^* = 0\}$ by $(B_1, B_2, i, i^*) \mapsto (\mathrm{tr}B_1, \mathrm{tr}B_2, i)$.*

(2) *If $N \geq 3$, μ is flat.*

(3) *If $N \geq 5$, $\mu^{-1}(0)$ is irreducible and normal. Hence, μ is irreducible and normal. If $N = 4$, $\mu^{-1}(0)$ is a reduced scheme and a union of two isomorphic irreducible components. If $N = 3$, it is irreducible, but non-reduced.*

(4) *If $N \leq 3$, $\mu^{-1}(0)^{\mathrm{reg}} = \emptyset$. If $N \geq 4$, $\mu^{-1}(0)^{\mathrm{reg}} = \mu^{-1}(0)^{\mathrm{sm}}$.*

(5) *If $N \leq 3$, the reduced scheme $\mu^{-1}(0)/\!\!/\mathrm{Sp}(V)_{\mathrm{red}}$ is isomorphic to $\mathbb{C}^2$.*

(6) *If $N \geq 4$, $\mu^{-1}(0)/\!\!/\mathrm{Sp}(V)$ is isomorphic to $\mathbb{C}^2 \times \mathbf{P}$. Moreover the isomorphism restricts to $\mu^{-1}(0)^{\mathrm{reg}}/\mathrm{Sp}(V) \cong \mathbb{C}^2 \times (\mathbf{P}\backslash 0)$.*

Theorem 4.2. (1) *Let $N = 2$ and $k \geq 2$. Then $\mu^{-1}(0)$ is not a complete intersection. Hence μ is not flat.*

(2) *Let $N = 2$ and $k = 4$. Then $\mu^{-1}(0)^{\mathrm{reg}} = \emptyset$.*

It is true that $\mu^{-1}(0)^{\mathrm{reg}} = \emptyset$ for $N = 2$ and any $k \geq 2$, as there is no nontrivial SO(2)-instantons. But the author does not find its proof in terms of the ADHM description.

Theorem 4.3. *Let $N = 3$ and $k = 4$. Then we have the following assertions.*

(1) *$\mu^{-1}(0)$ is a reduced complete intersection and a union of two irreducible components. Hence μ is flat.*

(2) *One irreducible component of $\mu^{-1}(0)$ is the closure of $\mu^{-1}(0)^{\mathrm{reg}}$.*

(3) *$\mu^{-1}(0)/\!\!/\mathrm{Sp}(V)$ is isomorphic to $\mathbb{C}^2 \times (\mathbf{P} \sqcup_0 \mathbf{P})$. Moreover the isomorphism restricts to $\mu^{-1}(0)^{\mathrm{reg}}/\mathrm{Sp}(V) \cong \mathbb{C}^2 \times (\mathbf{P}\backslash 0)$.*

Here, $\mathbf{P} \sqcup_0 \mathbf{P} := (\mathbf{P} \times \{0\}) \cup (\{0\} \times \mathbf{P})$ in $\mathbf{P} \times \mathbf{P}$. The proofs will appear in the subsequent sections.

Note that the first statement of Theorem 4.1(4), Theorem 4.2(2) and the second statement of Theorem 4.3(3) are either obvious or well-known in the context of instantons on S^4 in Section 1.3. (See also Remark 2.8.)

The properties of the moment map μ in Theorem 4.1 (2), (3), Theorem 4.2(1) and Theorem 4.3 (1) follow from the corresponding properties of $\mu^{-1}(0)$ by the method of associated cones [30, II.4.2], as in the proof of Theorem 3.1.

The author is planning to study on flatness of μ for $N \geq 4$ and normality of μ for $N \geq 5$ in the near future. At least the following is true for small k for a fixed N.

Remark 4.4. It is known by [32, Remark 11.3] that $\rho\colon \mathrm{Hom}(W, V) \to \mathfrak{sp}(V)$, $i \mapsto ii^*$, is flat if $N \geq 2k$. Moreover ρ is normal and irreducible if $N \geq 2k + 1$. By the base change argument used in the proof of Proposition 3.6, the same is true for μ.

5. Kraft–Procesi's classification theory of nilpotent pairs

In this section we review some geometry of $\mathrm{Hom}(W, V)$ following Kraft–Procesi [32], which will be used in the proofs of the theorems from the previous section.

5.1. Generalized eigenspaces and bilinear forms

Let V be a vector space with $(\ ,\ ) := (\ ,\ )_\varepsilon$, where $\varepsilon \in \{-1, +1\}$.

Definition 5.1. Let $X \in \mathfrak{gl}(V)$. Let $V_a^n := \mathrm{Ker}(X - a\mathrm{Id})^n$ for $n \geq 0$. We call $V_a := \cup_{n\geq 1} V_a^n$ the *generalized a-eigenspace of X*.

Lemma 5.2. (1) *If $X \in \mathfrak{p}(V)$ and $a \neq b$ then $V_a \perp V_b$.*

(2) *If $X \in \mathfrak{k}(V)$ and $a \neq -b$ then $V_a \perp V_b$.*

Proof. (1) Let $m, n \geq 0$. Let $v \in V_a^n$ and $w \in V_b^m$. We will show $(v, w) = 0$. We use the induction on m, n. If m or $n = 0$, the assertion is obvious. Suppose the assertion is true for $m - 1, n$ and $m, n - 1$. Let $v' := (X - a)v \in V_a^{m-1}$ and $w' := (X - a)w \in V_b^{n-1}$. We have $(Xv, w) = a(v, w)$ as $(v', w) = 0$ by our assumption. Similarly we also have $(v, Xw) = b(v, w)$. Thus we have $a(v, w) = b(v, w)$ which asserts $(v, w) = 0$.

(2) The proof is same if one uses $(Xv, w) = -(v, Xw)$. $\square$

Let $X \in \mathfrak{k}(V)$ or $\mathfrak{p}(V)$. Let us define a bilinear form $|\,,\,|$ on $\mathrm{Im}(X)$ by $|Xv, Xv'| := (v, Xv')$ (cf. [32, §4.1]).

Lemma 5.3. (1) *If $X \in \mathfrak{k}(V)$ then $|\,,\,|$ on $\mathrm{Im}(X)$ is a nondegenerate bilinear form of type $-\varepsilon$.*
(2) *If $X \in \mathfrak{p}(V)$ then $|\,,\,|$ on $\mathrm{Im}(X)$ is a nondegenerate bilinear form of type ε.*

Proof. (1) is obvious, as noted in [32, §4.1]. (2) is also clear. □

Proposition 5.4. *Let V be a 4-dimensional symplectic vector space. Let $X \in \mathfrak{p}(V)$. Then X has an eigenspace of dimension ≥ 2.*

Proof. By Lemma 5.2, *either* $V = V_a \oplus V_b$ for some $a \neq b$, *or* $V = V_a$ for some a. In the first case, V_a and V_b are 2-dimensional symplectic subspaces of V and thus $X|_{V_a} = a$ and $X|_{V_b} = b$ by Remark 2.1. In the second case, by Lemma 5.3, $\mathrm{Im}(X-a)$ is a symplectic subspace of dimension 0 or 2. If $\dim \mathrm{Im}(X-a) = 0$ then $X = a$. If $\dim \mathrm{Im}(X-a) = 2$ then $X|_{\mathrm{Im}(X-a)} = a$ by Remark 2.1 as $X \in \mathfrak{p}(\mathrm{Im}(X-a))$. We are done. □

Let W be a vector space with $(\,,\,)_{-\varepsilon}$. Recall that for $i \in \mathrm{Hom}(W, V)$, we have $ii^* \in \mathfrak{k}(V)$ and $i^*i \in \mathfrak{k}(W)$ (Section 2.3). Let V_a (resp. W_a) be the generalized a-eigenspace of ii^* (resp. i^*i).

Lemma 5.5. *We have $i^*(V_a) \subset W_a$ and $i(W_a) \subset V_a$. Moreover if $a \neq 0$ then i and i^* are isomorphisms between V_a and W_a.*

Proof. For any $a \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$, we have $i^*(ii^* - a)^n = (i^*i - a)^n i^*$ and $(ii^* - a)^n i = i(i^*i - a)^n$. If $v \in V_a$ then $(i^*i - a)^n i^* v = i^*(ii^* - a)^n v = 0$ for $n \gg 0$. Similarly, if $w \in W_a$ then $(ii^* - a)^n iw = i(i^*i - a)^n w = 0$ for $n \gg 0$. Therefore $i^*(V_a) \subset W_a$ and $i(W_a) \subset V_a$. This proves the first claim.

The restriction of ii^* to V_a is $a\mathrm{Id}_{V_a}$ plus a nilpotent endomorphism of V_a. So it is an isomorphism of V_a if $a \neq 0$. Similarly i^*i gives an isomorphism of W_a. Therefore i and i^* give isomorphism between V_a and W_a. □

5.2. Kraft–Procesi's results on nilpotent endomorphisms in $\mathfrak{k}$

Let $\mathrm{Sp} := \mathrm{Sp}(V)$, $\mathrm{O} := \mathrm{O}(W)$, $\mathfrak{sp} := \mathfrak{sp}(V)$ and $\mathfrak{o} := \mathfrak{o}(W)$, for short. Let X and Y be nilpotent elements of $\mathfrak{sp}(V)$ and $\mathfrak{o}(V)$ respectively.

A nilpotent orbit $\mathrm{Sp}.X$ corresponds uniquely to a partition $\eta = (\eta_1, \eta_2, \ldots)$ such that its transpose $(\hat{\eta}_1, \hat{\eta}_2, \ldots)$ is given by $\hat{\eta}_n := \dim \mathrm{Ker}(X^n)/\mathrm{Ker}(X^{n-1})$. We denote η by η_X. We represent η_X by the Young diagram with the boxes replaced by b (*b-diagram*). For instance, a partition $(5, 3, 3, 2, 2, 0, 0, \ldots)$ is represented by the b-diagram:

$$\begin{array}{c} bbbbb \\ bbb \\ bbb \\ bb \\ bb \end{array}$$

For a nilpotent orbit $\mathrm{O}.Y$, we use the symbol a instead of b, and the Young diagram is called an *a-diagram*.

Remark 5.6. By [33, IV 2.15], for $\eta_X = (\eta_1, \eta_2, \ldots)$ (resp. $\eta_Y = (\eta_1, \eta_2, \ldots)$), $\#\{n|\ \eta_n = m\}$ is even for any odd m (resp. for any even m).

Proposition 5.7 (*[32, Proposition 2.4]*). *For $\eta_X = \eta := (\eta_1, \eta_2, \ldots)$,*

$$\dim \mathrm{Sp}.X = \frac{1}{2}\left(|\eta|^2 + |\eta| - \sum \hat{\eta}_n^2 - \#\{n|\ \eta_n \text{ is odd}\}\right).$$

For $\eta_Y = \eta := (\eta_1, \eta_2, \ldots)$,

$$\dim \mathrm{O}.Y = \frac{1}{2}\left(|\eta|^2 - |\eta| - \sum \hat{\eta}_n^2 + \#\{n|\ \eta_n \text{ is odd}\}\right).$$

We say $\sigma \geq \eta$ for two partitions $\sigma = (\sigma_1, \sigma_2, \ldots)$ and $\eta = (\eta_1, \eta_2, \ldots)$ with $|\sigma| = |\eta|$ if $\sum_{1\leq i\leq j} \sigma_i \geq \sum_{1\leq i\leq j} \eta_i$ for any $j \geq 1$ (*dominance order* [34, §I.1]).

Proposition 5.8 (*[35, Theorem 3.10]*). *Let X and X' be nilpotent elements in $\mathfrak{sp}$. Then $\eta_X \geq \eta_{X'}$ if and only if $\mathrm{Sp}.X' \subset \overline{\mathrm{Sp}.X}$. The same is true for $\mathfrak{o}$.*

Corollary 5.9. (1) *If $\dim W \leq 2$, there is no nonzero nilpotent orbit in $\mathfrak{o}$.*
(2) *If $\dim W = 3$, there exists a unique nonzero nilpotent orbit, and it is given by the a-diagram aaa.*
(3) *If $\dim W \geq 4$, the minimal nilpotent orbit is given by the a-diagram*

$$\begin{array}{l} aa \\ aa \\ a \\ \vdots \\ a \end{array} \tag{5.1}$$

Proof. These assertions follow from Remark 5.6 and Proposition 5.8. □

Let us define two maps from $\mathrm{Hom}(W, V)$

$$\pi : \mathrm{Hom}(W, V) \to \mathfrak{o}, \quad i \mapsto i^* i,$$
$$\rho : \mathrm{Hom}(W, V) \to \mathfrak{sp}, \quad i \mapsto ii^*.$$

Theorem 5.10 *([32, Theorem 1.2]). π and ρ are the GIT quotient maps onto the images, by* Sp *and* O *respectively.*

We review Kraft–Procesi's classification theory of nilpotent pairs in [36] and [32]. A pair $(i, j) \in \mathrm{Hom}(W, V) \times \mathrm{Hom}(V, W)$ is a nilpotent pair if ij is a nilpotent endomorphism. As in the case of nilpotent endomorphisms, a Young diagram plays an important role in the classification of the $\mathrm{GL}(V) \times \mathrm{GL}(W)$-orbits of nilpotent pairs.

Definition 5.11 ([36, §§4.2–4.3]). By an *ab-diagram*, we mean a Young diagram whose rows consists of alternating a and b. E.g.,

$$\begin{array}{c} ababa \\ aba \\ aba \\ ab \\ ba \end{array} \tag{5.2}$$

An *ab*-diagram A gives a nilpotent pair as follows. Suppose the number of a (resp. b) in A is k (resp. N). Let us take any basis $\{b_1, b_2, \dots, b_k\}$ of V (resp. $\{a_1, a_2, \dots, a_N\}$ of W). We replace all the a and b in A by a_i and b_i. We define a nilpotent pair (i, j) such that i maps a_m to b_n in the right adjacent position or 0 if there is no such b_n and j maps b_m to a_n or 0 similarly. In the above example of *ab*-diagram, we have a nilpotent pair as follows:

$$\begin{array}{l} a_1 \mapsto b_1 \mapsto a_2 \mapsto b_2 \mapsto a_3 \mapsto 0 \\ \quad a_4 \mapsto b_3 \mapsto a_5 \mapsto 0 \\ \quad a_6 \mapsto b_4 \mapsto a_7 \mapsto 0 \\ \quad\quad a_8 \mapsto b_5 \mapsto 0 \\ \quad\quad b_6 \mapsto a_9 \mapsto 0 \end{array} \tag{5.3}$$

In the above correspondence, an *ab*-diagram does not determine uniquely a nilpotent pair as the bases of V and W can be changed. Therefore up to the change of the bases, we have the bijective correspondence ([36, §4.3])

$$\{ab\text{-diagram with } \#a = \dim V \text{ and } \#b = \dim W\}$$
$$\to \{(\mathrm{GL}(V) \times \mathrm{GL}(W)).(i, j) \mid (i, j) \text{ is a nilpotent pair}\}.$$

Suppose (i, i^*) is a nilpotent pair. From the *ab*-diagram of (i, i^*), the *a*-diagram of i^*i and the *b*-diagram of ii^* are obtained by removing b and a respectively. For example,

$$\begin{array}{ccccc} bb & & ababa & & aaa \\ b & & aba & & aa \\ b & \xleftarrow{\rho} & aba & \xrightarrow{\pi} & aa \\ b & & ab & & a \\ b & & ba & & a \end{array}$$

For an *ab*-diagram we define

$$\Delta_{ab} := \sum_{n:\,\mathrm{odd}} (\#\text{rows of length } n \text{ starting with } a) \cdot (\#\text{rows of length } n \text{ starting with } b). \tag{5.4}$$

Theorem 5.12 *([32, Theorem 6.5 and Proposition 7.1]). Let $i \in \mathrm{Hom}(W, V)$.*

(1) *The orbits $(\mathrm{Sp} \times \mathrm{O}).i$ such that $X := ii^*$ and $Y := i^*i$ are nilpotent, are in 1–1 correspondence with the ab-diagrams whose rows are of one of the types $\alpha, \beta, \gamma, \delta, \epsilon$ from Table 1 with $\#a = \dim W$, $\#b = \dim V$.*
(2) *For the ab-diagram associated to i,*

$$\dim(\mathrm{Sp} \times \mathrm{O}).i = \frac{1}{2}(\dim \mathrm{Sp}.X + \dim \mathrm{O}.Y + \dim V.\dim W - \Delta_{ab}). \tag{5.5}$$

6. Moduli spaces of SO(N)-data with $N \geq 2$ and $k = 2$

This section contains the proof of Theorem 4.1.

Let $\dim V = k = 2$ and $\dim W = N \geq 2$. Then $\mathfrak{p} := \mathfrak{p}(V)$ consists of scalars by Remark 2.1. Thus $(B_1, B_2, i, i^*) \in \mu^{-1}(0)$ implies $[B_1, B_2] = ii^* = 0$, and $\mu^{-1}(0) \cong \mathbb{C}^2 \times \rho^{-1}(0)$ by $(B_1, B_2, i, i^*) \mapsto (\mathrm{tr}(B_1), \mathrm{tr}(B_2), i)$. This proves (1).

Table 1
Rows of *ab*-diagrams.

Type	α_n	β_n	γ_n	δ_n	ϵ_n
ab-diagram	$aba\cdots ba$	$bab\cdots ab$	$aba\cdots ba$ $aba\cdots ba$	$bab\cdots ab$ $bab\cdots ab$	$aba\cdots ab$ $bab\cdots ba$
n			Odd	Even	
$\#a$	$2n+1$	$2n-1$	$2(n+1)$	$2n$	$2n$
$\#b$	$2n$	$2n$	$2n$	$2(n+1)$	$2n$

Let $i \in \rho^{-1}(0)$. Since $ii^* = 0$, b cannot appear twice in the same row in the ab-diagram of i. Looking at Table 1, we find that the ab-diagram of i is one of the following:

$$\begin{array}{ccc} aba & ab & b \\ aba & ba & b \\ a & a & a \\ \vdots & \vdots & \vdots \\ a & a & a \end{array} \tag{6.1}$$

where the left-most one actually occurs only when $N \geq 4$. We denote elements of $\mathrm{Hom}(W, V)$ corresponding to the above ab-diagrams by i_1, i_2 and $i_3(=0)$ respectively, where i_1 does not exist unless $N \geq 4$. By Theorem 5.12, $\dim(\mathrm{Sp}\times \mathrm{O}).i_1 = 2N-3$, $\dim(\mathrm{Sp}\times\mathrm{O}).i_2 = N$ and $\dim(\mathrm{Sp}\times\mathrm{O}).i_3 = 0$. Therefore $\rho^{-1}(0)$ is of the expected dimension. By the method of associated cones [30, II.4.2], ρ is flat. Thus μ is flat. This proves (2). See Remark 4.4 for the flatness when $N \geq 4$.

Suppose $N = 3$. Then $\mathrm{O} = \mathrm{SO} \sqcup -\mathrm{SO}$. Since $-\mathrm{Id}_V \in \mathrm{Sp}$ and $-\mathrm{Id}_W.i = -\mathrm{Id}_V.i$ for any $i \in \mathrm{Hom}(W,V)$, $(\mathrm{Sp}\times\mathrm{O}).i_2$ $(=(\mathrm{Sp}\times\mathrm{SO}).i_2)$ is irreducible and its Zariski closure is $\rho^{-1}(0)$. Let us check $\rho^{-1}(0)$ is not reduced. By [32, Remark 11.4], $\rho^{-1}(0)^{\mathrm{sm}} = \{i \in \rho^{-1}(0) \mid i \text{ is surjective}\}$. By (6.1) any $i \in \rho^{-1}(0)$ is not surjective, which implies $\rho^{-1}(0)$ is not reduced.

If $N = 4$ then by [32, §11.3], $\rho^{-1}(0)$ is reduced and consists of two irreducible components which are isomorphic by the action of an element of $\mathrm{O}\backslash\mathrm{SO}$.

If $N \geq 5$, [32, §11.3] asserts $\rho^{-1}(0)$ is normal and irreducible. This proves (3).

Let $x := (B_1, B_2, i, i^*) \in \mu^{-1}(0)$. By the ab-diagrams in (6.1), $\mathrm{Ker}(i_2^*)$ and $\mathrm{Ker}(i_3^*)$ are nonzero. Any nonzero vector in $\mathrm{Ker}(i_2^*)$ or $\mathrm{Ker}(i_3^*)$ is a common eigenvector of scalars B_1 and B_2. Thus, if $i = i_2$ or i_3 then x is not costable. Since $\mathrm{Ker}(i_1^*) = 0$, if $i = i_1$ then x is costable. So $\mu^{-1}(0)^{\mathrm{reg}} = \emptyset$ if $N \leq 3$. If $N \geq 4$, $\mu^{-1}(0)^{\mathrm{reg}} = (\mathrm{Sp}\times\mathrm{O}).i_1$.

We describe the smooth locus of $\mu^{-1}(0) = \mathbb{C}^2 \times \rho^{-1}(0)$ when $N \geq 4$. Since $T_x\rho^{-1}(0) = \mathrm{Ker}\, d\rho_x$ and $\dim\rho^{-1}(0) = \dim\mathrm{Hom}(W,V) - \dim\mathrm{Sp}$, $\rho^{-1}(0)^{\mathrm{sm}}$ is the locus of x such that $d\rho_x$ is surjective. Thus it is the locus of x such that ρ is smooth at x. By [32, Remark 11.4], $\rho^{-1}(0)^{\mathrm{sm}} = \{i \in \rho^{-1}(0) \mid i \text{ is surjective}\}$. Among i_1, i_2 and i_3, only i_1 is surjective. This proves (4).

Since Sp acts trivially on $\mathfrak{p}$, we have $\mu^{-1}(0)/\!\!/\mathrm{Sp} \cong \mathbb{C}^2 \times (\rho^{-1}(0)/\!\!/\mathrm{Sp})$. By Theorem 5.10 and the ab-diagrams in (6.1), the reduced scheme

$$\rho^{-1}(0)/\!\!/\mathrm{Sp}_{\mathrm{red}} \cong \begin{cases} 0 & \text{if } N \leq 3 \\ \mathbf{P} & \text{if } N \geq 4. \end{cases}$$

This proves the statement on $\mu^{-1}(0)/\!\!/\mathrm{Sp}$ in (5) and (6). By Theorem 5.10 and the a-diagrams coming from the ab-diagrams in (6.1), we have $\overline{(\mathrm{Sp}\times\mathrm{O}).i_2}/\!\!/\mathrm{Sp} = 0$. This proves the second claim of (6). □

7. Moduli spaces of SO(2)-data

This section contains the proof of Theorem 4.2(1). The proof of Theorem 4.2(2) will appear in Section 9.

Let $\dim V = k \geq 2$ and $\dim W = N = 2$. Let $m\colon \mathfrak{p}\times\mathfrak{p} \to \mathfrak{k}$, $(B, B') \mapsto [B, B']$.

We will show there exists a subvariety X in $\mu^{-1}(0)$ of dimension $> 2\dim\mathfrak{p} - \dim\mathfrak{k} + \dim\mathrm{Hom}(W,V)$. Let $X := \{(B, B', i, i^*) \mid [B, B'] = 0 = ii^*\} = m^{-1}(0) \times \rho^{-1}(0) \subset \mu^{-1}(0)$.

Let us estimate $\dim m^{-1}(0)$. Let $\mathfrak{p}^{(e)} := \{B \in \mathfrak{p} \mid B \text{ has distinct } e \text{ eigenvalues}\}$. As shown in Appendix C, $\mathfrak{p}^{(\leq e)}$ is a closed subvariety of $\mathfrak{p}$. By Lemma 5.2(1), if $e > k/2$, $\mathfrak{p}^{(e)} = \emptyset$. With respect to a symplectic basis of V, $\mathrm{diag}(a_1, a_2, \ldots, a_{k/2})^{\oplus 2} \in \mathfrak{p}^{(k/2)}$, where $a_1, a_2, \ldots, a_{k/2}$ are all distinct. Therefore $\mathfrak{p}^{(k/2)}$ is a Zariski dense open subset of $\mathfrak{p}$. Let $p\colon m^{-1}(0) \to \mathfrak{p}$ be the first projection. It is clear that p is surjective. And $p^{-1}(B_0) \cong \mathfrak{p}^{B_0}$ for any $B_0 \in \mathfrak{p}$ (see Section 3.1 for the notation). If $B_0 \in \mathfrak{p}^{(k/2)}$ then we claim $\dim\mathfrak{p}^{B_0} = k/2$. We have the (2-dimensional) eigenspace decomposition $V = \bigoplus_{i=1}^{k/2} V_{a_i}$ of B_0. Since $\mathfrak{gl}^{B_0} = \bigoplus_{i=1}^{k/2}\mathfrak{gl}(V_{a_i})$ we have $\mathfrak{p}^{B_0} = \bigoplus_{i=1}^{k/2}\mathfrak{p}(V_{a_i}) = \bigoplus_{i=1}^{k/2}\mathbb{C}$.

By the claim we obtain an estimate:

$$\dim m^{-1}(0) \geq \frac{k}{2} + \dim\mathfrak{p}.$$

Let us compute $\dim \rho^{-1}(0)$. Let $i \in \rho^{-1}(0)$. Since $\dim W = 2$ and i^*i is nilpotent, $i^*i = 0$ (Corollary 5.9(1)). By an argument similar to the one we used in Section 6, the ab-diagram of any nonzero i is

$$\begin{array}{c} ab \\ ba \\ b \\ \vdots \\ b \end{array}$$

Thus $\rho^{-1}(0)$ is the Zariski closure of $(\mathrm{Sp} \times \mathrm{O}).i$ and $\dim \rho^{-1}(0) = k$ by Theorem 5.12(2).

To sum up, $\dim X \geq \frac{3}{2}k + \dim \mathfrak{p} > 2\dim \mathfrak{p} - \dim \mathfrak{t} + 2k$ because $\dim \mathfrak{t} - \dim \mathfrak{p} = k$, which means $\mu^{-1}(0)$ is not a complete intersection in $\mathbf{N}$. $\square$

8. Moduli spaces of SO(3)-data with $k = 4$

This section contains the proof of Theorem 4.3.

Let $\dim V = k = 4$ and $\dim W = N = 3$. Let $\mathrm{SO} := \mathrm{SO}(W)$ for short.

8.1. Description of $\mu^{-1}(0)$

Let $\mathfrak{p}' := \{D \in \mathfrak{p} \,|\, \mathrm{tr} D = 0\}$. Let $\tilde{X} := \{(B_1, B_2, i, i^*) \in \mu^{-1}(0) \,|\, B_1, B_2 \in \mathfrak{p}'\}$. Then

$$\mu^{-1}(0) \cong \mathbb{C}^2 \times \tilde{X},$$
$$(B_1, B_2, i, i^*) \mapsto \left((\mathrm{tr}(B_1), \mathrm{tr}(B_2)), \left(B_1 - \frac{1}{4}\mathrm{tr}(B_1)\mathrm{Id}_V, B_2 - \frac{1}{4}\mathrm{tr}(B_2)\mathrm{Id}_V, i, i^*\right)\right).$$

Let $\mathbf{N}' := \mathfrak{p}'^{\oplus 2} \oplus \{(i, i^*) \,|\, i \in \mathrm{Hom}(W, V)\}$. Then $\tilde{X}$ is defined by $\mu = 0$ in $\mathbf{N}'$. Therefore it is enough to show the corresponding statements for $\tilde{X}$ to prove Theorem 4.3.

Let $\tilde{X}_1 := \{(B_1, B_2, i, i^*) \in \tilde{X} \,|\, [B_1, B_2] = 0\}$. Then $\tilde{X}_1 = \{(B_1, B_2) \in \mathfrak{p}' \times \mathfrak{p}' \,|\, [B_1, B_2] = 0\} \times \rho^{-1}(0)$.

Lemma 8.1. *$\rho^{-1}(0)$ consists of two irreducible strata which are $(\mathrm{Sp} \times \mathrm{O})$-orbits of dimension 5 and 0 respectively. Hence $\rho^{-1}(0)$, as a cone, is irreducible.*

Proof. By a similar argument as in Section 6, the possible ab-diagrams of i of $\rho^{-1}(0)$ are

$$\begin{array}{cc} & b \\ ab & b \\ ba & b \\ b & b \\ b & a \\ a & a \\ & a \end{array} \tag{8.1}$$

By Theorem 5.12, $\rho^{-1}(0)$ consists of two $(\mathrm{Sp} \times \mathrm{O})$-orbits associated to the above ab-diagrams of dimension 5 and 0 respectively.

Since $-\mathrm{Id}_W \in \mathrm{O} \backslash \mathrm{SO}$ and $-\mathrm{Id}_V \in \mathrm{Sp}$, $(\mathrm{Sp} \times \mathrm{O}).i = (\mathrm{Sp} \times \mathrm{SO}).i$ for any $i \in \mathrm{Hom}(W, V)$. So we have irreducibility. $\square$

Let e_1, e_2, e_3, e_4 be a basis of V such that $(e_1, e_2)_V = (e_3, e_4)_V = 1$ and $(e_l, e_m)_V = 0$ for other l, m with $l \leq m$. Let f_1, f_2, f_3 be an orthogonal basis of W so that $(f_i, f_j) = \delta_{ij}$.

Let I be the 2×2 identity matrix. Let

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{8.2}$$

We identify $\mathfrak{gl}(V) = \mathrm{Mat}_4$ with respect to $e_1, \ldots, e_4$. Then we can write the elements of $\mathfrak{t}$ and $\mathfrak{p}$ in matrix forms in 2×2 subminors:

$$\begin{aligned} \mathfrak{t} &:= \left\{ \begin{pmatrix} P & JQ \\ JQ^t & S \end{pmatrix} \in \mathfrak{gl}(V) \,\middle|\, \mathrm{tr}(P) = \mathrm{tr}(S) = 0 \right\}, \\ \mathfrak{p} &:= \left\{ \begin{pmatrix} aI & JR \\ -JR^t & bI \end{pmatrix} \in \mathfrak{gl}(V) \,\middle|\, a, b \in \mathbb{C} \right\}, \end{aligned} \tag{8.3}$$

where $P, Q, R, S \in \mathrm{Mat}_2$. Note that

$$\begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix}$$

are elements of $\mathfrak{p}'$ by letting $R = -JH$, $R = -JX$, $R = -JY$ and $R = -J$ respectively. Therefore we obtain a basis of $\mathfrak{p}'$ as

$$\begin{aligned} v_1 &:= \frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad v_2 := \begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix}, \qquad v_3 := \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}, \\ v_4 &:= \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix}, \qquad v_5 := \frac{1}{2}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \end{aligned} \tag{8.4}$$

By direct computation, the Lie brackets of pairs of basis elements of $\mathfrak{p}'$ are

$$\begin{aligned} &[v_1, v_2] = \begin{pmatrix} 0 & H \\ H & 0 \end{pmatrix}, \quad [v_1, v_3] = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}, \quad [v_1, v_4] = \begin{pmatrix} 0 & Y \\ Y & 0 \end{pmatrix}, \quad [v_1, v_5] = \frac{1}{2}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \\ &[v_2, v_3] = -2\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}, \quad [v_2, v_4] = 2\begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix}, \quad [v_2, v_5] = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}, \\ &[v_3, v_4] = -\begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad [v_3, v_5] = \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}, \quad [v_4, v_5] = \begin{pmatrix} Y & 0 \\ 0 & -Y \end{pmatrix}. \end{aligned} \tag{8.5}$$

These form a basis of $\mathfrak{t}$.

Let us define a linear map $F\colon \wedge^2 \mathfrak{p}' \to \mathfrak{t}$ by setting $F(v_i \wedge v_j) := [v_i, v_j]$, where $1 \le i < j \le 5$. Then we have a commuting diagram

$$\begin{tikzcd} \mathfrak{p}' \times \mathfrak{p}' \arrow[r, "\omega"] \arrow[dr, "{[\,,\,]}"'] & \wedge^2 \mathfrak{p}' \arrow[d, "F"] \\ & \mathfrak{t}. \end{tikzcd} \tag{8.6}$$

By (8.5), $\mathrm{Im}(F)$ contains the basis of $\mathfrak{t}$ and $\dim \mathfrak{t} = \dim \wedge^2 \mathfrak{p}' = 10$. Thus F is an isomorphism. In particular, for $B_1, B_2 \in \mathfrak{p}'$, $[B_1, B_2] = 0$ if and only if $B_1 \wedge B_2 = 0$. This proves the following lemma.

Lemma 8.2. $\{(B_1, B_2) \in \mathfrak{p}' \times \mathfrak{p}' \mid [B_1, B_2] = 0\} = \{(aB, bB) \mid a, b \in \mathbb{C},\ B \in \mathfrak{p}'\}$ *and it is irreducible.* □

Corollary 8.3. (1) $\tilde{X}_1$ *is irreducible of dimension* 11.

(2) *Any element of* $\tilde{X}_1$ *is an unstable quiver representation.*

Proof. (1) follows from Lemmas 8.1 and 8.2.

(2) The non-costability amounts to the existence of a common eigenvector v of B_1 and B_2 such that $i^*(v) = 0$. From (8.1), we have $\dim \mathrm{Ker}(i^*) \ge 3$. Let $B_1 = a_1 B$ and $B_2 = a_2 B$ for some $a_1, a_2 \in \mathbb{C}$ and $B \in \mathfrak{p}'$ (Lemma 8.2). By Proposition 5.4, we have an eigenspace of B of dimension ≥ 2. By the dimension reason it has a nonzero vector contained in the eigenspace and $\mathrm{Ker}(i^*)$. □

On the other hand, if $\sigma \in \mathrm{Im}\,\omega$ is nonzero then we have

$$\omega^{-1}(\sigma) \cong \mathrm{SL}(2), \tag{8.7}$$

since $\mathrm{Im}\,\omega \backslash 0$ is the set of 2-dimensional subspaces S of $\mathfrak{p}'$ with a volume form of S. This isomorphism can be described in more detail as follows. Define an SL(2)-action on $\mathfrak{p}' \times \mathfrak{p}'$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.(B_1, B_2) := (aB_1 + bB_2, cB_1 + dB_2). \tag{8.8}$$

The SL(2)-action on $\omega^{-1}(\wedge^2 \mathfrak{p}' \backslash 0)$ is free. The SL(2)-action on $\wedge^2 \mathfrak{p}'$ is trivial and $\omega|_{\omega^{-1}(\wedge^2 \mathfrak{p}' \backslash 0)}$ is SL(2)-equivariant. Now (8.7) is nothing but the identification of a free SL(2)-orbit.

Definition 8.4. Let $G := \mathrm{SL}(2) \times \mathrm{Sp} \times \mathrm{O}$. Then we have a G-action on $\tilde{X}$ by

$$(g_1, g_2, g_3).((B_1, B_2), i, i^*) := (g_1.(g_2 B_1 g_2^{-1}, g_2 B_2 g_2^{-1}), g_2 i g_3^{-1}, (g_2 i g_3^{-1})^*).$$

Lemma 8.5. *Let* $B_1, B_2 \in \mathfrak{p}'$. *Then* $([B_1, B_2])^2$ *is a scalar endomorphism.*

Proof. This follows from a tedious, but direct computation. □

Let $\tilde{X}_2 := \tilde{X} \backslash \tilde{X}_1 = \{(B_1, B_2, i, i^*) \in \tilde{X} \mid [B_1, B_2] \neq 0\}$. Let $p\colon \tilde{X}_2 \to \mathrm{Hom}(W, V)$ be the projection.

Corollary 8.6. *Let $i \in p(\tilde{X}_2)$. Then $(ii^*)^2 = 0$.*

Proof. Write $i = p(B_1, B_2, i, i^*)$ for some $B_1, B_2 \in \mathfrak{p}'$. By Lemma 8.5, $[B_1, B_2]^2 = (ii^*)^2$ is a scalar endomorphism. Since $\operatorname{rank} i \le 3$, the scalar is 0. $\square$

Corollary 8.7. *Let $i \in p(\tilde{X}_2)$. Then the ab-diagram of i is one of the following:*

$$\text{(I)}\ \begin{matrix} bab \\ bab \\ a \end{matrix} \qquad \text{(II)}\ \begin{matrix} ababa \\ b \\ b \end{matrix} \qquad \text{(III)}\ \begin{matrix} bab \\ ba \\ ab \end{matrix} \qquad \text{(IV)}\ \begin{matrix} bab \\ b \\ b \\ a \\ a \end{matrix} \tag{8.9}$$

Proof. Since $(ii^*)^2 = 0$ by Corollary 8.6 and $ii^* \neq 0$, the maximal length of rows of the b-diagram of ii^* is 2. Thus the b-diagram of ii^* is one of the following:

$$\begin{matrix} bb \\ bb \end{matrix} \qquad \begin{matrix} bb \\ b \\ b \end{matrix} \tag{8.10}$$

From Table 1, we get the list of ab-diagrams as above. $\square$

Lemma 8.8. (1) *Let $B_1 := v_5$ and $B_2 := v_1 - \frac{1}{2}v_2$. Then $[B_1, B_2]$ is a nilpotent matrix whose b-diagram is*

$$\begin{matrix} bb \\ bb \end{matrix} \tag{8.11}$$

(1) *Let $B_1 := v_3$ and $B_2 := \sqrt{-1}v_1 - v_5$. Then $[B_1, B_2]$ is a nilpotent matrix whose b-diagram is*

$$\begin{matrix} bb \\ b \\ b \end{matrix} \tag{8.12}$$

Proof. (1) By direct calculation we have

$$[B_1, B_2] = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

which maps

$$e_1 \mapsto \frac{1}{2}(e_1 + e_3), \qquad \frac{1}{2}(e_1 + e_3) \mapsto 0, \qquad e_2 \mapsto -\frac{1}{2}(e_2 - e_4), \qquad \frac{1}{2}(e_2 - e_4) \mapsto 0. \tag{8.13}$$

(2) By direct calculation we have

$$[B_1, B_2] = \begin{pmatrix} 0 & -1 & 0 & -\sqrt{-1} \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{-1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which maps

$$e_2 \mapsto -(e_1 + \sqrt{-1}e_3), \qquad e_1 + \sqrt{-1}e_3 \mapsto 0, \qquad e_1 \mapsto 0, \qquad e_2 + \sqrt{-1}e_4 \to 0. \quad \square \tag{8.14}$$

We denote B_1, B_2 in Lemma 8.8(1) by $B_1^{(\mathrm{I})}, B_2^{(\mathrm{I})}$ respectively. We denote B_1, B_2 in Lemma 8.8(2) by $B_1^{(\mathrm{II})}, B_2^{(\mathrm{II})}$ respectively. Let $B_n^{(\mathrm{III})}$ and $B_n^{(\mathrm{IV})}$ be $B_n^{(\mathrm{II})}$ $(n = 1, 2)$. For each $A = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$ there exists $i^{(A)} \in \operatorname{Hom}(W, V)$ whose ab-diagram is of type (A) in (8.9) and $[B_1^{(A)}, B_2^{(A)}] + i^{(A)}i^{(A)*} = 0$ by Theorem 5.12(1).

Let us define $i^{(\mathrm{I})}$ in a matrix form. We take a basis $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4\}$ of V by setting $\widetilde{e}_1 := e_1, \widetilde{e}_2 := e_2 - e_4, \widetilde{e}_3 := e_1 + e_3$ and $\widetilde{e}_4 := e_4$. We take a basis $\{\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3\}$ of W by setting $\widetilde{f}_1 := \frac{1}{\sqrt{2}}(f_1 + \sqrt{-1}f_2), \widetilde{f}_2 := \frac{1}{\sqrt{2}}(f_1 - \sqrt{-1}f_2)$ and $\widetilde{f}_3 := f_3$. Let

$$i^{(\mathrm{I})} := \frac{1}{\sqrt{-2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4\}$ and $\{\widetilde{f}_1, \widetilde{f}_2, \widetilde{f}_3\}$.

Let us define $i^{(\mathrm{II})}$, $i^{(III)}$ and $i^{(\mathrm{IV})}$ in matrix forms respectively. We take a basis $\{\widetilde{e}'_1, \widetilde{e}'_2, \widetilde{e}'_3, \widetilde{e}'_4\}$ of V by setting $\widetilde{e}'_1 := e_2$, $\widetilde{e}'_2 := -(e_1 + \sqrt{-1}e_3)$, $\widetilde{e}'_3 := e_3$ and $\widetilde{e}'_4 := -\sqrt{-1}e_2 + e_4$. We take a basis $\{\widetilde{f}'_1, \widetilde{f}'_2, \widetilde{f}'_3\}$ of W by setting $\widetilde{f}'_1 := \frac{1}{\sqrt{2}}(f_1 + \sqrt{-1}f_3)$, $\widetilde{f}'_2 := f_2$ and $\widetilde{f}'_3 := \frac{1}{\sqrt{2}}(f_1 - \sqrt{-1}f_3)$. Let

$$i^{(\mathrm{II})} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad i^{(\mathrm{III})} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad i^{(\mathrm{IV})} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to $\{\widetilde{e}'_1, \widetilde{e}'_2, \widetilde{e}'_3, \widetilde{e}'_4\}$ and $\{\widetilde{f}'_1, \widetilde{f}'_2, \widetilde{f}'_3\}$.

Corollary 8.9. *$\tilde{X}_2$ is the union of four G-orbits through $(B_1^{(A)}, B_2^{(A)}, i^{(A)}, i^{(A)*})$ for $A = \mathrm{I, II, III, IV}$. And each G-orbit is irreducible of dimension* 12, 12, 11 *and* 9 *respectively.*

Proof. Let $\tilde{X}'_2$ be the union of four G-orbits. Since $\tilde{X}'_2 \subset \tilde{X}_2$, we need to show the opposite inclusion. By Corollary 8.7, we have $p(\tilde{X}_2) \subset \bigcup_{A=\mathrm{I,II}} \mathrm{Sp}.i^{(A)} = p(\tilde{X}'_2)$. Thus $p(\tilde{X}_2) = \bigcup_{A=\mathrm{I,II}} \mathrm{Sp}.i^{(A)} = p(\tilde{X}'_2)$. On the other hand, $p^{-1}(p(x)) \cong \mathrm{SL}(2)$ for $x \in \tilde{X}_2$ by (8.7). Therefore $\tilde{X}_2 \subset p^{-1}(p(\tilde{X}_2)) = p^{-1}(p(\tilde{X}'_2)) = \tilde{X}'_2$.

By Theorem 5.12(2), the explicit value of $\dim G.(B_1^{(A)}, B_2^{(A)}, i^{(A)}, i^{(A)*}) = 3 + \dim(\mathrm{Sp} \times \mathrm{O}).i^{(A)}$ for each A, is computed as above.

The irreducibility comes from the fact that the above G-orbits are the $(\mathrm{SL}(2) \times \mathrm{Sp} \times \mathrm{SO})$-orbits. To see this we observe that $-\mathrm{Id}_W \in \mathrm{O}$ acts on $\mathbf{N}$ in the same way as $-\mathrm{Id}_V \in \mathrm{Sp}$. □

Lemma 8.10. *Let $x \in G.(B_1^{(A)}, B_2^{(A)}, i^{(A)}, i^{(A)*})$ for $A = \mathrm{I, II, III, IV}$. Then we have the following assertions.*
(1) *If $A = \mathrm{I}$, x is unstable and $\mathfrak{sp}^x$ is trivial.*
(2) *If $A = \mathrm{II}$, x is stable (hence Sp^x is trivial).*
(3) *If $A = \mathrm{III, IV}$, x is unstable.*
(4) *$\tilde{X}$ is a reduced complete intersection.*

Proof. (1) It is direct to check $e_1 + e_3$ is a common eigenvector of $B_1^{(\mathrm{I})}$ and $B_2^{(\mathrm{I})}$. We have $i^{(\mathrm{I})*}(e_1 + e_3) = i^{(\mathrm{I})*}(\widetilde{e}_3) = 0$. So $e_1 + e_3$ violates the costability of x.

We check $\mathfrak{sp}^x$ is trivial. Use (8.3). Then we have

$$\begin{aligned} \mathfrak{sp}^{B_1^{(\mathrm{I})}} &= \left\{ \begin{pmatrix} P & JQ^t \\ JQ & S \end{pmatrix} \in \mathfrak{sp} \,\middle|\, \left[\begin{pmatrix} P & JQ^t \\ JQ & S \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right] = 0 \right\} \\ &= \left\{ \begin{pmatrix} P & R \\ R & P \end{pmatrix} \,\middle|\, \mathrm{tr}P = 0,\ R = JQ,\ Q = Q^t \right\}, \\ \mathfrak{sp}^{B_1^{(\mathrm{I})}} \cap \mathfrak{sp}^{B_2^{(\mathrm{I})}} &= \left\{ \begin{pmatrix} P & R \\ R & P \end{pmatrix} \,\middle|\, \left[\begin{pmatrix} P & R \\ R & P \end{pmatrix}, \begin{pmatrix} I & H \\ -H & -I \end{pmatrix} \right] = 0,\ \mathrm{tr}P = 0 \right\} \\ &= \mathbb{C} \left\langle \begin{pmatrix} X & -X \\ -X & X \end{pmatrix}, \begin{pmatrix} Y & Y \\ Y & Y \end{pmatrix}, \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} \right\rangle. \end{aligned}$$

On the other hand, $\mathrm{Im}(i^{(\mathrm{I})}) = \mathbb{C}\langle \widetilde{e}_2, \widetilde{e}_3 \rangle = \mathbb{C}\langle e_1 + e_3, e_2 - e_4 \rangle$. So

$$\mathfrak{sp}^{i^{(\mathrm{I})}} = \{ g \in \mathfrak{sp} \,|\, g(e_1 + e_3) = g(e_2 - e_4) = 0 \}.$$

We claim $\mathfrak{sp}^{B_1^{(\mathrm{I})}} \cap \mathfrak{sp}^{B_2^{(\mathrm{I})}} \cap \mathfrak{sp}^{i^{(\mathrm{I})}} = 0$. The solution $a, b, c \in \mathbb{C}$ of the following equations

$$\left(a \begin{pmatrix} X & -X \\ -X & X \end{pmatrix} + b \begin{pmatrix} Y & Y \\ Y & Y \end{pmatrix} + c \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} \right) (e_1 + e_3) = 0$$
$$\left(a \begin{pmatrix} X & -X \\ -X & X \end{pmatrix} + b \begin{pmatrix} Y & Y \\ Y & Y \end{pmatrix} + c \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} \right) (e_2 - e_4) = 0$$

is trivial. So the claim is proven.

(2) We show there does not exist a common eigenvector v of $B_1^{(\mathrm{II})}$ and $B_2^{(\mathrm{II})}$ such that $i^{(\mathrm{II})*}(v) = 0$. By direct computation we have $\mathrm{Ker}(B_1^{(\mathrm{II})}) = \mathbb{C}\langle e_1, e_3 \rangle$ and $\mathrm{Ker}(B_2^{(\mathrm{II})}) = \mathbb{C}\langle e_1 + \sqrt{-1}e_3, e_2 + \sqrt{-1}e_4 \rangle$. So we have $\mathrm{Ker}(B_1^{(\mathrm{II})}) \cap \mathrm{Ker}(B_2^{(\mathrm{II})}) = \mathbb{C}\langle e_1 + \sqrt{-1}e_3 \rangle$. On the other hand $i^{(\mathrm{II})*}(e_1 + \sqrt{-1}e_3) = -i^{(\mathrm{II})*}(\widetilde{e}'_2) \neq 0$. This proves (2).

(3) In (2) we checked that $e_1 + \sqrt{-1}e_3$ is a unique common eigenvector of $B_1^{(A)}$ and $B_2^{(A)}$ for $A = \mathrm{III, IV}$ up to constant. On the other hand $i^{(A)*}(e_1 + \sqrt{-1}e_3) = -i^{(A)*}(\widetilde{e}'_2) = 0$ for $A = \mathrm{III, IV}$. So $e_1 + \sqrt{-1}e_3$ violates the costability of x.

(4) By Corollaries 8.3 and 8.9 and the dimension reason, $\tilde{X}$ is a complete intersection of dimension 12. By (1) and (2), $\tilde{X}$ is smooth along the two 12-dimensional G-orbits since $\mathfrak{sp}^x = 0$ implies that $d\mu_x \colon T_x\mathbf{N}' \to \mathfrak{sp}$ is surjective. Therefore $\tilde{X}$ is reduced along the two 12-dimensional G-orbits. Hence $\tilde{X}$ itself is reduced by [37, Prop. 5.8.5]. □

8.2. Description of $\mu^{-1}(0)/\!\!/\mathrm{Sp}$

In the previous subsection we proved

(1) $\tilde{X}$ is a reduced complete intersection with two irreducible components $\overline{\mathbf{S}}$ and $\overline{\mathbf{U}}$, where $\mathbf{S}$ is the stable locus and $\mathbf{U}$ is a G-orbit $G.(B_1^{(\mathrm{I})}, B_2^{(\mathrm{I})}, i^{(\mathrm{I})}, i^{(\mathrm{I})*})$.
(2) $\mathbf{S}$ is a G-orbit $G.(B_1^{(\mathrm{II})}, B_2^{(\mathrm{II})}, i_4, i_4^*)$.
(3) The Sp-action on $\overline{\mathbf{U}}$ is locally free (i.e., $\mathfrak{sp}^x = 0$ for any $x \in \mathbf{U}$).
(4) $\tilde{X}/\!\!/\mathrm{Sp} = \overline{\mathbf{S}}/\!\!/\mathrm{Sp} \cup \overline{\mathbf{U}}/\!\!/\mathrm{Sp}$ as varieties (as $\tilde{X}$ is reduced).

From now on we fix an orthogonal basis of W. Then $\mathfrak{o}$ is the set of anti-symmetric matrices. Let us identify

$$\mathfrak{o} \cong \mathbb{C}^3, \qquad \begin{pmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{pmatrix} \mapsto (e, f, g). \tag{8.15}$$

The characteristic polynomial of $A := \begin{pmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{pmatrix}$ in t is $t^3 + (e^2 + f^2 + g^2)t$. Therefore A is nilpotent if and only if $e^2 + f^2 + g^2 = 0$. Since any nonzero nilpotent element x in $\mathfrak{o}$ has the a-diagram aaa, the minimal nilpotent orbit is $\mathrm{O}.x$. Hence, $\mathbf{P}$ is the quadric surface in $\mathbb{C}^3$ defined by $e^2 + f^2 + g^2 = 0$, which also equals the nilpotent variety.

Lemma 8.11. *The map* $[(x, y)] \mapsto (x^2, \sqrt{-1}xy, y^2)$ *gives an isomorphism* $\mathbb{C}^2/\mathbb{Z}_2 \cong \mathbf{P}$, *and hence* $\mathbf{P}$ *is an irreducible normal variety. Moreover,* $\mathbf{P}^{\mathrm{rank}2} := \{A \in \mathbf{P} \mid \mathrm{rank} A = 2\} = \mathbf{P}\backslash 0 \cong (\mathbb{C}^2\backslash 0)/\mathbb{Z}_2$.

Proof. The first isomorphism is well-known in invariant theory. Since $\mathbb{C}^2$ is irreducible and normal, so is $\mathbb{C}^2/\mathbb{Z}_2$.
We prove the second assertion. The a-diagram of a nilpotent matrix $A \in \mathfrak{o}$ is either

$$\begin{array}{l} a \\ a \quad \text{or} \quad aaa \\ a \end{array} \tag{8.16}$$

Therefore $A \neq 0$ means $\mathrm{rank} A = 2$. $\square$

Lemma 8.12. $\Phi_{\tilde{X}_1}: \tilde{X}_1 \to \mathbf{P}$, $(B_1, B_2, i, i^*) \mapsto (\mathrm{tr}(B_1^2), \sqrt{-1}\mathrm{tr}(B_1B_2), \mathrm{tr}(B_2^2))$, *is the GIT quotient by* Sp.

Proof. Let $x := (B_1, B_2, i, i^*) \in \tilde{X}_1$. By Corollary 8.3(2), x is unstable. Suppose $\mathrm{Sp}.x$ is closed in $\mathbf{N}$. Since $x \notin \mu^{-1}(0)^{\mathrm{reg}}$, we have $x^s = 0$ by Theorems 2.2 and 4.1(4). Thus $i = 0$. Let $T := \{(aB, bB, 0, 0) \mid B \in \mathfrak{p}', \ a, b \in \mathbb{C}\}$. Let $\phi := \Phi_{\tilde{X}_1}|_T$. It is enough to show that ϕ is the GIT quotient by Sp by Lemma 8.2.
Since the Sp-action on $\mathbf{P}$ is trivial and $\mathbf{P}$ is normal, we need to show that $\phi/\!\!/\mathrm{Sp}: T/\!\!/\mathrm{Sp} \to \mathbf{P}$ is bijective by Zariski's main theorem.
Since $\mathrm{tr}(v_1^2) \neq 0$, ϕ is surjective and thus so is $\phi/\!\!/\mathrm{Sp}$.
To show injectivity, it is enough to show that $\phi^{-1}(c)$ is an Sp-orbit for any $c \in \mathbf{P}\backslash 0$ and that 0 is the unique closed Sp-orbit in $\phi^{-1}(0)$.
Let $c \in \mathbf{P}\backslash 0$. Then $c = (a^2, \sqrt{-1}ab, b^2)$ for some $a, b \in \mathbb{C}$. Then we have $\phi^{-1}(1, 0, 0) \cong \phi^{-1}(c)$, $(B, 0, 0, 0) \mapsto (aB, bB, 0, 0)$. Thus $\phi^{-1}(c)$ is an irreducible variety of dimension 4 since $\dim T = 6$ and $\dim \mathbf{P} = 2$.
On the other hand, we have $\mathfrak{p}'^B = \mathbb{C}\langle B\rangle$ for any $B \in \mathfrak{p}'\backslash 0$ by Lemma 8.2. So $\dim \mathfrak{p}^B = 2$. By [27, Prop. 5] we have $\dim \mathfrak{k}^B = 6$ and $\dim \mathrm{Sp}.B = 4$. This means $\phi^{-1}(c)$ is a Sp-orbit by irreducibility and the dimension reason. This proves the first item.
By a similar argument we have $\varphi/\!\!/\mathrm{Sp}: \mathfrak{p}'/\!\!/\mathrm{Sp} \to \mathbb{C}$ is a birational surjective morphism, where $\varphi: \mathfrak{p}' \to \mathbb{C}$ is given by $B \mapsto \mathrm{tr}B^2$. Since both $\mathfrak{p}'/\!\!/\mathrm{Sp}$ and $\mathbb{C}$ are irreducible normal varieties of dimension 1, $\varphi/\!\!/\mathrm{Sp}$ is an isomorphism by Zariski's main theorem. Therefore 0 is the unique closed Sp-orbit in $\varphi^{-1}(0)$. This proves the second item. $\square$

Lemma 8.13. $\Phi_{\overline{\mathbf{S}}}: \overline{\mathbf{S}} \to \mathbf{P}$, $(B_1, B_2, i, i^*) \mapsto i^*i$, *is the GIT quotient by* Sp.

Proof. Let $x := (B_1, B_2, i, i^*) \in \mathbf{S}$. By the ab-diagram (8.9) of i, we have $i^*i \in \mathbf{P}^{\mathrm{rank}2}$. So $\Phi_{\overline{\mathbf{S}}}(\mathbf{S}) = \mathbf{P}^{\mathrm{rank}2}$ and $\Phi_{\overline{\mathbf{S}}}$ is well-defined.
We claim that $\Phi_{\overline{\mathbf{S}}}^{-1}(a)$ is an Sp-orbit for any $a \in \mathbf{P}\backslash 0 = \mathbf{P}^{\mathrm{rank}2}$. Note that since $\mathbf{S}$ is irreducible, so is $\Phi_{\overline{\mathbf{S}}}^{-1}(a)$. Since x is stable, Sp^x is trivial. Thus $\dim \mathrm{Sp}.x = 10$. On the other hand, $\Phi_{\overline{\mathbf{S}}}^{-1}(i^*i) = (\mathrm{SL}(2) \times \mathrm{Sp}).x$ by Theorem 5.10 and (8.7). Thus $\Phi_{\overline{\mathbf{S}}}^{-1}(i^*i)$ is an irreducible 10-dimensional variety. If $\mathrm{Sp}.x \subsetneq (\mathrm{SL}(2) \times \mathrm{Sp}).x$ then for $y \in (\mathrm{SL}(2) \times \mathrm{Sp}).x\backslash \mathrm{Sp}.x$, we have $\dim \mathrm{Sp}.y = 10$. But then $\Phi_{\overline{\mathbf{S}}}^{-1}(i^*i)$ contains two disjoint locally closed subvarieties $\mathrm{Sp}.x$ and $\mathrm{Sp}.y$ of dimension 10. This contradicts the irreducibility of $\Phi_{\overline{\mathbf{S}}}^{-1}(i^*i)$. Therefore $\mathrm{Sp}.x = \Phi_{\overline{\mathbf{S}}}^{-1}(i^*i)$ as desired.

Since $\mathbf{S}$ consists of Sp-closed orbits (Theorem 2.2), $\mathbf{S}/\mathrm{Sp}$ is Zariski open in $\overline{\mathbf{S}}/\!\!/\mathrm{Sp}$. By Luna's slice theorem [38], $\mathbf{S}/\mathrm{Sp}$ is a smooth variety. By the above claim and Zariski's main theorem, $\overline{\Phi_{\overline{\mathbf{S}}}}|_{\mathbf{S}/\mathrm{Sp}}: \mathbf{S}/\mathrm{Sp} \to \mathbf{P}^{\mathrm{rank}2}$ is an isomorphism, where $\overline{\Phi_{\overline{\mathbf{S}}}}: \overline{\mathbf{S}}/\!\!/\mathrm{Sp} \to \mathbf{P}$ is the induced morphism. Let us finish the proof of the lemma. Let $f \in \mathbb{C}[\overline{\mathbf{S}}]^{\mathrm{Sp}}$. Then $f|_S \in \Phi^*\Gamma(\mathcal{O}_{\mathbf{P}\backslash 0})$. By the normality of $\mathbf{P}$, $\Gamma(\mathcal{O}_{\mathbf{P}\backslash 0}) = \mathbb{C}[\mathbf{P}]$. Thus $f \in \Phi_{\overline{\mathbf{S}}}^*\mathbb{C}[\mathbf{P}]$. This means $\mathbb{C}[\overline{\mathbf{S}}]^{\mathrm{Sp}} = \Phi_{\overline{\mathbf{S}}}^*\mathbb{C}[\mathbf{P}]$, equivalently Φ is the GIT quotient by Sp as desired. $\square$

Lemma 8.14. *$\tilde{X}_1 \subset \overline{\mathbf{U}}$ and $\tilde{X}_1/\!\!/\mathrm{Sp} = \overline{\mathbf{U}}/\!\!/\mathrm{Sp}$.*

Proof. Since $\mathbf{U}$ is an irreducible reduced locally free G-orbit of dimension 12, we have $\overline{\mathbf{U}}/\!\!/\mathrm{Sp}$ is an irreducible reduced variety of dimension ≤ 2. Now the second assertion follows from the first by Lemma 8.12.

Let us prove the first assertion. Suppose $\tilde{X}_1 \not\subseteq \overline{\mathbf{U}}$. Then $\tilde{X}_1 \subset \overline{\mathbf{S}}$ since $\tilde{X}_1$ is irreducible by Corollary 8.3(1). Let $T := \{(aB, bB) \in \mathfrak{p}' \times \mathfrak{p}' \mid a, b \in \mathbb{C},\ B \in \mathfrak{p}'\}$. By Lemma 8.1 and (8.1), $\tilde{X}_1$ is the closure of $T \times (\mathrm{Sp} \times \mathrm{O}).i$ for a nonzero $i \in \mathrm{Hom}(W, V)$ with $i^*i = 0$. By Lemma 8.13, $\Phi_{\overline{\mathbf{S}}}(\tilde{X}_1) = 0$. This contradicts Lemma 8.12. □

Definition 8.15. Define $\Phi: \tilde{X} \to (\mathbf{P} \times 0) \cup (0 \times \mathbf{P})\ (\subset \mathbf{P} \times \mathbf{P})$, $(B_1, B_2, i, i^*) \mapsto ((\mathrm{tr}(B_1^2), \sqrt{-1}\mathrm{tr}(B_1B_2), \mathrm{tr}(B_2^2)), i^*i)$.

To see Φ is well-defined morphism one notices that $i^{(\mathrm{I})*}i^{(\mathrm{I})} = i^{(\mathrm{II})*}i^{(\mathrm{II})} = i^{(\mathrm{III})*}i^{(\mathrm{III})} = 0$ and $\mathrm{tr}({B_1^{(\mathrm{II})}}^2) = \sqrt{-1}\mathrm{tr}(B_1^{(\mathrm{II})}B_2^{(\mathrm{II})}) = \mathrm{tr}({B_2^{(\mathrm{II})}}^2) = 0$, which come from Corollary 8.7 and the direct computation respectively.

Theorem 8.16. *Φ is the GIT quotient by Sp onto $(\mathbf{P} \times 0) \cup (0 \times \mathbf{P})$. Hence, $\mu^{-1}(0)/\!\!/\mathrm{Sp} \cong \mathbb{C}^2 \times ((\mathbf{P} \times 0) \cup (0 \times \mathbf{P}))$.*

Proof. By Lemmas 8.12–8.14, Φ factors through $\tilde{X}/\!\!/\mathrm{Sp}$. The canonical morphism given by the composite

$$(\mathbf{P} \times 0) \cup (0 \times \mathbf{P}) \xrightarrow{f} \tilde{X}_1/\!\!/\mathrm{Sp} \coprod_{[0]} \overline{\mathbf{S}}/\!\!/\mathrm{Sp} \longrightarrow \tilde{X}/\!\!/\mathrm{Sp}$$

is the inverse of $\Phi/\!\!/\mathrm{Sp}$, where $f := (\Phi|_{\tilde{X}_1}/\!\!/\mathrm{Sp})^{-1} \coprod_{[0]} (\Phi|_{\overline{\mathbf{S}}}/\!\!/\mathrm{Sp})^{-1}$ and $\Phi/\!\!/\mathrm{Sp}: \tilde{X}/\!\!/\mathrm{Sp} \to (\mathbf{P} \times 0) \times (0 \times \mathbf{P})$ is the induced morphism. □

9. Moduli spaces of SO(2)-data with $k = 4$

This section will be devoted to the proof of Theorem 4.2(2).

Let $\dim V = k = 4$ and $\dim W = N = 2$. Let $\mathfrak{p}' := \{B \in \mathfrak{p} \mid \mathrm{tr}(B) = 0\}$. Let $\tilde{X} := \{(B_1, B_2, i, i^*) \in \mu^{-1}(0) \mid B_1, B_2 \in \mathfrak{p}'\}$. Then as in Section 8.1, $\mu^{-1}(0) \cong \mathbb{C}^2 \times \tilde{X}$. Let $p: \tilde{X} \to \mathrm{Hom}(W, V)$ be the projection.

Lemma 9.1. *Let $i \in \mathrm{Im}(p)$. Then ii^* is nilpotent and the ab-diagram of i is one of the following:*

$$\begin{array}{cccc} & b & & \\ ab & b & & bab \\ ba & b & bab & b \\ b & b & bab & b \\ b & a & & a \\ & a & & \end{array} \tag{9.1}$$

Proof. Let $(B_1, B_2, i, i^*) \in \tilde{X}$. By Lemma 8.5, $([B_1, B_2])^2 = (ii^*)^2$ is a scalar. Since $\mathrm{rank}\, i \leq 2$ and $\dim V = 4$, the scalar is 0. Since there is no nontrivial nilpotent element in $\mathfrak{o}$, we have $i^*i = 0$. By a similar argument as in Section 6, the ab-diagram of i is one of (9.1) from Table 1. □

Let $\tilde{X}_1 := \{(B_1, B_2) \in \mathfrak{p}' \times \mathfrak{p}' \mid [B_1, B_2] = 0\} \times \{i \in \mathrm{Hom}(W, V) \mid ii^* = 0\}$. Let $\tilde{X}_2 := \tilde{X} \backslash \tilde{X}_1$.

Theorem 9.2. *$\mu^{-1}(0)^{\mathrm{reg}} = \emptyset$.*

Proof. Let $x := (B_1, B_2, i, i^*) \in \tilde{X}$. We will prove that x is not costable.

Suppose $x \in \tilde{X}_1$. The ab-diagram of i is either the first or the second in (9.1). Therefore $\dim \mathrm{Ker}(i^*) \geq 3$. As in the proof of Corollary 8.3(2), we see that x is not costable.

Suppose $x \in \tilde{X}_2$. As in the proofs of Lemma 8.10(1) and (3), we deduce that x is not costable. □

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Appendix A. Finite dimensionality of weight spaces

The main purpose of this section is to prove that each weight space of $\mathbb{C}[\mu^{-1}(0)]^{G(V)}$ with respect to T is finite-dimensional.

Let $\epsilon(m) = 0$ (resp. $\epsilon(m) = 1$) if m is even integer (resp. odd integer). If $\varepsilon = -1$ then using a symplectic basis of V, we identify $V = \mathbb{C}^k$. If $\varepsilon = +1$ then using the basis

$$\{f_1 \pm \sqrt{-1}f_2, f_3 \pm \sqrt{-1}f_4, \ldots, f_{k-\epsilon(k)-1} \pm \sqrt{-1}f_{k-\epsilon(k)}, (f_k)\}$$

we identify $V = \mathbb{C}^k$, where $\{f_1, f_2, \ldots, f_k\}$ is an orthogonal basis of V. Here the notation (f_k) denotes f_k only when k is odd (vacuous otherwise). By a similar way we identify $W = \mathbb{C}^N$. Let $\lfloor a \rfloor$ be the maximal integer in $\mathbb{Z}_{\leq a}$, where $a \in \mathbb{R}$. We fix maximal tori of $G(V)$ and $G(W)$ as

$$T_{G(V)} = \{\mathrm{diag}(z_1, z_2, \ldots, z_{\lfloor k/2 \rfloor}) \oplus \mathrm{diag}(z_1^{-1}, z_2^{-1}, \ldots, z_{\lfloor k/2 \rfloor}^{-1}) | \, z_1, z_2, \ldots, z_{\lfloor k/2 \rfloor} \in \mathbb{C}^*\},$$
$$T_{G(W)} = \{\mathrm{diag}(t_1, t_2, \ldots, t_{\lfloor N/2 \rfloor}) \oplus \mathrm{diag}(t_1^{-1}, t_2^{-1}, \ldots, t_{\lfloor N/2 \rfloor}^{-1}) | \, t_1, t_2, \ldots, t_{\lfloor N/2 \rfloor} \in \mathbb{C}^*\}$$

respectively.

Now we identify the rings of characters of $T_{G(V)}, T_{G(W)}$ and $(\mathbb{C}^*)^2$

$$R(T_{G(V)}) = \mathbb{Z}[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_{\lfloor k/2 \rfloor}^{\pm 1}],$$
$$R(T_{G(W)}) = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_{\lfloor N/2 \rfloor}^{\pm 1}]$$
$$R((\mathbb{C}^*)^2) = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$$

respectively.

Let us prove

$$\mathcal{X}_{\mathbb{C}[\mu^{-1}(0)]^{G(V)}} \in \hat{R}(T) := R(T_{G(W)})[[q_1^{-1}, q_2^{-1}]] \tag{A.1}$$

where $T = T_{G(W)} \times (\mathbb{C}^*)^2$ and $\mathcal{X}_{\mathbb{C}[\mu^{-1}(0)]^{G(V)}}$ is the formal T-character. The idea is to use the following two:

(1) $\mu^{-1}(0)/\!\!/G(V)$ is a closed $(\mathbb{C}^*)^2 \times G(W)$-subscheme of $\mathbf{N}/\!\!/G(V)$;
(2) the (surjective) GIT quotient $\mathbf{N} \to \mathbf{N}/\!\!/G(V)$ is $(\mathbb{C}^*)^2 \times G(W)$-equivariant.

So each weight space of $\mathbb{C}[\mu^{-1}(0)]^{G(V)}$ is a subquotient of the weight space of the same weight of $\mathbb{C}[\mathbf{N}]$. So the proof will be done if we show a stronger claim:

$$\mathcal{X}_{\mathbb{C}[\mathbf{N}]} \in \hat{R}(T).$$

The claim will follow from an even stronger claim:

$$\mathcal{X}_{\mathbb{C}[\mathbf{N}]}^{(\mathbb{C}^*)^2} \in \mathbb{Z}[[q_1^{-\frac{1}{2}}, q_2^{-\frac{1}{2}}]]$$

where $\mathcal{X}_{\mathbb{C}[\mathbf{N}]}^{(\mathbb{C}^*)^2}$ denotes the formal $(\mathbb{C}^*)^2$-character of $\mathbb{C}[\mathbf{N}]$. This is because all the monomials in $\mathcal{X}_{\mathbb{C}[\mathbf{N}]}$ have nonnegative integer coefficients.

Let us check the last claim. The decomposition $\mathbf{N} = \mathfrak{p}(V) \oplus \mathfrak{p}(V) \oplus \mathrm{Hom}(W, V)$ is the weight decomposition with respect to $(\mathbb{C}^*)^2$. The first two direct summands $\mathfrak{p}(V)$ of $\mathbf{N}$ are of weight q_1 and q_2 respectively. The last summand $\mathrm{Hom}(W, V)$ is of weight $(q_1q_2)^{\frac{1}{2}}$. Since $\mathbb{C}[\mathbf{N}] = S(\mathfrak{p}(V)^\vee) \otimes S(\mathfrak{p}(V)^\vee) \otimes S(\mathrm{Hom}(W, V)^\vee)$, where S denotes the symmetric product, we have

$$\mathcal{X}_{\mathbb{C}[\mathbf{N}]}^{(\mathbb{C}^*)^2} = \left(\sum_{n \geq 0} q_1^{-n}\right)^{\frac{1}{2}k(k+\varepsilon)} \left(\sum_{n \geq 0} q_2^{-n}\right)^{\frac{1}{2}k(k+\varepsilon)} \left(\sum_{n \geq 0} (q_1q_2)^{-\frac{n}{2}}\right)^{kN}.$$

This proves the last claim.

To complete the proof of (A.1) we recall from Section 1.3 that $i \in \mathrm{Hom}(W, V)$ always appears together with i^* in $\mathbb{C}[\mathbf{N}]^{G(W)}$. Hence any monomials with non-integer exponents in $\mathcal{X}_{\mathbb{C}[\mu^{-1}(0)]^{G(W)}}$ have coefficient 0. This proves (A.1).

Remark A.1. (1) $\hat{R}(T)$ is a ring.

(2) The origin 0 is the unique $(\mathbb{C}^*)^2$-fixed point of $\mathbf{N}$.

Appendix B. Scheme structure of moduli space of framed vector bundles with symplectic and orthogonal structure

Let V, W be vector spaces of dimension k, N with $(\ ,\ )_\varepsilon, (\ ,\ )_{-\varepsilon}$ respectively, where $\varepsilon = \pm 1$. The above pairings give isomorphisms $a_\varepsilon: V \to V^\vee$ and $b_{-\varepsilon}: W \to W^\vee$ given by $v \mapsto (v, \bullet)_\varepsilon$ and $w \mapsto (w, \bullet)_{-\varepsilon}$ respectively. Then, $a_\varepsilon^\vee = \varepsilon a_\varepsilon$ and ${b_{-\varepsilon}}^\vee = -\varepsilon b_{-\varepsilon}$.

We use the notations $\mathbf{M}$ and $*_{\mathbf{M}}$ in Section 2.4. Let $* := *_{\mathbf{M}}$ and $x^* := *(x)$ for simplicity. Let $\mu_{\mathbf{M}}: \mathbf{M} \to \mathfrak{gl}(V)$ be the moment map given by $(B_1, B_2, i, j) \mapsto [B_1, B_2] + ij$. It is obvious that $\mu = \mu_{\mathbf{M}}|_{\mathbf{N}}$.

We define an involution $\overline{*}: \mathbf{M}^{\mathrm{reg}}/\mathrm{GL}(V) \to \mathbf{M}^{\mathrm{reg}}/\mathrm{GL}(V)$ by $\mathrm{GL}(V).x \mapsto \mathrm{GL}(V).x^*$. Then the fixed locus $\mathbf{M}^{\mathrm{reg}}/\mathrm{GL}(V)^{\overline{*}}$ (resp. $\mu_{\mathbf{M}}^{-1}(0)^{\mathrm{reg}}/\mathrm{GL}(V)^{\overline{*}}$) is a smooth subscheme of $\mathbf{M}^{\mathrm{reg}}/\mathrm{GL}(V)$ (resp. $\mu_{\mathbf{M}}^{-1}(0)^{\mathrm{reg}}/\mathrm{GL}(V)$).

Since $\mathbf{N}$ is the fixed locus of $\mathbf{M}$, we have a canonical embedding $\iota\colon \mu^{-1}(0)^{\mathrm{reg}}/G(V) \to \mu_{\mathbf{M}}{}^{-1}(0)^{\mathrm{reg}}/\mathrm{GL}(V)^{\overline{*}}$. We will see ι is surjective.

Let $\mathcal{G}$ be the vector bundle locus of the Gieseker moduli scheme of framed torsion-free sheaves E with rank N and $c_2(E) = k$. By Barth's correspondence [39], there is an isomorphism $F'\colon \mu_{\mathbf{M}}{}^{-1}(0)^{\mathrm{reg}}/\mathrm{GL}(V) \to \mathcal{G}$ (cf. [23, §2]). We denote by the same notation $\overline{*}$ the induced involution on $\mathcal{G}$ via F'. Thus we have the isomorphism between the fixed loci $F\colon \mu_{\mathbf{M}}{}^{-1}(0)^{\mathrm{reg}}/\mathrm{GL}(V)^{\overline{*}} \to \mathcal{G}^{\overline{*}}$ by restriction.

Donaldson's argument [16] asserts that the image of the composite $F\iota$ is the set of the isomorphism classes of framed vector bundles E which admits an isomorphism $\phi\colon E \to E^{\vee}$ with $\phi^{\vee} = -\varepsilon\phi$ and $\phi|_x = b_{-\varepsilon}$ (after identifying $E|_x = W$ via the given framing for any $x \in l_{\infty}$). See also [24].

We claim that the image of F is contained in that of $F\iota$ given as above. So our claim will assert that ι is surjective. The proof itself goes along Donaldson's argument, so will be sketchy. Let $x \in \mu_{\mathbf{M}}^{-1}(0)^{\mathrm{reg}}$ such that $g.x = x^*$ for some $g \in \mathrm{GL}(V)$. By taking $*$ to the both sides of $g.x = x^*$, we obtain $g.x = g^*.x$. By the stability of x, we have $g = g^*$. The cohomology sheaf of the monad associated to x^* is isomorphic to $E^{\vee}$. The induced maps by g, g^* between the monads are explicitly written in terms of linear maps in $\mathrm{End}(V^{\oplus 2} \oplus W)$. By diagram-chasing and passing to the cohomology sheaves, the constraint $g = g^*$ gives an isomorphism $\overline{g}\colon E \to E^{\vee}$ with $\overline{g}^{\vee} = -\varepsilon\overline{g}$ and $\overline{g}|_x = b_{-\varepsilon}$. This finishes the proof of the claim.

By Zariski's main theorem we obtain the isomorphism $\mathcal{M}_n^K \cong \mathcal{G}^{\overline{*}}$, which was used in Section 1.4.

Appendix C. Proof of Proposition 3.7(2)

We prove Proposition 3.7(2) in this section: the codimension of $m^{-1}(X)\backslash(\pi_1^{-1}(\mathfrak{p}_k) \cup \pi_2^{-1}(\mathfrak{p}_k))$ in $m^{-1}(X)$ is larger than 1.

Lemma C.1 (*[40, Theorem XI.4]*). *Let $B \in \mathfrak{p}$. Then $\mathrm{O}(V).B = \mathfrak{p} \cap \mathrm{GL}(V).B$, where $\mathrm{O}(V).B$ and $\mathrm{GL}(V).B$ are the orbits by conjugation.* □

Let $E_{\mathfrak{gl}}\colon \mathfrak{gl} \to S^k\mathbb{C}$ be the morphism mapping B to the unordered set of eigenvalues of B. Here, S_k is the symmetric group of k-letters acting on $\mathbb{C}^k$ by permutation of coordinates, so that $S^k\mathbb{C} := \mathbb{C}^k/S_k$. Let $E := E_{\mathfrak{gl}}|_{\mathfrak{p}}$ and $E_l := E|_{\mathfrak{p}_l}$. To construct $E_{\mathfrak{gl}}$ explicitly, let $P\colon S^k\mathbb{C} \to \mathbb{C}^k$ be the isomorphism by $[(a_1, \ldots, a_k)] \mapsto (p_1(a), \ldots, p_k(a))$, where $a := (a_1, \ldots, a_k)$ and $p_i(a) = a_1^i + \cdots + a_k^i$ (the ith power sum). Let $E'_{\mathfrak{gl}}\colon \mathfrak{gl} \to \mathbb{C}^k$ by $B \mapsto (\mathrm{tr}B, \mathrm{tr}B^2, \ldots, \mathrm{tr}B^k)$. Let $E_{\mathfrak{gl}} := P^{-1} \circ E'_{\mathfrak{gl}}$.

Let $\mathfrak{p}_l^{(e)} := \{B \in \mathfrak{p}_l \mid B \text{ has } e \text{ distinct eigenvalues}\}$. Let $\mathfrak{p}_l^{(\le e)} := \bigsqcup_{e' \le e} \mathfrak{p}_l^{(e')}$. Then $\mathfrak{p}_l^{(\le e)}$ is a closed subvariety of $\mathfrak{p}$. Indeed, let $\Delta^{(e)} \subset S^k\mathbb{C}$ be the locus of all the unordered sets of e distinct points. Let $\Delta^{(\le e)} := \bigsqcup_{e' \le e} \Delta^{(e')}$. Then $\Delta^{(\le e)}$ is a closed subvariety of $S^k\mathbb{C}$. Therefore $E^{-1}(\Delta^{(\le e)})$ is a closed subvariety of $\mathfrak{p}$. In particular, $\mathfrak{p}_l^{(e)} = \mathfrak{p}_l \cap E^{-1}(\Delta^{(e)})$ is locally closed in $\mathfrak{p}$ by Lemma 3.5. It is manifest that if $e > k$ then for any l, $\mathfrak{p}_l^{(e)} = \emptyset$ and that if $e = k$ then $\mathfrak{p}_l^{(e)} \neq \emptyset$ if and only if $l = k$.

Lemma C.2. (1) *If $\mathfrak{p}_l^{(e)} \neq \emptyset$ then $\dim \mathfrak{p}_l^{(e)} \le \dim \mathfrak{p} - l + e$. In particular, if $l > k$ then $\dim \mathfrak{p}_l^{(e)} \le \dim \mathfrak{p} - 2$.*

(2) *If $e = k - 1$ then $\mathfrak{p}_l^{(e)} \neq \emptyset$ if and only if $l \in \{k, k+1\}$.*

(3) *$\mathfrak{p}_{k+1}^{(k-1)}$ consists of $B \in \mathfrak{p}$ conjugate by $\mathrm{O}(V)$ to $\mathrm{diag}(a_1, a_1, a_2, \ldots, a_{k-1})$, where $a_1, a_2, \ldots, a_{k-1}$ are distinct in $\mathbb{C}$.*

Proof. (1) The image $E_l|\colon \mathfrak{p}_l^{(e)} \to S^k\mathbb{C}$ is contained in $\Delta^{(e)}$. Any nonempty fibre of E_l is a union of $\mathrm{O}(V).B$ for finitely many $B \in \mathfrak{p}_l^{(e)}$ (Lemma C.1), so that its dimension is $\dim \mathrm{O}(V).B = \dim \mathrm{O}(V) - \dim \mathrm{O}(V)^B = \dim \mathfrak{k} - \dim \mathfrak{k}^B = \dim \mathfrak{p} - \mathfrak{p}^B = \dim \mathfrak{p} - l$, where the third identity comes from Lemma 3.4. Therefore $\dim \mathfrak{p}_l^{(e)} \le \dim \mathfrak{p} - l + \dim \Delta^{(e)}$. Since $\dim \Delta^{(e)} = e$, we have proven (1).

(2) Let $e = k - 1$. Let $B \in \mathfrak{p}_l^{(e)}$. Let $a_1, \ldots, a_{k-1}$ be the (distinct) eigenvalues of B. We may assume that only the a_1-eigenspace of B is 2-dimensional while the other ones are all 1-dimensional. The Jordan normal form of B is either $\begin{pmatrix} a_1 & 1 \\ 0 & a_1 \end{pmatrix} \oplus \mathrm{diag}(a_2, \ldots, a_{k-1})$ or $\mathrm{diag}(a_1, a_1, \ldots, a_{k-1})$. Both cases actually happen, since $\begin{pmatrix} a_1 & 1 \\ 0 & a_1 \end{pmatrix}$ is conjugate by $\mathrm{GL}(2)$ to a symmetric matrix $\begin{pmatrix} a_1 + \sqrt{-1} & 1 \\ 1 & a_1 - \sqrt{-1} \end{pmatrix}$. We have $\dim \mathfrak{gl}^B = k$ and $k+2$ respectively. Using $\mathfrak{gl}^B = \mathfrak{p}^B \oplus \mathfrak{k}^B$ (by (2.2)) and Lemma 3.4, we have $\dim \mathfrak{p}^B = k$ and $k+1$ respectively. This proves (2).

(3) Follows from Lemma C.1. □

Lemma C.3. *Let $X \in \mathfrak{k}$. Let $i \in \{1, 2\}$. Suppose $\pi_i^{-1}(\mathfrak{p}_l^{(e)}) \cap m^{-1}(X) \neq \emptyset$. Then $\dim \pi_i^{-1}(\mathfrak{p}_l^{(e)}) \cap m^{-1}(X) \le \dim \mathfrak{p} + e$. In particular, if $e \le k - 2$ then $\dim \pi_i^{-1}(\mathfrak{p}_l^{(e)}) \cap m^{-1}(X) \le \dim \mathfrak{p} + k - 2 = \dim m^{-1}(X) - 2$.*

Proof. We claim that any nonempty fibre of $\pi_i|\colon \pi_i^{-1}(\mathfrak{p}_l^{(e)}) \cap m^{-1}(X) \to \mathfrak{p}_l^{(e)}$ is of dimension l. Take $B \in \mathfrak{p}^{(e)}$ and identify $\pi_i^{-1}(B)$ with $\mathfrak{p}$. Then $\pi_i^{-1}(B) \cap m^{-1}(X)$, unless empty, is an affine space isomorphic to $\mathfrak{p}^B$, since for any $B' \in \pi_i^{-1}(B) \cap m^{-1}(X)$, $B' - B \in \mathfrak{p}^B$. The base dimension $\dim \mathfrak{p}_l^{(e)}$ is estimated in Lemma C.2. Thus the lemma is proven. □

Now we are ready to estimate the codimension of $m^{-1}(X)\backslash(\pi_1^{-1}(\mathfrak{p}_k) \cup \pi_2^{-1}(\mathfrak{p}_k))$ in $m^{-1}(X)$. By Lemmas C.2 and C.3(2), to check the codimension ≥ 2, it suffices to check that so is the codimension of $\pi_i^{-1}(\mathfrak{p}_{k+1}^{(k-1)}) \cap \pi_j^{-1}(\mathfrak{p}\backslash\mathfrak{p}_k) \cap m^{-1}(X)$ in $m^{-1}(X)$, whenever $\{i, j\} = \{1, 2\}$. By Lemma C.3, $\pi_i^{-1}(\mathfrak{p}_{k+1}^{(k-1)}) \cap m^{-1}(X)$ is of codimension ≥ 1 in $m^{-1}(X)$. It remains to prove $\pi_i^{-1}(\mathfrak{p}_{k+1}^{(k-1)}) \cap \pi_j^{-1}(\mathfrak{p}_k) \cap m^{-1}(X)$ is Zariski (open) dense in $\pi_i^{-1}(\mathfrak{p}_{k+1}^{(k-1)}) \cap m^{-1}(X)$. Let $B_1 \in \mathfrak{p}_{k+1}^{(k-1)}$. This is reduced to check that

$$\pi_i^{-1}(B_1) \cap \pi_j^{-1}(\mathfrak{p}_k) \cap m^{-1}(X) \neq \emptyset \text{ provided } \pi_i^{-1}(B_1) \cap m^{-1}(X) \neq \emptyset \tag{C.1}$$

since $\pi_i^{-1}(B_1) \cap m^{-1}(X) \cong \mathfrak{p}^{B_1}$ irreducible (see the proof of Lemma C.3). Let $B_0 \in \pi_i^{-1}(B_1) \cap m^{-1}(X) \subset \mathfrak{p}$, where $\pi_i^{-1}(B_1)$ are canonically identified with $\mathfrak{p}$. Let us write $B_1 = g.\mathrm{diag}(a_1, a_1, a_2, \dots, a_{k-1})$, where $g \in \mathrm{O}(V)$ and $a_1, \dots, a_{k-1}$ are distinct (Lemmas C.1 and C.2(3)). Let $B_2 := g.\mathrm{diag}(b_1, \dots, b_k)$, where $b_1, \dots, b_k$ are distinct so that $B_2 \in \mathfrak{p}_k$. By the Zariski openness of $\mathfrak{p}_k$ in $\mathfrak{p}$, there exists $u \in \mathbb{C}\backslash\{1\}$ such that $(1 - u)B_0 + uB_2 \in \mathfrak{p}_k$ since for $u = 1$, $B_2 \in \mathfrak{p}_k$. Therefore $B_0 + \frac{u}{1-u}B_2 \in \mathfrak{p}_k$. Now we have $(B_1, B_0 + \frac{u}{1-u}B_2)$ or $(B_0 + \frac{u}{1-u}B_2, B_1) \in \pi_i^{-1}(B_1) \cap \pi_j^{-1}(\mathfrak{p}_k) \cap m^{-1}(X)$, which shows (C.1). This completes the proof of Proposition 3.7(2).

References

[1] N. Nekrasov, Seiberg-Witten prepotential from instanton counting, Adv. Theor. Math. Phys. 7 (5) (2003) 831–864. arXiv:hep-th/0206161.

[2] N. Seiberg, E. Witten, Electric-magnetic duality, monopole condensation, and confinement in N = 2 supersymmetric Yang-Mills theory, Nuclear Phys. B 426 (1) (1994) 19–52; Nuclear Phys. B 430 (2) (1994) 485–486. Erratum: Electric-magnetic duality, monopole condensation, and confinement in N = 2 supersymmetric Yang-Mills theory. arXiv:math/9407087v1.

[3] H. Nakajima, K. Yoshioka, Lectures on instanton counting, in: Algebraic Structures and Moduli Spaces, in: CRM Proc. Lecture Notes, vol. 38, Amer. Math. Soc., Providence, RI, 2004, pp. 31–101. arXiv:math/0311058.

[4] N. Nekrasov, A. Okounkov, Seiberg-Witten prepotential and random partitions, in: The Unity of Mathematics, in: Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006, pp. 525–596. arXiv:hep-th/0306238.

[5] A. Braverman, P. Etingof, Instanton counting via affine Lie algebras II: from Whittaker vectors to the Seiberg-Witten prepotential, in: Studies in Lie theory, in: Progr. Math., vol. 243, Birkhäuser, Boston, 2006, pp. 61–78. arXiv:math/0409441.

[6] L. Göttsche, H. Nakajima, K. Yoshioka, Instanton counting and Donaldson invariants, J. Differential Geom. 80 (3) (2008) 343–390.

[7] L. Göttsche, H. Nakajima, K. Yoshioka, Donaldson = Seiberg-Witten from Mochizuki's formula and instanton counting, Publ. Res. Inst. Math. Sci. 47 (1) (2011) 307–359.

[8] N. Nekrasov, Five-dimensional gauge theories and relativistic integrable systems, Nuclear Phys. B 531 (1–3) (1998) 323–344. arXiv:math/9609219v3.

[9] H. Nakajima, K. Yoshioka, Instanton counting on blowup. II. K-theoretic partition function, Transform. Groups 10 (2005) 489–519. arXiv:math/0505553.

[10] L. Göttsche, H. Nakajima, K. Yoshioka, K-theoretic Donaldson invariants via instanton counting, Pure Appl. Math. 5 (2009) 1029–1110. arXiv:math/0611945.

[11] N. Nekrasov, S. Shadchin, ABCD of instantons, Comm. Math. Phys. 252 (1–3) (2004) 359–391. arXiv:math/0404225v2.

[12] M.F. Atiyah, V.G. Drinfeld, N.J. Hitchin, Y.I. Manin, Construction of instantons, Phys. Lett. A 65 (3) (1978) 185–187.

[13] W. Crawley-Boevey, Normality of Marsden-Weinstein reductions for representations of quivers, Math. Ann. 325 (1) (2003) 55–79.

[14] S.K. Donaldson, P.B. Kronheimer, The geometry of four-manifolds, in: Oxford Mathematical Monographs, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1990, x+440 pp.

[15] M.F. Atiyah, N.J. Hitchin, I.M. Singer, Self-duality in four-dimensional Riemannian geometry, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 362 (1711) (1978) 425–461.

[16] S.K. Donaldson, Instantons and geometric invariant theory, Comm. Math. Phys. 93 (4) (1984) 453–460.

[17] H. Weyl, The Classical Groups, Princeton university press, Princeton, New Jersey, 1973, the eighth printing.

[18] S. Benvenuti, A. Hanany, N. Mekareeya, The Hilbert series of the one instanton moduli space, J. High Energy Phys. (6) (2010) 100, 40 pp.

[19] L. Hollands, C. Keller, J. Song, From SO/Sp instantons to W-algebra blocks, J. High Energy Phys. (3) (2011) 053, 79 pp.

[20] W. Crawley-Boevey, Geometry of the moment map for representations of quivers, Compos. Math. 126 (3) (2001) 257–293.

[21] D.I. Panyushev, The Jacobian modules of a representation of a Lie algebra and geometry of commuting varieties, Compos. Math. 94 (2) (1994) 181–199.

[22] L. Le Bruyn, C. Procesi, Semisimple representations of quivers, Trans. AMS 317 (2) (1990) 585–598.

[23] H. Nakajima, Lectures on Hilbert schemes of points on surfaces, in: Univ. Lect. Ser., vol. 18, AMS, 1999.

[24] J. Bryan, M. Sanders, Instantons on S^4 and CP^2, rank stabilization, and Bott periodicity, Topology 39 (2) (2000) 331–352.

[25] A. Braverman, M. Finkelberg, D. Gaitsgory, Uhlenbeck spaces via affine Lie algebras, in: The Unity of Mathematics, in: Progr. Math., vol. 244, Birkhauser Boston, Boston, MA, 2006, pp. 17–135.

[26] J.P. Brennan, Normality of the commuting variety of symmetric matrices, Comm. Algebra 22 (15) (1994) 6409–6415.

[27] B. Kostant, S. Rallis, Orbits and representations associated with symmetric spaces, Amer. J. Math. 93 (1971) 753–809.

[28] D. Mumford, J. Fogarty, F. Kirwan, Geometric invariant theory, third ed., in: Ergebnisse der Mathematik und ihrer Grenzgebiete (2), vol. 34, Springer-Verlag, Berlin, 1994, xiv+292.

[29] R. Hartshorne, Algebraic geometry, in: Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York-Heidelberg, 1977.

[30] H. Kraft, Geometrische Methoden in der Invariantentheorie (German), Aspects of Mathematics, D1. Friedr. Vieweg & Sohn, Braunschweig, 1984.

[31] D.H. Collingwood, W.M. McGovern, Nilpotent Orbits in Semisimple Lie Algebras, Van Nostrand Reinhold, New York, 1993.

[32] H. Kraft, C. Procesi, On the geometry of conjugacy classes in classical groups, Comment. Math. Helv. 57 (1982) 539–602.

[33] T.A. Springer, R. Steinberg, Conjugacy classes, Seminar on algebraic groups and related finite groups, in: Springer Lecture Notes, vol. 131, Springer Verlag, Berlin-Heidelberg-New York, 1970.

[34] I.G. Macdonald, Symmetric functions and Hall polynomials, second ed., in: Oxford Math. Monographs, Oxford Univ. Press, 1995.

[35] W. Hesselink, Singularties in the nilpotent scheme of a classical group, Trans. Amer. Math. Soc. 222 (1976) 1–32.

[36] H. Kraft, C. Procesi, Closures of conjugacy classes of matrices are normal, Invent. Math. 53 (1979) 227–247.

[37] A. Grothendieck, Elements de geometrie algebrique IV. Etude locale des schemas et des morphismes de schemas. II (French), Publ. Math. Inst. Hautes Études Sci. (24) (1965) 231.

[38] D. Luna, Slices étales, Bull. Soc. Math. France 33 (1973) 81–105.

[39] R. Barth, Moduli of bundles on the projecive plane, Invent. Math. 42 (1977) 63–91.

[40] F.R. Gantmacher, The Theory of Matrices,Vols. 1, 2 (K. A. Hirsch, Trans.), Chelsea Publishing Co., New York, 1959.