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Subnormality of unbounded composition operators over one-circuit directed graphs: Exotic examples ☆

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ABSTRACT

A recent example of a non-hyponormal injective composition operator in an L^2-space generating Stieltjes moment sequences, invented by three of the present authors, was built over a non-locally finite directed tree. The main goal of this paper is to solve the problem of whether there exists such an operator over a locally finite directed graph and, in the affirmative case, to find the simplest possible graph with these properties (simplicity refers to local valency). The problem is solved affirmatively for the locally finite directed graph $\mathscr{G}_{2,0}$, which consists of two branches and one loop. The only simpler directed graph for which the problem remains unsolved consists of one branch and one loop. The consistency condition, the only efficient tool for verifying subnormality of unbounded composition operators, is intensively studied in the context of $\mathscr{G}_{2,0}$, which leads to a constructive method of solving the problem. The method itself is partly based on

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Hamburger and Stieltjes moment sequences

transforming the Krein and the Friedrichs measures coming either from shifted Al-Salam–Carlitz q-polynomials or from a quartic birth and death process.

Contents

1. Preliminaries

1.1. Introduction

The theory of bounded subnormal operators was initiated by Halmos (see [32]). The definition and the first characterization of their unbounded counterparts were given independently by Bishop (see [9]) and Foiaş (see [29]). The foundations of the theory of unbounded (i.e., not necessarily bounded) subnormal operators were developed by the fourth-named author and Szafraniec (see [63–66]). The study of this topic turned out to be highly successful. It led to a number of challenging problems and nontrivial results in various branches of mathematics including functional analysis and mathematical physics (see, e.g., [23–25] for the case of bounded operators and [50,43,26,45–47,44] for unbounded ones). This area of interest still plays a vital role in operator theory.

The first characterization of bounded subnormal operators was given by Halmos himself. It was successively simplified by Bram (see [10]), Embry (see [28]) and Lambert (see [48]; see also [64, Theorem 7] where the assumption of injectivity was removed). The Lambert characterization states that a bounded Hilbert space operator is subnormal if and only if it generates Stieltjes moment sequences (see Section 1.2 for definitions). It turns out that this characterization also works for unbounded operators which have sufficiently many analytic vectors (see [64, Theorem 7]). However, it is no longer true for arbitrary unbounded operators (see [12, Section 3.2]). Recall that subnormal operators with dense set of C^∞-vectors always generate Stieltjes moment sequences (see [15, Proposition 3.2.1]). It is also worth pointing out that subnormal composition operators in L^2-spaces, as opposed to abstract subnormal operators, are always injective (see [15, Corollary 6.3]). Hence, there arises the question whether or not composition operators in L^2-spaces generating Stieltjes moment sequences are injective (see Problem 3.3.6).

In a recent paper [16], we have developed a completely new, even in the bounded case, approach to studying subnormality of composition operators (in L^2-spaces over σ-finite measure spaces) which involves measurable families of probability measures satisfying the so-called consistency condition. This approach provides a criterion (read: sufficient condition) for subnormality of composition operators, which does not refer to the density of domains of powers. The corresponding technique for weighted shifts on directed trees worked out in [14] (see also [18]) enabled us to construct an unexpected example of a subnormal composition operator whose square has trivial domain (see [17]).

As shown in [42, Example 4.2.1], there are unbounded injective operators generating Stieltjes moment sequences which are not even hyponormal, and thus not subnormal. In fact, it was proved there that if $\mathscr{T}$ is a leafless directed tree which has exactly one branching vertex and if the branching vertex itself has infinite valency, then there exists a non-hyponormal injective weighted shift on $\mathscr{T}$ with nonzero weights generating Stieltjes moment sequences, where the valency of a vertex v is understood as the number of outgoing edges at v. Up to isomorphism, there is only one rootless directed tree of this kind, denoted in [41, p. 67] by $\mathscr{T}_{\infty,\infty}$. A weighted shift on $\mathscr{T}_{\infty,\infty}$ with nonzero weights is unitarily equivalent to an injective composition operator in an L^2-space over a discrete measure space (see [42, Lemma 4.3.1] and [41, Theorem 3.2.1]). Since the directed graph induced by the symbol of such a composition operator coincides with $\mathscr{T}_{\infty,\infty}$ (see Section 3.2 for the definition), it is *not locally finite*. This raises the question as to whether there exists a non-hyponormal injective composition operator over a *locally finite* connected directed graph generating Stieltjes moment sequences, and, if this is the case, how *simple* such a directed graph can be, where simplicity is understood with respect to local valency (see Remark 3.2.2); here, saying that a composition operator C is *over* a directed graph $\mathscr{G}$ means that $\mathscr{G}$ is induced by the symbol of C. The present paper addresses both of these questions. Taking into account the simplicity leads to considering directed graphs induced by self-maps whose vertices, all but one, say ω, have valency one, and the valency of ω is greater than or equal to 1. Such directed graphs are described in Theorem 3.2.1 and Remark 3.2.2. In view of part 1) of Remark 3.2.2, the situation in which the valency of ω is equal to one is excluded by an unbounded variant of Herrero's characterization of subnormal injective bilateral weighted shifts (see [64, Theorem 5]; see also [13, Theorem 3.2] for a recent approach). Recall that the Herrero result (see [36]; see also [27]) is a bilateral analogue of the Berger–Gellar–Wallen characterization of bounded subnormal injective unilateral weighted shifts (see [33,31]). If the valency of ω is strictly greater than one, then, by Theorem 3.2.1, we have two cases. The first, which is described in Theorem 3.2.1(ii-b), reduces to the directed tree $\mathscr{T}_{\infty,\infty}$, the case studied in [42]. Unfortunately, the method invented in [42] does not give any hope of answering our questions. In the second case, which is described in Theorem 3.2.1(ii-a), the directed graph under consideration has exactly one circuit of length $\kappa+1$ starting at ω and η branches of infinite length attached to ω, where $\eta \in \{1,2,3,\dots\} \cup \{\infty\}$ and $\kappa \in \{0,1,2,\dots\}$ (see Fig. 2 with $\omega = x_\kappa$); denote it by $\mathscr{G}_{\eta,\kappa}$. The culminating result of the present paper, Theorem 5.5.2, shows that there exists a non-hyponormal injective composition operator over the locally finite directed graph $\mathscr{G}_{2,0}$ generating Stieltjes moment sequences. This answers our first question in the affirmative. Regarding simplicity, the only simpler directed graph which potentially may admit a composition operator with the above-mentioned properties is $\mathscr{G}_{1,0}$ (the subnormality over $\mathscr{G}_{1,0}$ was studied in [16, Section 3.4]). However, so far this particular case remains unsolved because composition operators over $\mathscr{G}_{1,0}$ obtained by our method are automatically subnormal (see Theorem 5.4.2(iv)).

A large part of the present paper is devoted to the study of subnormality of composition operators over the directed graph $\mathscr{G}_{\eta,\kappa}$. They all have the same symbol $\phi_{\eta,\kappa}$ whereas masses attached to vertices that define the underlying L^2-space are subject to changes. Note that general criteria for subnormality of unbounded operators (see [9,29,67,68]) seem hardly to be applicable to composition operators. The only known efficient criterion for subnormality of unbounded composition operators relies on the consistency condition (CC) (see [16, Theorem 9]). This is why we begin by characterizing families of Borel probability measures (on the positive half-line) indexed by the vertices of $\mathscr{G}_{\eta,\kappa}$ which satisfy (CC) (see Theorem 4.1.1). This enables us to model all such families via collections of measures indexed by the set $\{x_{i,1}\colon i \in J_\eta\}$ which satisfy some natural conditions (see Procedure 4.2.1), where $\{x_{i,1}\colon i \in J_\eta\}$ are ends of edges outgoing from $\omega = x_\kappa$ not lying on the circuit (see Fig. 2) and J_η is the set of all positive integers less than or equal to η. The end x_0 of the edge that outgoes from x_κ and lies on the circuit also plays an important role in our considerations. Namely, assuming both that the Radon–Nikodym derivatives $\{\mathsf{h}_{\phi^n}\}_{n=0}^\infty$ (see Section 3.1) calculated at x_0 and $x_{i,1}$, $i \in J_\eta$, form Stieltjes moment sequences and that appropriate sequences coming from $\{\mathsf{h}_{\phi^n}(x_0)\}_{n=0}^\infty$ are S-determinate, we show that the corresponding composition operator over $\mathscr{G}_{\eta,\kappa}$ is subnormal (see Theorem 4.3.3). The case of $\mathscr{G}_{1,\kappa}$ does not require any determinacy assumption and may be written purely in terms of the Hankel matrices $[\mathsf{h}_{\phi^{i+j}}(x_0)]_{i,j=0}^\infty$ and $[\mathsf{h}_{\phi^{i+j+1}}(x_0)]_{i,j=0}^\infty$ (see Proposition 4.3.4). The proofs of

Theorem 4.3.3 and Proposition 4.3.4 rely on constructing families of measures satisfying (CC). These two results are in the spirit of Lambert's characterization of subnormality of bounded composition operators (see [49]) which is no longer true for unbounded operators (see [42, Theorem 4.3.3] and [15, Section 11]). The case of bounded composition operators over $\mathscr{G}_{\eta,\kappa}$, which is also covered by Lambert's criterion, follows easily from Theorem 4.3.3 (see Proposition 4.3.6). The optimality of the assumptions of Propositions 4.3.4 and 4.3.6 is illustrated by Examples 4.3.5 and 4.3.7.

It follows from [16, Theorems 9 and 17] (see also Theorem 3.1.3) that under the assumption that h_{ϕ^n} takes finite values for all positive integers n, any family of Borel probability measures satisfying (CC) consists of representing measures of Stieltjes moment sequences $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$, where x varies over the vertices of $\mathscr{G}_{\eta,\kappa}$. In Section 4.4 we discuss the question of extending a given family $\{P(x_{i,1},\cdot)\}_{i\in J_\eta}$ of Borel probability measures to a wider one (indexed by $\mathscr{G}_{\eta,\kappa}$) satisfying the consistency condition (CC). According to Theorem 4.4.1, such extension exists if and only if for every $i \in J_\eta$, $\{\mathsf{h}_{\phi^n}(x_{i,1})\}_{n=0}^\infty$ is a Stieltjes moment sequence represented by a measure $P(x_{i,1},\cdot)$ satisfying the conditions (i-b), (i-c) and (i-d) of this theorem. The condition (i-b) refers to moments of the measures $P(x_{i,1},\cdot)$, $i \in J_\eta$. The remaining two are of different nature, namely (i-c) is a system of κ equations (the case of $\kappa = 0$ is not excluded), while (i-d) is a single inequality. In Theorem 4.4.2 we introduce the condition (i-d′) which is a weaker version of (i-d). This turns out to be the key idea that leads to constructing exotic examples. Assuming the S-determinacy of the sequence $\{\mathsf{h}_{\phi^n}(x_\kappa) + c\}_{n=0}^\infty$ for any $c \in (0,\infty)$, it is proved in Theorem 4.4.2 that the conditions (i-d) and (i-d′) are equivalent (provided the remaining ones (i-a), (i-b) and (i-c) are satisfied). However, this is no longer true if the S-determinacy assumption is dropped. We show this by using Procedure 5.2.1 that heavily depends on the existence of a pair of N-extremal measures satisfying some constraints (see Lemma 5.3.1). The task of finding such a pair is challenging. It is realized by transforming via special homotheties the Krein and the Friedrichs measures (which are particular instances of N-extremal measures). The crucial properties of these transformations are described in Lemma 5.3.2. The proof of the existence of the gap between (i-d) and (i-d′) is brought to completion in Theorems 5.4.1 (the case of $\eta \geqslant 2$) and 5.4.2 (the case of $\eta = 1$). Adapting the above technique, we show in Theorem 5.5.2 that for any integer $\eta \geqslant 2$, there exists a non-hyponormal injective composition operator over $\mathscr{G}_{\eta,0}$ which generates Stieltjes moment sequences. The case of $\eta = \infty$ is treated in Theorem 5.5.1. The parallel question of determinacy of moment sequences $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$, $x \in \{x_\kappa\} \cup \{x_{i,j} \colon i \in J_\eta,\ j \in \mathbb{N}\}$, is studied in Section 5 by using the index of H-determinacy introduced by Berg and Duran in [5].

As noted above, the proofs of the main results of the present paper (Theorems 5.4.1, 5.4.2, 5.5.1 and 5.5.2) essentially depend on subtle properties of N-extremal measures. The question of determinacy of moment sequences is of considerable importance in our study as well. Therefore, for the sake of completeness, we collect in Section 2 basic concepts of the classical theory of moments and include some new results in this field. Using [5, Theorem 3.6], we show that a measure which comes from an N-extremal measure by removing an infinite number of its atoms has infinite index of H-determinacy (see Theorem 2.4.1). The Carleman condition, which always guarantees the H-determinacy of Stieltjes moment sequences, is investigated in Section 2.5. The process of transforming moment sequences and their representing measures, including N-extremal ones, via homotheties is described in Section 2.2 (the particular case of transformations induced by translations has already been studied via different approaches in [52,57]). Particular attention is paid to transforming the Krein and the Friedrichs measures (see Theorem 2.2.3). As a consequence, a new way of parametrizing N-extremal measures of H-indeterminate Stieltjes moment sequences is invented (see Theorem 2.2.5) and a trichotomy property of N-extremal measures of H-indeterminate Hamburger moment sequences is proven (see Theorem 2.2.6). The N-extremal measures used in the proofs of Theorems 5.4.1, 5.4.2 and 5.5.1 are derived from the Krein and the Friedrichs measures of an S-indeterminate Stieltjes moment sequence, first by scaling them and then by transforming them via carefully chosen homotheties (see Lemma 5.3.2). As for the proof of Theorem 5.5.2, the above method requires the usage of the Krein and the Friedrichs measures coming from shifted Al-Salam–Carlitz q-polynomials (or, alternatively, from a quartic birth and death process, see Remark 5.5.3). The existence, determinacy and explicit form of orthogonalizing measures for Al-Salam–Carlitz q-polynomials $\{V_n^{(a)}(x;q)\}_{n=0}^\infty$ are discussed in Section 2.3. It is worth mentioning that explicit examples of N-extremal measures such as those used in the present paper are to the best of our knowledge very rare (see, e.g., [39,38]).

The necessary facts concerning composition operators in L^2-spaces over discrete measure spaces, including criteria for their hyponormality and subnormality, are recapitulated in Section 3.1. A variety of relations between Radon–Nikodym derivatives $\{\mathsf{h}_{\phi^n}\}_{n=0}^\infty$ calculated in different vertices of the directed graph $\mathscr{G}_{\eta,\kappa}$ are established in Section 3.4.

1.2. Notation and terminology

Denote by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ the sets of complex numbers, real numbers, nonnegative real numbers, integers, nonnegative integers and positive integers, respectively. Set $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$ and $\mathbb{N}_2 = \mathbb{N} \setminus \{1\}$. Given $k \in \mathbb{Z}_+ \cup \{\infty\}$, we write $J_k = \{i \in \mathbb{N}\colon i \leqslant k\}$ (clearly $J_0 = \emptyset$). The identity map on a set X is denoted by id_X. We write $\mathrm{card}(X)$ for the cardinality of a set X and $\chi_\varDelta$ for the characteristic function of a subset $\varDelta$ of X. The symbol "$\bigsqcup$" denotes the disjoint union of sets. A mapping from X to X is called a *self-map* of X. The σ-algebra of all Borel subsets of a topological space X is denoted by $\mathfrak{B}(X)$. All measures considered in this paper are positive. Since any finite Borel measure ν on $\mathbb{R}$ is automatically regular (see [55, Theorem 2.18]), we can consider its closed support; we denote it by $\mathrm{supp}(\nu)$. Given $t \in \mathbb{R}$, we write δ_t for the Borel probability measure on $\mathbb{R}$ such that $\mathrm{supp}(\delta_t) = \{t\}$. In this paper we will use the notation $\int_a^b$ and $\int_a^\infty$ in place of $\int_{[a,b]}$ and $\int_{[a,\infty)}$ respectively ($a, b \in \mathbb{R}$). We also use the convention that

$0^0 = 1$. The ring of all complex polynomials in one real variable t (which in the context of L^2-spaces are regarded as equivalence classes) is denoted by $\mathbb{C}[t]$.

Let A be an operator in a complex Hilbert space $\mathcal{H}$ (all operators considered in this paper are linear). Denote by $\mathcal{D}(A)$ the domain of A. If A is closable, then the closure of A is denoted by $\bar{A}$. Set $\mathcal{D}^\infty(A) = \bigcap_{n=0}^\infty \mathcal{D}(A^n)$ with $A^0 = I$, where I is the identity operator on $\mathcal{H}$. We say that A is *positive* if $\langle Af, f\rangle \geqslant 0$ for all $f \in \mathcal{D}(A)$. A is said to be *normal* if it is densely defined, $\mathcal{D}(A) = \mathcal{D}(A^*)$ and $\|A^*f\| = \|Af\|$ for all $f \in \mathcal{D}(A)$ (or, equivalently, if and only if A is closed, densely defined and $A^*A = AA^*$, see [69, Proposition on p. 125]). A is called *subnormal* if A is densely defined and there exists a normal operator N in a complex Hilbert space $\mathcal{K}$ with $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding) such that $\mathcal{D}(A) \subseteq \mathcal{D}(N)$ and $Af = Nf$ for all $f \in \mathcal{D}(A)$. A is said to be *hyponormal* if it is densely defined, $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $\|A^*f\| \leqslant \|Af\|$ for all $f \in \mathcal{D}(A)$. Following [42], we say that A *generates Stieltjes moment sequences* if $\mathcal{D}^\infty(A)$ is dense in $\mathcal{H}$ and $\{\|A^nf\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $f \in \mathcal{D}^\infty(A)$ (see Section 2.1 below for the definition and basic properties of Stieltjes moment sequences). It is known that if A is subnormal and $\mathcal{D}^\infty(A)$ is dense in $\mathcal{H}$, then A generates Stieltjes moment sequences (see [12, Proposition 3.2.1]). However, the reverse implication is not true in general (see [60]; see also [42]).

In what follows $\boldsymbol{B}(\mathcal{H})$ stands for the C^*-algebra of all bounded operators in $\mathcal{H}$ whose domains are equal to $\mathcal{H}$.

2. Determinacy in moment problems

2.1. Basic concepts

Denote by $\mathscr{M}$ the set of all Borel measures ν on $\mathbb{R}$ such that $\int_{\mathbb{R}} |t|^n \mathrm{d}\nu(t) < \infty$ for all $n \in \mathbb{Z}_+$. Set $\mathscr{M}^+ = \{\nu \in \mathscr{M} : \operatorname{supp}(\nu) \subseteq \mathbb{R}_+\}$. A sequence $\boldsymbol{\gamma} = \{\gamma_n\}_{n=0}^\infty \subseteq \mathbb{R}$ is said to be a *Hamburger* (resp. *Stieltjes*) *moment sequence* if there exists $\nu \in \mathscr{M}$ (resp. $\nu \in \mathscr{M}^+$) such that

$$\gamma_n = \int_{\mathbb{R}} t^n \mathrm{d}\nu(t), \quad n \in \mathbb{Z}_+;$$

the set of all such measures, called *H-representing* (resp. *S-representing*) measures of $\boldsymbol{\gamma}$, is denoted by $\mathscr{M}(\boldsymbol{\gamma})$ (resp. $\mathscr{M}^+(\boldsymbol{\gamma})$). A Hamburger (resp. Stieltjes) moment sequence $\boldsymbol{\gamma}$ is said to be *H-determinate* (resp. *S-determinate*) if $\operatorname{card}(\mathscr{M}(\boldsymbol{\gamma})) = 1$ (resp. $\operatorname{card}(\mathscr{M}^+(\boldsymbol{\gamma})) = 1$); otherwise, we call it *H-indeterminate* (resp. *S-indeterminate*). We say that a measure $\nu \in \mathscr{M}$ (resp. $\nu \in \mathscr{M}^+$) is H-determinate (resp. S-determinate) if the sequence $\{\int_{\mathbb{R}} t^n \mathrm{d}\nu(t)\}_{n=0}^\infty$ is H-determinate (resp. S-determinate). Similarly, we define H-indeterminacy and S-indeterminacy of measures. Clearly, an S-indeterminate Stieltjes moment sequence is H-indeterminate. It is well-known that a Hamburger moment sequence which has a compactly supported H-representing measure is H-determinate (see [30]). Note that H-determinacy and S-determinacy coincide for Stieltjes moment sequences having S-representing measures vanishing on $\{0\}$ (see [21, Corollary on p. 481]; see also [42, Lemma 2.2.5]).

Let $\boldsymbol{\gamma} = \{\gamma_n\}_{n=0}^{\infty}$ be a Hamburger moment sequence. A measure $\nu \in \mathscr{M}(\boldsymbol{\gamma})$ is called an *N-extremal* measure of $\boldsymbol{\gamma}$ if $\boldsymbol{\gamma}$ is H-indeterminate and $\mathbb{C}[t]$ is dense in $L^2(\nu)$. We say that $\nu \in \mathscr{M}$ is an N-extremal measure if ν is an N-extremal measure of the Hamburger moment sequence $\{\int_{\mathbb{R}} t^n \mathrm{d}\nu(t)\}_{n=0}^{\infty}$. Denote by $\mathscr{M}_e(\boldsymbol{\gamma})$ the set of all N-extremal measures of $\boldsymbol{\gamma}$ and put $\mathscr{M}_e^+(\boldsymbol{\gamma}) = \mathscr{M}_e(\boldsymbol{\gamma}) \cap \mathscr{M}^+$.

Note that if $\boldsymbol{\gamma} = \{\gamma_n\}_{n=0}^{\infty}$ is an H-indeterminate Hamburger moment sequence, then $\operatorname{card}(\mathscr{M}_e(\boldsymbol{\gamma})) = \mathfrak{c}$ (see [57, Theorem 4 and Remark on p. 96]). Moreover, we have:

Lemma 2.1.1 *([57, Theorems 5 and 4.11]). If $\boldsymbol{\gamma} = \{\gamma_n\}_{n=0}^{\infty}$ is an H-indeterminate Hamburger moment sequence, then $\mathbb{R} = \bigsqcup_{\nu \in \mathscr{M}_e(\boldsymbol{\gamma})} \operatorname{supp}(\nu)$, and the closed support of any $\nu \in \mathscr{M}_e(\boldsymbol{\gamma})$ is countably infinite with no accumulation point in $\mathbb{R}$.*

Now we state the M. Riesz characterizations of H-determinacy and N-extremality (see [54, p. 223] or [30, Theorem on p. 58]) and the Berg–Thill characterization of S-determinacy (see [6, Theorem 3.8] or [7, Proposition 1.3]).

Lemma 2.1.2. (i) *A measure $\nu \in \mathscr{M}$ is H-determinate* (*resp. N-extremal*) *if and only if $\mathbb{C}[t]$ is dense in $L^2((1+t^2)\mathrm{d}\nu(t))$* (*resp. $\mathbb{C}[t]$ is dense in $L^2(\nu)$ and not dense in $L^2((1+t^2)\mathrm{d}\nu(t))$*). (ii) *A measure $\nu \in \mathscr{M}^+$ is S-determinate if and only if $\mathbb{C}[t]$ is dense in both $L^2((1+t)\mathrm{d}\nu(t))$ and $L^2(t(1+t)\mathrm{d}\nu(t))$.*

The above enables us to formulate a comparison test for determinacy.

Proposition 2.1.3 *(Comparison test). Let ρ and ν be Borel measures on $\mathbb{R}$ such that $\nu \in \mathscr{M}$* (*resp. $\nu \in \mathscr{M}^+$*) *and $\rho(\sigma) \leqslant M\,\nu(\sigma)$ for every $\sigma \in \mathfrak{B}(\mathbb{R})$ and for some $M \in \mathbb{R}_+$. Then $\rho \in \mathscr{M}$* (*resp. $\rho \in \mathscr{M}^+$*). *Moreover, if ν is H-determinate* (*resp. S-determinate*), *then ρ is H-determinate* (*resp. S-determinate*).

Proof. We deal only with the case of H-determinacy; the other case can be treated similarly. The standard measure-theoretic argument implies that $\rho \in \mathscr{M}$. Since $\rho \leqslant M\,\nu$, we deduce from [55, Theorem 3.13] that $L^2((1+t^2)\mathrm{d}\nu(t)) \ni f \mapsto f \in L^2((1+t^2)\mathrm{d}\rho(t))$ is a well-defined bounded operator with dense range. Hence, applying Lemma 2.1.2 completes the proof. $\square$

Corollary 2.1.4. *Suppose that $\{\gamma_{1,n}\}_{n=0}^{\infty}$ and $\{\gamma_{2,n}\}_{n=0}^{\infty}$ are Hamburger* (*resp. Stieltjes*) *moment sequences such that $\{\gamma_{1,n} + \gamma_{2,n}\}_{n=0}^{\infty}$ is H-determinate* (*resp. S-determinate*). *Then both $\{\gamma_{1,n}\}_{n=0}^{\infty}$ and $\{\gamma_{2,n}\}_{n=0}^{\infty}$ are H-determinate* (*resp. S-determinate*).

Remark 2.1.5. It follows from Corollary 2.1.4 that if $\{\gamma_n\}_{n=0}^{\infty}$ is a Stieltjes moment sequence such that $\{\gamma_n + c\}_{n=0}^{\infty}$ is S-determinate for some $c \in (0, \infty)$, then $\{\gamma_n\}_{n=0}^{\infty}$ is

S-determinate. This may suggest that if $\{\gamma_n\}_{n=0}^\infty$ is an S-determinate Stieltjes moment sequence, then so is $\{\gamma_n + c\}_{n=0}^\infty$ for some $c \in (0, \infty)$. However, in general, this is not true. In fact, one can show more; namely there exists an H-determinate Stieltjes moment sequence $\{\gamma_n\}_{n=0}^\infty$ such that $\{\gamma_n + c\}_{n=0}^\infty$ is S-indeterminate for all $c \in (0, \infty)$. Indeed, as noticed by C. Berg (private communication), if ν is an N-extremal measure of an S-indeterminate Stieltjes moment sequence such that $\inf \operatorname{supp}(\nu) = 1$ (e.g., the orthogonalizing measure $\beta^{(a;q)}$ for the Al-Salam–Carlitz q-polynomials $\{V_n^{(a)}(x;q)\}_{n=0}^\infty$ with $0 < q < a \leqslant 1$ is N-extremal and its closed support is equal to $\{q^{-n}\}_{n=0}^\infty$, see Section 2.3), then the measure $\mu := \nu - \nu(\{1\})\delta_1 \in \mathscr{M}^+$ is H-determinate and for every $c \in (0, \infty)$, the measure $\mu + c\delta_1$ is N-extremal (see [5, Theorem 3.6 and Lemma 3.7]) and consequently, since $\inf \operatorname{supp}(\mu + c\delta_1) > 0$, it is S-indeterminate (see [21, Corollary on p. 481] or [42, Lemma 2.2.5]).

The following lemma will be used in Section 4.3.

Lemma 2.1.6. *If $\{\gamma_n\}_{n=0}^\infty \subseteq \mathbb{R}_+$, then the following conditions are equivalent:*

(i) *$\{\gamma_n\}_{n=0}^\infty$ is a Stieltjes moment sequence which has an S-representing measure vanishing on $[0, 1)$,*
(ii) *$0 \leqslant \sum_{i,j=0}^n \gamma_{i+j}\lambda_i\bar{\lambda}_j \leqslant \sum_{i,j=0}^n \gamma_{i+j+1}\lambda_i\bar{\lambda}_j$ for all finite sequences $\{\lambda_i\}_{i=0}^n$ of complex numbers,*
(iii) *$\{\gamma_n\}_{n=0}^\infty$ is a Stieltjes moment sequence and $\sum_{i,j=0}^n (\gamma_{i+j+1} - \gamma_{i+j})\lambda_i\bar{\lambda}_j \geqslant 0$ for all finite sequences $\{\lambda_i\}_{i=0}^n$ of complex numbers.*

Proof. (i)⇒(ii) Obvious.

(ii)⇔(iii) Apply the Stieltjes theorem (see [4, Theorem 6.2.5]).

(ii)⇒(i) Let $\Lambda\colon \mathbb{C}[t] \to \mathbb{C}$ be a linear functional such that $\Lambda(t^n) = \gamma_n$ for all $n \in \mathbb{Z}_+$. Take $p \in \mathbb{C}[t]$ which is nonnegative on $[1, \infty)$. Since $p(t+1)$ is nonnegative on $[0, \infty)$, there exist $q_1, q_2 \in \mathbb{C}[t]$ such that $p = (t-1)|q_1|^2 + |q_2|^2$ (see [53, Problem 45 on p. 78]). Hence $\Lambda(p) \geqslant 0$. Applying the Riesz–Haviland theorem (see [35]) completes the proof. □

The question of when $\mathscr{M}_e^+(\boldsymbol{\gamma})$ is nonempty has the following answer.

Lemma 2.1.7. *Suppose $\boldsymbol{\gamma}$ is an H-indeterminate Stieltjes moment sequence. Then $\mathscr{M}_e^+(\boldsymbol{\gamma}) \neq \emptyset$. Moreover, $\boldsymbol{\gamma}$ is S-determinate if and only if* $\operatorname{card}(\mathscr{M}_e^+(\boldsymbol{\gamma})) = 1$.

Proof. Let A be a symmetric operator in a complex Hilbert space $\mathcal{H}$ and $e \in \mathcal{D}^\infty(A)$ be such that $\mathcal{D}(A)$ is the linear span of $\{A^n e\colon n \in \mathbb{Z}_+\}$, and $\gamma_n = \langle A^n e, e\rangle$ for all $n \in \mathbb{Z}_+$ (see [57, (1.10)]). By assumption, A is a positive operator with deficiency indices $(1, 1)$ (see [57, Theorem 2 and Corollary 2.9]). Hence, the Friedrichs extension S of A differs from $\bar{A}$ (see [69, Theorem 5.38]). As a consequence, $\langle E(\cdot)e, e\rangle \in \mathscr{M}_e^+(\boldsymbol{\gamma})$, where E is the spectral measure of S (see [42, p. 3951]). This also proves necessity in the "moreover" part. The sufficiency is a direct consequence of [57, Theorem 4]. □

Recall that if $\boldsymbol{\gamma} = \{\gamma_n\}_{n=0}^\infty$ is an S-indeterminate Stieltjes moment sequence, then $\operatorname{card}(\mathscr{M}_e^+(\boldsymbol{\gamma})) = \mathfrak{c}$ and there exist distinct measures $\alpha, \beta \in \mathscr{M}_e^+(\boldsymbol{\gamma})$ (uniquely determined by (2.1.1)) such that for every $\rho \in \mathscr{M}_e^+(\boldsymbol{\gamma}) \setminus \{\alpha, \beta\}$,

$$0 = \inf \operatorname{supp}(\alpha) < \inf \operatorname{supp}(\rho) < \inf \operatorname{supp}(\beta); \tag{2.1.1}$$

α and β are called the *Krein* and the *Friedrichs* measures of $\boldsymbol{\gamma}$, respectively. These two particular N-extremal measures come from the Krein and the Friedrichs extensions of a positive operator attached to $\boldsymbol{\gamma}$. The reader is referred to [52] for the case of Friedrichs extensions and to [57] for a complete and up-to-date operator approach to moment problems (see also [42, Section 2]).

2.2. Transforming moments via homotheties

In this section we investigate transformations acting on real sequences induced by homotheties of $\mathbb{R}$. Such transformations are shown to preserve many properties of Hamburger and Stieltjes moment sequences. The particular case of transformations induced by translations has been considered in [52, Section 3] and [57, p. 96] (with different approaches).

Fix $\vartheta \in (0, \infty)$ and $a \in \mathbb{R}$. Let us define the self-map $\psi_{\vartheta,a}$ of $\mathbb{R}$ by

$$\psi_{\vartheta,a}(t) = \vartheta(t + a), \quad t \in \mathbb{R}. \tag{2.2.1}$$

Note that $\psi_{\vartheta,a}$ is a strictly increasing homeomorphism of $\mathbb{R}$ onto itself such that

$$\psi_{1,0} = \mathrm{id}_{\mathbb{R}}, \quad \psi_{\tilde{\vartheta},\tilde{a}} \circ \psi_{\vartheta,a} = \psi_{\tilde{\vartheta}\vartheta, \frac{\tilde{a}}{\vartheta}+a}, \quad \psi_{\vartheta,a}^{-1} = \psi_{\frac{1}{\vartheta}, -a\vartheta} \tag{2.2.2}$$

for all $\tilde{\vartheta} \in (0, \infty)$ and $\tilde{a} \in \mathbb{R}$. Next, we define the linear self-map $T_{\vartheta,a}$ of $\mathbb{R}^{\mathbb{Z}_+}$ by

$$(T_{\vartheta,a}\boldsymbol{\gamma})_n = \sum_{j=0}^n \binom{n}{j} a^{n-j}\vartheta^n \gamma_j, \quad n \in \mathbb{Z}_+, \ \boldsymbol{\gamma} = \{\gamma_n\}_{n=0}^\infty \subseteq \mathbb{R}. \tag{2.2.3}$$

The proof of Lemma 2.2.1 below, being elementary, is omitted.

Lemma 2.2.1. *The following hold for all $\vartheta, \tilde{\vartheta} \in (0, \infty)$ and $a, \tilde{a} \in \mathbb{R}$:*

$$\begin{gathered} T_{\vartheta,a} \text{ is a bijection of } \mathbb{R}^{\mathbb{Z}_+} \text{ onto itself,} \\ T_{1,0} = \mathrm{id}_{\mathbb{R}^{\mathbb{Z}_+}}, \quad T_{\tilde{\vartheta},\tilde{a}} T_{\vartheta,a} = T_{\tilde{\vartheta}\vartheta, \frac{\tilde{a}}{\vartheta}+a}, \quad T_{\vartheta,a}^{-1} = T_{\frac{1}{\vartheta}, -a\vartheta}. \end{gathered} \tag{2.2.4}$$

In view of (2.2.2) and Lemma 2.2.1, the correspondence $\psi_{\vartheta,a} \mapsto T_{\vartheta,a}$ defines a faithful representation of the group of all strictly increasing homotheties of $\mathbb{R}$. Moreover, by (2.2.3) and (2.2.4), we have, for all $\vartheta \in (0, \infty)$ and $a \in \mathbb{R}$,

$$(T_{\vartheta,a}^{-1}\boldsymbol{\gamma})_n = \sum_{j=0}^{n} \binom{n}{j} (-a)^{n-j} \vartheta^{-j} \gamma_j, \quad n \in \mathbb{Z}_+, \ \boldsymbol{\gamma} = \{\gamma_n\}_{n=0}^{\infty} \subseteq \mathbb{R}.$$

In Lemma 2.2.2 and Theorem 2.2.3 below we state properties of $T_{\vartheta,a}$ which are relevant for further considerations. If ν is a Borel measure on $\mathbb{R}$ and φ is a homeomorphism of $\mathbb{R}$ onto itself, then $\nu \circ \varphi$ is the Borel measure on $\mathbb{R}$ given by

$$\nu \circ \varphi(\sigma) = \nu(\varphi(\sigma)), \quad \sigma \in \mathfrak{B}(\mathbb{R}). \tag{2.2.5}$$

Lemma 2.2.2. *Let $\vartheta \in (0, \infty)$ and $a \in \mathbb{R}$. Then*

(i) *$T_{\vartheta,a}$ is a self-bijection on the set of all Hamburger moment sequences,*
(ii) *if $\boldsymbol{\gamma}$ is a Hamburger moment sequence, then the mapping*

$$\mathscr{M}(\boldsymbol{\gamma}) \ni \nu \mapsto \nu \circ \psi_{\vartheta,a}^{-1} \in \mathscr{M}(T_{\vartheta,a}\boldsymbol{\gamma}) \tag{2.2.6}$$

is a well-defined bijection with the inverse given by

$$\mathscr{M}(T_{\vartheta,a}\boldsymbol{\gamma}) \ni \nu \mapsto \nu \circ \psi_{\vartheta,a} \in \mathscr{M}(\boldsymbol{\gamma});$$

in particular, $\boldsymbol{\gamma}$ is H-determinate if and only if $T_{\vartheta,a}\boldsymbol{\gamma}$ is H-determinate,
(iii) *if $\boldsymbol{\gamma}$ is an H-indeterminate Hamburger moment sequence, then so is $T_{\vartheta,a}\boldsymbol{\gamma}$ and the mapping defined by (2.2.6) maps $\mathscr{M}_e(\boldsymbol{\gamma})$ onto $\mathscr{M}_e(T_{\vartheta,a}\boldsymbol{\gamma})$,*
(iv) *if $\boldsymbol{\gamma}$ is a nonzero Hamburger moment sequence and $\nu \in \mathscr{M}(\boldsymbol{\gamma})$, then*

$$\operatorname{supp}(\nu \circ \psi_{\vartheta,a}^{-1}) = \psi_{\vartheta,a}(\operatorname{supp}(\nu)), \tag{2.2.7}$$

$$\inf \operatorname{supp}(\nu \circ \psi_{\vartheta,a}^{-1}) = \psi_{\vartheta,a}(\inf \operatorname{supp}(\nu)), \tag{2.2.8}$$

with convention that $\psi_{\vartheta,a}(-\infty) = -\infty$,
(v) *if $a \geqslant 0$, $\boldsymbol{\gamma}$ is an H-indeterminate Stieltjes moment sequence and $\nu \in \mathscr{M}_e^+(\boldsymbol{\gamma})$, then $T_{\vartheta,a}\boldsymbol{\gamma}$ is an H-indeterminate Stieltjes moment sequence and $\nu \circ \psi_{\vartheta,a}^{-1} \in \mathscr{M}_e^+(T_{\vartheta,a}\boldsymbol{\gamma})$.*

Proof. (i)&(ii) If $\boldsymbol{\gamma}$ is a Hamburger moment sequence and $\nu \in \mathscr{M}(\boldsymbol{\gamma})$, then, by the measure transport theorem (see [3, Theorem 1.6.12]), we have

$$(T_{\vartheta,a}\boldsymbol{\gamma})_n = \int_{\mathbb{R}} \big(\psi_{\vartheta,a}(t)\big)^n \mathrm{d}\nu(t) = \int_{\mathbb{R}} t^n \mathrm{d}\nu \circ \psi_{\vartheta,a}^{-1}(t), \quad n \in \mathbb{Z}_+,$$

which means that $T_{\vartheta,a}\boldsymbol{\gamma}$ is a Hamburger moment sequence and $\nu \circ \psi_{\vartheta,a}^{-1} \in \mathscr{M}(T_{\vartheta,a}\boldsymbol{\gamma})$. The above combined with (2.2.2) and (2.2.4) completes the proof of (i) and (ii).

(iii) Let $\boldsymbol{\gamma}$ be a Hamburger moment sequence and $\nu \in \mathscr{M}(\boldsymbol{\gamma})$. Since

$$\sup_{t\in\mathbb{R}}(1+\varphi(t)^2)/(1+t^2) < \infty, \quad \varphi \in \big\{\psi_{\vartheta,a}, \psi_{\vartheta,a}^{-1}\big\},$$

we deduce from the measure transport theorem that the mapping

$$W\colon L^2((1+t^2)\mathrm{d}\nu\circ\psi_{\vartheta,a}^{-1}(t))\ni f\to f\circ\psi_{\vartheta,a}\in L^2((1+t^2)\mathrm{d}\nu(t))$$

is a well-defined linear homeomorphism (with the inverse $g\to g\circ\psi_{\vartheta,a}^{-1}$) such that $W(\mathbb{C}[t])=\mathbb{C}[t]$. Similarly, $V\colon L^2(\nu\circ\psi_{\vartheta,a}^{-1})\ni f\to f\circ\psi_{\vartheta,a}\in L^2(\nu)$ is a unitary isomorphism such that $V(\mathbb{C}[t])=\mathbb{C}[t]$. This, (ii) and Lemma 2.1.2(i) yield (iii).

(iv) The equality (2.2.7) is a direct consequence of the definition of the closed support of a measure (see also [62, Lemma 3.2] for a more general result). Clearly, (2.2.7) implies (2.2.8).

(v) Apply (iii) and (iv). □

Theorem 2.2.3. *Let $\vartheta\in(0,\infty)$ and $a\in\mathbb{R}$. Suppose $\boldsymbol{\gamma}=\{\gamma_n\}_{n=0}^\infty$ is an S-indeterminate Stieltjes moment sequence and β is its Friedrichs measure. Then $T_{\vartheta,a}\boldsymbol{\gamma}$ is an H-indeterminate Hamburger moment sequence and the following holds*:

(i) *if $c>0$, then $T_{\vartheta,a}\boldsymbol{\gamma}$ is an S-indeterminate Stieltjes moment sequence and $\beta\circ\psi_{\vartheta,a}^{-1}$ is the Friedrichs measure of $T_{\vartheta,a}\boldsymbol{\gamma}$,*
(ii) *if $c=0$, then $T_{\vartheta,a}\boldsymbol{\gamma}$ is an S-determinate Stieltjes moment sequence,*
(iii) *if $c<0$, then $T_{\vartheta,a}\boldsymbol{\gamma}$ is not a Stieltjes moment sequence,*

where $c:=\psi_{\vartheta,a}\big(\inf\operatorname{supp}(\beta)\big)$.

Proof. By Lemma 2.2.2, $T_{\vartheta,a}\boldsymbol{\gamma}$ is an H-indeterminate Hamburger moment sequence such that

$$\beta\circ\psi_{\vartheta,a}^{-1}\in\mathscr{M}_e(T_{\vartheta,a}\boldsymbol{\gamma})\ \text{and}\ \inf\operatorname{supp}(\beta\circ\psi_{\vartheta,a}^{-1})=\psi_{\vartheta,a}(\inf\operatorname{supp}(\beta))=c.\tag{2.2.9}$$

Note that

$$\inf\operatorname{supp}(\beta\circ\psi_{\vartheta,a}^{-1})\geqslant\inf\operatorname{supp}(\rho),\quad \rho\in\mathscr{M}_e(T_{\vartheta,a}\boldsymbol{\gamma}).\tag{2.2.10}$$

Indeed, otherwise, by (2.2.9) and Lemma 2.2.2, $\inf\operatorname{supp}(\beta)<\inf\operatorname{supp}(\rho\circ\psi_{\vartheta,a})$ and $\rho\circ\psi_{\vartheta,a}\in\mathscr{M}_e^+(\boldsymbol{\gamma})$, which contradicts (2.1.1).

(i) By (2.2.9) and [21, Corollary on p. 481] (see also [42, Lemma 2.2.5]), $T_{\vartheta,a}\boldsymbol{\gamma}$ is an S-indeterminate Stieltjes moment sequence. Denote by ρ its Friedrichs measure. If $\beta\circ\psi_{\vartheta,a}^{-1}\neq\rho$, then by (2.1.1) and (2.2.9), $\inf\operatorname{supp}(\beta\circ\psi_{\vartheta,a}^{-1})<\inf\operatorname{supp}(\rho)$, which would contradict (2.2.10).

(ii) By (2.2.9), $T_{\vartheta,a}\boldsymbol{\gamma}$ is a Stieltjes moment sequence and $\beta\circ\psi_{\vartheta,a}^{-1}\in\mathscr{M}_e^+(T_{\vartheta,a}\boldsymbol{\gamma})$. If $T_{\vartheta,a}\boldsymbol{\gamma}$ was S-indeterminate and ρ was its Friedrichs measure, then by (2.2.9), $\beta\circ\psi_{\vartheta,a}^{-1}\neq\rho$, which, as in (i), would contradict (2.2.10).

(iii) If $T_{\vartheta,a}\boldsymbol{\gamma}$ was a Stieltjes moment sequence, then by Lemma 2.1.7, there would exist $\rho\in\mathscr{M}_e^+(T_{\vartheta,a}\boldsymbol{\gamma})$, and thus by (2.2.9), $\inf\operatorname{supp}(\beta\circ\psi_{\vartheta,a}^{-1})<\inf\operatorname{supp}(\rho)$, which would contradict (2.2.10). □

Corollary 2.2.4. *Let $\vartheta, a \in (0, \infty)$. Suppose $\boldsymbol{\gamma}$ is an S-indeterminate Stieltjes moment sequence and α and β are its Krein and Friedrichs measures, respectively. Then $T_{\vartheta,a}\boldsymbol{\gamma}$ is an S-indeterminate Stieltjes moment sequence, $\alpha \circ \psi_{\vartheta,a}^{-1} \in \mathscr{M}_e^+(T_{\vartheta,a}\boldsymbol{\gamma})$, $\beta \circ \psi_{\vartheta,a}^{-1}$ is the Friedrichs measure of $T_{\vartheta,a}\boldsymbol{\gamma}$ and*

$$0 < \inf \operatorname{supp}(\alpha \circ \psi_{\vartheta,a}^{-1}) < \inf \operatorname{supp}(\beta \circ \psi_{\vartheta,a}^{-1}). \tag{2.2.11}$$

In particular, $\alpha \circ \psi_{\vartheta,a}^{-1}$ is neither the Krein nor the Friedrichs measure of $T_{\vartheta,a}\boldsymbol{\gamma}$.

Proof. In view of (2.1.1) and Lemma 2.2.2, (2.2.11) holds and $\alpha \circ \psi_{\vartheta,a}^{-1} \in \mathscr{M}_e^+(T_{\vartheta,a}\boldsymbol{\gamma})$. This together with (2.1.1) and Theorem 2.2.3 completes the proof. $\square$

The particular case of Theorem 2.2.3 with $\vartheta = 1$ (without the statement that $\beta \circ \psi_{1,a}^{-1}$ is the Friedrichs measure of $T_{1,a}\boldsymbol{\gamma} = \boldsymbol{\gamma}(a)$) appeared in [57, Theorem 3.3] with a very brief outline of the proof based on the von Neumann theory of selfadjoint extensions of symmetric operators. In turn, the particular case of Corollary 2.2.4 with $\vartheta = 1$ (without any statement on the Krein measure) appeared in [52, Proposition 3.5] with another approach based on the Nevanlinna parametrization.

It follows from Corollary 2.2.4 that if $\vartheta, a > 0$, then the transformation $T_{\vartheta,a}$ preserves S-indeterminate Stieltjes moment sequences, and the corresponding mapping defined by (2.2.6) preserves the Friedrichs measures (but never the Krein ones). The situation changes drastically when $a < 0$ (for example, when we consider $T_{\vartheta,b}^{-1}$ with $b > 0$; see (2.2.4)). This is because the quantity $c = \psi_{\vartheta,a}(\inf \operatorname{supp}(\beta))$ may happen to be negative (see Theorem 2.2.3).

The above-mentioned properties of self-maps $T_{\vartheta,a}$ enable us to parameterize N-extremal measures of H-indeterminate Stieltjes moment sequences in a new way.

Theorem 2.2.5. *Suppose $\boldsymbol{\gamma} = \{\gamma_n\}_{n=0}^\infty$ is an H-indeterminate Stieltjes moment sequence. Set $t_0 = \inf \operatorname{supp}(\beta)$, where β is either the Friedrichs measure of $\boldsymbol{\gamma}$ if $\boldsymbol{\gamma}$ is S-indeterminate, or β is the unique S-representing measure of $\boldsymbol{\gamma}$ otherwise. Then*

(i) *if γ is S-determinate, then $t_0 = 0$ and $\beta \in \mathscr{M}_e^+(\boldsymbol{\gamma})$,*
(ii) *if γ is S-indeterminate, then $t_0 > 0$,*
(iii) *for every $t \in (-\infty, t_0]$ there exists a unique $\nu_t \in \mathscr{M}_e(\boldsymbol{\gamma})$ such that*

$$\inf \operatorname{supp}(\nu_t) = t, \tag{2.2.12}$$

(iv) *the mapping $(-\infty, t_0] \ni t \mapsto \nu_t \in \mathscr{M}_e(\boldsymbol{\gamma})$ is a bijection,*
(v) *$\mathscr{M}_e^+(\boldsymbol{\gamma}) = \{\nu_t \colon t \in [0, t_0]\}$,*
(vi) *the closed support of each N-extremal measure of $\boldsymbol{\gamma}$ is bounded from below,*
(vii) *$\operatorname{supp}(\nu_t) \cap (-\infty, t_0] = \{t\}$ for every $t \in (-\infty, t_0]$,*
(viii) *$[0, t_0] \subsetneq \bigcup_{\mu \in \mathscr{M}_e^+(\boldsymbol{\gamma})} \operatorname{supp}(\mu)$ and $\operatorname{card}\big(\mathbb{R}_+ \setminus \bigcup_{\mu \in \mathscr{M}_e^+(\boldsymbol{\gamma})} \operatorname{supp}(\mu)\big) = \mathfrak{c}$.*

Proof. Assume that $\boldsymbol{\gamma}$ is S-determinate. Then by [21, Corollary on p. 481] (see also [42, Lemma 2.2.5]), $0 \in \operatorname{supp}(\beta)$ and thus $t_0 = 0$. In view of Lemma 2.1.7, $\beta \in \mathscr{M}_e^+(\boldsymbol{\gamma})$ and the measure $\nu_0 := \beta$ satisfies (2.2.12). Now take $t \in (-\infty, 0)$. Then by Lemma 2.2.2, the sequence $T_{1,-t}\boldsymbol{\gamma}$ is H-indeterminate, $\beta \circ \psi_{1,-t}^{-1} \in \mathscr{M}_e(T_{1,-t}\boldsymbol{\gamma})$ and $\inf \operatorname{supp}(\beta \circ \psi_{1,-t}^{-1}) = |t| > 0$. Applying [21, Corollary on p. 481] again, we see that $T_{1,-t}\boldsymbol{\gamma}$ is S-indeterminate. Let ρ_t be the Krein measure of $T_{1,-t}\boldsymbol{\gamma}$. Since $\boldsymbol{\gamma} = T_{1,t}T_{1,-t}\boldsymbol{\gamma}$, we infer from Lemma 2.2.2 and (2.1.1) that $\nu_t := \rho_t \circ \psi_{1,t}^{-1} \in \mathscr{M}_e(\boldsymbol{\gamma})$ and $\inf \operatorname{supp}(\nu_t) = t$.

Assume now that $\boldsymbol{\gamma}$ is S-indeterminate. Let $t \in (-\infty, t_0)$. Since

$$c := \psi_{1,-t}(\inf \operatorname{supp}(\beta)) = t_0 - t > 0,$$

we deduce from Theorem 2.2.3(i) that $T_{1,-t}\boldsymbol{\gamma}$ is S-indeterminate. Taking the Krein measure ρ_t of $T_{1,-t}\boldsymbol{\gamma}$ and arguing as in the previous paragraph, we see that $\nu_t := \rho_t \circ \psi_{1,t}^{-1} \in \mathscr{M}_e(\boldsymbol{\gamma})$ satisfies (2.2.12). If $t = t_0$, then $\nu_{t_0} := \beta$ does the job.

In both cases, S-determinate and S-indeterminate, the uniqueness of $\nu_t \in \mathscr{M}_e(\boldsymbol{\gamma})$ satisfying (2.2.12) follows from Lemma 2.1.1. Altogether this proves (i), (ii) and (iii). Clearly, by (2.2.12), the mapping $(-\infty, t_0] \ni t \mapsto \nu_t \in \mathscr{M}_e(\boldsymbol{\gamma})$ is injective. To prove its surjectivity, take $\nu \in \mathscr{M}_e(\boldsymbol{\gamma})$. By Lemma 2.1.1 and (2.1.1), there exists $t \in (-\infty, t_0] \cap \operatorname{supp}(\nu)$. Then $t \in \operatorname{supp}(\nu_t) \cap \operatorname{supp}(\nu)$ and so, by Lemma 2.1.1, $\nu_t = \nu$. This proves (iv) and consequently (v) and (vi). The condition (vii) is a direct consequence of (iii) and Lemma 2.1.1. To prove (viii) take any $t \in (-\infty, 0)$. By (iii) and Lemma 2.1.1, $\operatorname{supp}(\nu_t) \cap (t_0, \infty)$ is a nonempty (in fact, a countably infinite) subset of $\mathbb{R}_+$ which is disjoint with $\bigcup_{\mu \in \mathscr{M}_e^+(\boldsymbol{\gamma})} \operatorname{supp}(\mu)$ and the latter set being unbounded properly contains $[0, t_0]$. It follows from (iv) and Lemma 2.1.1 that the sets $\operatorname{supp}(\nu_t) \cap (t_0, \infty)$, $t \in (-\infty, 0)$, are nonempty and disjoint. This completes the proof of (viii) and the theorem. □

We conclude this section by stating the following trichotomy property of N-extremal measures of H-indeterminate Hamburger moment sequences.

Theorem 2.2.6 *(Trichotomy). Suppose that $\boldsymbol{\gamma} = \{\gamma_n\}_{n=0}^\infty$ is an H-indeterminate Hamburger moment sequence. Then exactly one of the following three conditions holds*:

(i) *for every $\nu \in \mathscr{M}_e(\boldsymbol{\gamma})$, $\inf \operatorname{supp}(\nu) = -\infty$ and $\sup \operatorname{supp}(\nu) = \infty$,*
(ii) *for every $\nu \in \mathscr{M}_e(\boldsymbol{\gamma})$, $\inf \operatorname{supp}(\nu) > -\infty$ and $\sup \operatorname{supp}(\nu) = \infty$,*
(iii) *for every $\nu \in \mathscr{M}_e(\boldsymbol{\gamma})$, $\inf \operatorname{supp}(\nu) = -\infty$ and $\sup \operatorname{supp}(\nu) < \infty$.*

Proof. Suppose that (i) does not hold. Then there exists $\nu \in \mathscr{M}_e(\boldsymbol{\gamma})$ such that either $s := \inf \operatorname{supp}(\nu) > -\infty$ or $\sup \operatorname{supp}(\nu) < \infty$. In the former case, $T_{1,-s}\boldsymbol{\gamma}$ is an H-indeterminate Stieltjes moment sequence (see Lemma 2.2.2), and thus applying Theorem 2.2.5 to $T_{1,-s}\boldsymbol{\gamma}$ and returning to $\boldsymbol{\gamma}$ via the self-map $T_{1,s}$ we get (ii). Considering the H-indeterminate Hamburger moment sequence $\widetilde{\boldsymbol{\gamma}} := \{(-1)^n\gamma_n\}_{n=0}^\infty$, one can show that the latter case leads to (iii) (as in the proof of Lemma 2.2.2, we verify that the mapping $\mathscr{M}(\boldsymbol{\gamma}) \ni \nu \mapsto \nu \circ \psi^{-1} \in$

$\mathscr{M}(\widetilde{\gamma})$ is a bijection which maps $\mathscr{M}_e(\gamma)$ onto $\mathscr{M}_e(\widetilde{\gamma})$ and $\operatorname{supp}(\nu \circ \psi^{-1}) = \psi(\operatorname{supp}(\nu))$ for every $\nu \in \mathscr{M}(\gamma)$, where ψ is a self-map of $\mathbb{R}$ given by $\psi(t) = -t$ for $t \in \mathbb{R}$). This completes the proof. $\square$

We refer the reader to [39, pages 93 and 94] (see also [38, Theorem 21.5.3]), where all N-extremal measures of the H-indeterminate Hamburger moment sequence arising from q^{-1}-Hermite polynomials with $q \in (0,1)$ are explicitly calculated. Since their closed supports are bounded neither from below nor from above, we see that none of the conditions (i), (ii) and (iii) of Theorem 2.2.6 is redundant.

It is worth mentioning that, by using the Nevanlinna parametrization and methods of orthogonal polynomials (see [8, Lemma 2.2.1 and Remark 2.2.2]), one can provide alternative proofs of Theorems 2.2.5 and 2.2.6.

2.3. The Al-Salam–Carlitz moment problem

Orthogonal q-polynomials introduced by Al-Salam and Carlitz in [1] give rise to examples of S-indeterminate Stieltjes moment sequences for which some particular N-extremal measures are explicitly known. These special measures help us to show that the directed graph $\mathscr{G}_{2,0}$ admits a non-hyponormal composition operator generating Stieltjes moment sequences (see Theorem 5.5.2).

We begin by recalling the definition of q-Pochhammer symbol (called also q-shifted factorial). For $z, q \in \mathbb{C}$, we write

$$(z;q)_n = \begin{cases} 1 & \text{if } n = 0, \\ \prod_{j=1}^{n}(1 - zq^{j-1}) & \text{if } n \in \mathbb{N}, \\ \prod_{j=1}^{\infty}(1 - zq^{j-1}) & \text{if } n = \infty,\ |q| < 1. \end{cases} \tag{2.3.1}$$

(See [56, Section VII.1] for more on infinite products.) Following [22, Section VI.10], we extend the original definition of q-polynomials of Al-Salam and Carlitz to cover the case of $|q| > 1$. Given $a \in \mathbb{C}$ and $q \in \mathbb{C} \setminus \{0\}$, we define $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$, the sequence of complex polynomials in one variable x, by the recurrence formula

$$\begin{gathered} V_{n+1}^{(a)}(x;q) = \Big(x - \frac{1+a}{q^n}\Big) V_n^{(a)}(x;q) - a\frac{1-q^n}{q^{2n-1}} V_{n-1}^{(a)}(x;q), \quad n \in \mathbb{Z}_+, \\ V_{-1}^{(a)}(x;q) = 0, \qquad V_0^{(a)}(x;q) = 1. \end{gathered} \tag{2.3.2}$$

The generating function for $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$ can be described as follows (see [1] and [22, Section VI.10]). If $a, q, z \in \mathbb{C}$, $x \in \mathbb{R}$ and $q \neq 0$, then

$$\sum_{n=0}^{\infty} V_n^{(a)}(x;q) \frac{(-1)^n q^{\frac{n(n-1)}{2}} z^n}{(q;q)_n} = \begin{cases} \frac{(xz;q)_\infty}{(z;q)_\infty (az;q)_\infty} & \text{if } |z| < r_a, |q| < 1, \\ \frac{(z\frac{1}{q};\frac{1}{q})_\infty (z\frac{a}{q};\frac{1}{q})_\infty}{(z\frac{x}{q};\frac{1}{q})_\infty} & \text{if } |z| < \frac{|q|}{|x|}, |q| > 1, \end{cases} \tag{2.3.3}$$

where $r_a = \min\{1, \frac{1}{|a|}\}$ with the convention that $\frac{1}{0} = \infty$. The function of one complex variable z given by the right-hand side of (2.3.3) is called the generating function for $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$. Clearly, it is meromorphic and has simple poles because $(z;q)_\infty = 0$ if and only if $1 - zq^n = 0$ for some (unique) $n \in \mathbb{Z}_+$.

By (2.3.2), $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$ satisfies the following general recurrence relation

$$\begin{aligned} P_{n+1}(x) = (x - c_n)P_n(x) - \lambda_n P_{n-1}(x), \quad n \in \mathbb{Z}_+, \\ P_{-1}(x) = 0, \qquad P_0(x) = 1, \end{aligned} \tag{2.3.4}$$

where $\{c_n\}_{n=0}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ are sequences of complex numbers (λ_0 can be chosen arbitrarily) and $\{P_n(x)\}_{n=0}^{\infty}$ are polynomials. Suppose that (2.3.4) holds. Then $\{P_n(x)\}_{n=0}^{\infty}$ is a Hamel basis of $\mathbb{C}[x]$ and thus there exists a unique linear functional $L\colon \mathbb{C}[x] \to \mathbb{C}$ such that $L(x^0) = 1$ and $L(P_n) = 0$ for all $n \in \mathbb{N}$ (or equivalently if and only if $L(x^0) = 1$ and $L(P_m P_n) = 0$ for all $m, n \in \mathbb{Z}_+$ such that $m \neq n$). We say that $\mu \in \mathscr{M}$ is an *orthogonalizing measure* for $\{P_n(x)\}_{n=0}^{\infty}$ if $L(P) = \int_{\mathbb{R}} P \mathrm{d}\mu$ for all $P \in \mathbb{C}[x]$. If this is the case, then $\{L(x^n)\}_{n=0}^{\infty}$ is a Hamburger moment sequence and μ is its H-representing measure (clearly $\mu(\mathbb{R}) = 1$). By Favard's theorem (see [22, Theorems I.4.4 and II.3.1]), the polynomials $\{P_n(x)\}_{n=0}^{\infty}$ have an orthogonalizing measure if and only if $c_n \in \mathbb{R}$ and $\lambda_{n+1} > 0$ for all $n \in \mathbb{Z}_+$.

Applying the above, we obtain the following statement.

The polynomials $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$ have an orthogonalizing measure if and only if either $a < 0$ and $q \in (-1, 0) \cup (1, \infty)$, or $a > 0$ and $q \in (0, 1)$. (2.3.5)

As in [1] we concentrate on the case of $q \in (0, 1)$. Then, by (2.3.5), the polynomials $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$ have an orthogonalizing measure if and only if $a > 0$. For such a and q, the question of determinacy of orthogonalizing measures for $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$ can be answered completely. This is done below. Known orthogonalizing measures for $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$ are discussed in detail as well. We refer the reader to Fig. 1 which illustrates how determinacy depends on the parameters a and q.

Al-Salam and Carlitz showed in [1] that if $a > 0$ and $aq < 1$, then the measure

$$\beta^{(a;q)} := (aq;q)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(aq;q)_n (q;q)_n} \delta_{q^{-n}}, \tag{2.3.6}$$

is an orthogonalizing measure for $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$ (if $aq \geqslant 1$, then the right-hand side of (2.3.6) either does not make sense or does not define a positive measure), and

$$\mathsf{m}_n[\beta^{(a;q)}] := \int_{\mathbb{R}} t^n \mathrm{d}\beta^{(a;q)}(t) = \sum_{k=0}^{n} \frac{(q;q)_n q^{k(k-n)}}{(q;q)_k (q;q)_{n-k}} a^k, \quad n \in \mathbb{Z}_+.$$

(That $\beta^{(a;q)}$ is a probability measure was proved in [37, Theorem 5.1].) In turn, Chihara essentially proved that if $0 < q < a \leqslant 1$, then $\{\mathsf{m}_n[\beta^{(a;q)}]\}_{n=0}^{\infty}$ is an S-indeterminate

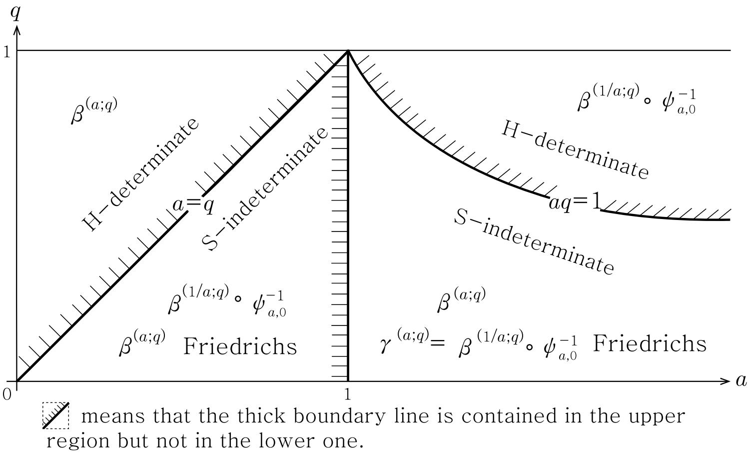

Fig. 1. Determinacy of known orthogonalizing measures for $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$.

Stieltjes moment sequence and $\beta^{(a;q)}$ is its Friedrichs measure (see [21]). This can be also deduced from Lemma 2.2.2, [8, Proposition 4.5.1], [21, Corollary on p. 481] (see also [42, Lemma 2.2.5]), [21, p. 483, (b)] and (2.1.1) by considering the measure $\beta^{(a;q)}\circ\psi_{1,-1}^{-1}$ which coincides with the measure $\nu_0^{(a)}$ appearing in [8, Proposition 4.5.1]. A similar reasoning shows that if $1<a<q^{-1}$, then $\{\mathsf{m}_n[\beta^{(a;q)}]\}_{n=0}^{\infty}$ is an S-indeterminate Stieltjes moment sequence (this was first noticed by Chihara in [21]) and $\beta^{(a;q)}$ is its N-extremal measure which is neither the Krein nor the Friedrichs measure. Consider now the measure

$$\gamma^{(a;q)} := (q/a;q)_\infty \sum_{n=0}^{\infty} \frac{a^{-n}q^{n^2}}{(q/a;q)_n(q;q)_n}\delta_{aq^{-n}}, \quad 1<a<q^{-1}. \tag{2.3.7}$$

Applying Lemmata 2.2.1 and 2.2.2, [8, Proposition 4.5.1] and Theorem 2.2.3, the latter to the measure $\gamma^{(a;q)}\circ\psi_{1,-1}^{-1}$ which coincides with the measure $\nu_{-1/\xi(a)}^{(a)}$ appearing in [8, Proposition 4.5.1] (consult also Remark 5.5.4), we deduce that if $1<a<q^{-1}$, then $\mathsf{m}_n[\beta^{(a;q)}]=\mathsf{m}_n[\gamma^{(a;q)}]$ for all $n\in\mathbb{Z}_+$, $\gamma^{(a;q)}$ is an orthogonalizing measure for $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$ and $\gamma^{(a;q)}$ is the Friedrichs measure of $\{\mathsf{m}_n[\beta^{(a;q)}]\}_{n=0}^{\infty}$. Finally, as shown in [21], if $0<a\leqslant q<1$ or $1<q^{-1}\leqslant a$, then the polynomials $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$ have an H-determinate orthogonalizing measure. Clearly, in the former case this measure coincides with $\beta^{(a;q)}$ defined by (2.3.6). To find the orthogonalizing measure for $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$ in the latter case, we follow an idea due to Ismail, which in fact can be applied to wider set of parameters (see [37, page 592]). For this, note that if $a,\tilde{q}\in\mathbb{C}\setminus\{0\}$, then the polynomials $\{a^nV_n^{(1/a)}(x/a;\tilde{q})\}_{n=0}^{\infty}$ satisfy the same recurrence relation as the polynomials $\{V_n^{(a)}(x;\tilde{q})\}_{n=0}^{\infty}$ (see (2.3.2)), and thus

$$V_n^{(a)}(x;\tilde{q}) = a^nV_n^{(1/a)}(x/a;\tilde{q}), \quad n\in\mathbb{Z}_+,\ a,\tilde{q}\in\mathbb{C}\setminus\{0\}. \tag{2.3.8}$$

Now suppose that $a \in (0,\infty)$ and $a^{-1}q < 1$ (recall that $q \in (0,1)$). Then using the measure transport theorem and the fact that $\beta^{(1/a;q)}$ is the orthogonalizing measure for $\{V_n^{(1/a)}(x;q)\}_{n=0}^{\infty}$, we get

$$\begin{aligned}
&\int\limits_{\mathbb{R}} V_m^{(a)}(x;q)V_n^{(a)}(x;q)\mathrm{d}\beta^{(1/a;q)} \circ \psi_{a,0}^{-1}(x) \\
&\quad\overset{(2.3.8)}{=} \int\limits_{\mathbb{R}} a^m V_m^{(1/a)}(\psi_{a,0}^{-1}(x);q)a^n V_n^{(1/a)}(\psi_{a,0}^{-1}(x);q)\mathrm{d}\beta^{(1/a;q)} \circ \psi_{a,0}^{-1}(x) \\
&\quad= a^{m+n}\int\limits_{\mathbb{R}} V_m^{(1/a)}(x;q)V_n^{(1/a)}(x;q)\mathrm{d}\beta^{(1/a;q)}(x) = 0, \quad m,n \in \mathbb{Z}_+,\ m \neq n.
\end{aligned}$$

Since $\beta^{(1/a;q)} \circ \psi_{a,0}^{-1}$ is a probability measure, we deduce that it is an orthogonalizing measure for $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$. If additionally $aq < 1$, then $\beta^{(a,q)}$ is another orthogonalizing measure for $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$. Now there are three possibilities. If $a = 1$, then the measures $\beta^{(a;q)}$ and $\beta^{(1/a;q)} \circ \psi_{a,0}^{-1}$ coincide. If $a < 1$, then $\{\mathsf{m}_n[\beta^{(a;q)}]\}_{n=0}^{\infty}$ is an S-indeterminate Stieltjes moment sequence, $\beta^{(a;q)}$ is its Friedrichs measure (because $0 < q < a < 1$) and $\beta^{(1/a;q)} \circ \psi_{a,0}^{-1}$ is its N-extremal measure which is neither the Krein nor the Friedrichs measure (because of $1 < a^{-1} < q^{-1}$, (2.1.1) and Lemma 2.2.2). In turn, if $a > 1$, then $\{\mathsf{m}_n[\beta^{(a;q)}]\}_{n=0}^{\infty}$ is an S-indeterminate Stieltjes moment sequence, $\beta^{(1/a;q)} \circ \psi_{a,0}^{-1}$ is its Friedrichs measure (because of $0 < q < a^{-1} < 1$ and Theorem 2.2.3) which coincides with $\gamma^{(a;q)}$ defined by (2.3.7), and $\beta^{(a;q)}$ is its N-extremal measure which is neither the Krein nor the Friedrichs measure (because $1 < a < q^{-1}$).

Finally, we note that if $a \in (0,\infty)$ and $1 < q^{-1} \leqslant a$ (i.e., $aq \geqslant 1$), then $a^{-1}q < 1$, and thus, by the above considerations, the measure $\beta^{(1/a;q)}\circ\psi_{a,0}^{-1}$ is the unique orthogonalizing measure for $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$.

One more observation is at hand. Namely, using the equality

$$(aq;q)_\infty = (aq;q)_n(aq^{n+1};q)_\infty,$$

we get

$$\beta^{(a;q)} = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}(aq^{n+1};q)_\infty}{(q;q)_n}\delta_{q^{-n}}, \quad a > 0,\ aq < 1.$$

Now it is easily seen that for every $(q,a) \in (0,1)\times(0,\infty)$ such that $aq \geqslant 1$, the right-hand side of the above equality defines the signed measure (understood as in [3]), call it $\tilde{\beta}^{(a;q)}$, which is positive if and only if $aq^k = 1$ for some $k \in \mathbb{N}$. Moreover, standard calculations show that if $aq^k = 1$ for some $k \in \mathbb{N}$, then $\mathrm{supp}(\tilde{\beta}^{(a;q)}) = \{q^{-j} : j \geqslant k\}$, which together with (2.3.6) implies that $\tilde{\beta}^{(a;q)} = \beta^{(1/a;q)}\circ\psi_{a,0}^{-1}$. In a sense, this means that all singularities appearing in (2.3.6) can be removed.

The H-determinacy of the Hamburger moment problem associated with the polynomials $\{V_n^{(a)}(x;q)\}_{n=0}^{\infty}$ for $q \in (-1,0) \cup (0,1)$ was discussed by Ismail in [37].

2.4. Index of H-determinacy

Following [5], we define $\mathrm{ind}_z(\rho) \in \mathbb{Z}_+ \cup \{\infty\}$, the *index of H-determinacy* of an H-determinate measure ρ at a point $z \in \mathbb{C}$, by

$$\mathrm{ind}_z(\rho) = \sup\{k \in \mathbb{Z}_+ \colon |t-z|^{2k}\mathrm{d}\rho(t) \text{ is H-determinate}\}.$$

By the index of H-determinacy of an H-determinate Stieltjes moment sequence $\boldsymbol{\gamma} = \{\gamma_n\}_{n=1}^\infty$ at a point $z \in \mathbb{C}$ we understand the index of H-determinacy of a unique H-representing measure of $\boldsymbol{\gamma}$ at the point z. Note that if ρ is an H-determinate measure such that $\mathrm{ind}_{z_0}(\rho) = \infty$ for some $z_0 \in \mathbb{C}$, then $\mathrm{ind}_z(\rho) = \infty$ for all $z \in \mathbb{C}$ (see [5, Corollary 3.4(1)]). If this is the case, then we say that ρ (or $\boldsymbol{\gamma}$) has *infinite index of H-determinacy*. Note also that the following holds.

$$\begin{aligned}&\textit{If } \rho \in \mathscr{M}^+ \textit{ is an H-determinate measure such that } \mathrm{ind}_z(\rho) = \infty\\ &\textit{for all } z \in \mathbb{C}, \textit{ then the measure } t^k \mathrm{d}\rho(t) \textit{ is H-determinate for all } k \in \mathbb{Z}_+.\end{aligned} \tag{2.4.1}$$

Indeed, this can be deduced from Proposition 2.1.3 and the inequality

$$\int_\sigma t^k \mathrm{d}\rho(t) \leqslant \int_\sigma |t-\mathrm{i}|^{2l}\mathrm{d}\rho(t), \quad \sigma \in \mathfrak{B}(\mathbb{R}),\, k,l \in \mathbb{Z}_+,\, k \leqslant 2l.$$

The following result can be thought of as a complement to [5, Theorem 3.6].

Theorem 2.4.1. *Let ν be an N-extremal measure and $\varOmega'$ be an infinite subset of $\varOmega :=$ $\mathrm{supp}(\nu)$. Set $\rho = \sum_{\lambda\in\varOmega\setminus\varOmega'} \nu(\{\lambda\})\delta_\lambda$ (with $\rho = 0$ if $\varOmega' = \varOmega$). Then ρ is an H-determinate measure such that $\mathrm{ind}_z(\rho) = \infty$ for all $z \in \mathbb{C}$.*

Proof. Fix $n \in \mathbb{N}$. Take any subset $\varOmega_n$ of $\varOmega'$ such that $\mathrm{card}(\varOmega_n) = n$. Set $\rho_n = \sum_{\lambda\in\varOmega\setminus\varOmega_n} \nu(\{\lambda\})\delta_\lambda$. Note that for every Borel function $f \colon \mathbb{R} \to \overline{\mathbb{R}}_+$,

$$\int_{\mathbb{R}} f \mathrm{d}\rho = \int_{\varOmega\setminus\varOmega'} f\mathrm{d}\nu \leqslant \int_{\varOmega\setminus\varOmega_n} f\mathrm{d}\nu = \int_{\mathbb{R}} f\mathrm{d}\rho_n. \tag{2.4.2}$$

By [5, Theorem 3.6], ρ_n is H-determinate and $n \geqslant \mathrm{ind}_z(\rho_n) \geqslant n-1$ for all $z \in \mathbb{C}$. Using Proposition 2.1.3 and applying the inequality (2.4.2) first to $f = \chi_\sigma$ and then to $f(t) = \chi_\sigma(t)|t-z|^{2k}$, we deduce that ρ is H-determinate and $\mathrm{ind}_z(\rho) \geqslant \mathrm{ind}_z(\rho_n)$ for all $z \in \mathbb{C}$. This completes the proof. $\square$

2.5. The Carleman condition

Suppose $\{\gamma_n\}_{n=0}^\infty$ is a Stieltjes moment sequence. Clearly, the shifted sequence $\{\gamma_{n+1}\}_{n=0}^\infty$ is a Stieltjes moment sequence. It turns out that if $\{\gamma_{n+1}\}_{n=0}^\infty$ is S-determinate, then so is $\{\gamma_n\}_{n=0}^\infty$ (see [57, Proposition 5.12]; see also [12, Lemma 2.4.1]).

The reverse implication is not true in general (see [42]). However, it is true if $\{\gamma_n\}_{n=0}^\infty$ satisfies the Carleman condition (see Proposition 2.5.1 below). Recall that a sequence $\{\gamma_n\}_{n=0}^\infty \subseteq \mathbb{R}_+$ satisfies the *Carleman condition* if $\sum_{n=1}^\infty \frac{1}{\gamma_n^{1/2n}} = \infty$ with the convention that $\frac{1}{0} = \infty$.

Below we collect some properties of Stieltjes moment sequences that satisfy the Carleman condition.

Proposition 2.5.1. *Let $\boldsymbol{\gamma} = \{\gamma_n\}_{n=0}^\infty$ be a Stieltjes moment sequence. Then*

(i) *if $\boldsymbol{\gamma}$ satisfies the Carleman condition, then $\boldsymbol{\gamma}$ is H-determinate,*
(ii) *$\boldsymbol{\gamma}$ satisfies the Carleman condition if and only if $\{\gamma_{n+1}\}_{n=0}^\infty$ satisfies the Carleman condition,*
(iii) *if $\boldsymbol{\gamma}$ satisfies the Carleman condition, then so does $\{\gamma_n + c\}_{n=0}^\infty$ for every $c \in (0, \infty)$,*
(iv) *$\boldsymbol{\gamma}$ satisfies the Carleman condition if and only if $\{\gamma_{jp}\}_{j=0}^\infty$ satisfies the Carleman condition for every $p \in \mathbb{N}$ (equivalently: for some $p \in \mathbb{N}$).*

Proof. (i) See [57, Corollary 4.5].

(ii) This can be deduced from the equivalence (d)⇔(e) in [55, Theorem 19.11] (the equivalence follows from the Carleman inequality, see [20, p. 105]).

(iii) Let ν be an S-representing measure of $\{\gamma_n\}_{n=0}^\infty$. Set $a = \nu((1, \infty))$. Since the case of $a = 0$ is obvious, we can assume that $a > 0$. Then

$$\gamma_n + c \leqslant \gamma_n + \frac{c}{a} \int\limits_{(1,\infty)} t^n \mathrm{d}\nu(t) \leqslant \Big(1 + \frac{c}{a}\Big)\gamma_n, \quad n \in \mathbb{Z}_+,$$

which implies that $\{\gamma_n + c\}_{n=0}^\infty$ satisfies the Carleman condition.

(iv) This follows from [63, Section 1] because, by the Cauchy–Schwarz inequality, $\gamma_n^2 \leqslant \gamma_k \gamma_l$ for all nonnegative integers k, l such that $k + l = 2n$. □

Note that in view of Proposition 2.5.1, a Stieltjes moment sequence which satisfies the Carleman condition has infinite index of H-determinacy because its index of H-determinacy at 0 is infinite.

3. Composition operators over one-circuit directed graphs

3.1. Criteria for hyponormality and subnormality

In this paper we study composition operators in L^2-spaces over discrete measure spaces. By a *discrete measure* on a (nonempty) set X we understand a σ-finite measure μ on the σ-algebra 2^X such that $\mu(x) > 0$ for every $x \in X$, with the convention

$$\mu(x) := \mu(\{x\}), \quad x \in X. \tag{3.1.1}$$

Note that if μ is a discrete measure on X, then X is at most countable and $\mu(x) < \infty$ for every $x \in X$. Moreover, any discrete measure μ on X is determined by a function $\mu\colon X \to (0, \infty)$ via $\mu(\{x\}) = \mu(x)$. This one-to-one correspondence between discrete measures and positive functions is used frequently in the present paper.

Let μ be a discrete measure on a set X and let ϕ be a self-map of X. Then the operator C_ϕ in $L^2(\mu)$ given by

$$\mathcal{D}(C_\phi) = \{f \in L^2(\mu)\colon f \circ \phi \in L^2(\mu)\} \text{ and } C_\phi f = f \circ \phi \text{ for } f \in \mathcal{D}(C_\phi),$$

is called a *composition operator* in $L^2(\mu)$ with a *symbol* ϕ. Since the measure $\mu \circ \phi^{-1}$ given by $(\mu \circ \phi^{-1})(\varDelta) = \mu(\phi^{-1}(\varDelta))$ for $\varDelta \in 2^X$ is absolutely continuous with respect to μ, we can consider the Radon–Nikodym derivative $\mathsf{h}_\phi = \mathrm{d}\mu \circ \phi^{-1}/\mathrm{d}\mu$. Clearly

$$\mathsf{h}_\phi(x) = \frac{\mu(\phi^{-1}(\{x\}))}{\mu(x)}, \quad x \in X. \tag{3.1.2}$$

Hence, $\mathsf{h}_\phi(x) > 0$ for every $x \in X$ if and only if $\phi(X) = X$. It is easily seen that (see [51])

$$\begin{aligned}&C_\phi \in \boldsymbol{B}(L^2(\mu)) \textit{ if and only if } \sup_{x\in X} \mathsf{h}_\phi(x) < \infty, \textit{ and if this}\\ &\textit{is the case, then } \|C_\phi^n\|^2 = \sup_{x\in X} \mathsf{h}_{\phi^n}(x) \textit{ for every } n \in \mathbb{Z}_+.\end{aligned} \tag{3.1.3}$$

Note also that

$$\|C_\phi^n \chi_{\{u\}}\|^2 = \mu(u)\mathsf{h}_{\phi^n}(u) \text{ whenever } u \in X,\ \chi_{\{u\}} \in \mathcal{D}(C_\phi^n) \text{ and } n \in \mathbb{Z}_+. \tag{3.1.4}$$

Applying [15, Proposition 3.2] and the assertions (ii) and (iv) of [15, Proposition 4.1], we get a new criterion for the nth power of C_ϕ to be densely defined.

$$\begin{aligned}&\textit{If } n \in \mathbb{N}, \textit{ then } C_\phi^n \textit{ is densely defined if and only if } \mathsf{h}_{\phi^n}(x) < \infty\\ &\textit{for every } x \in X.\end{aligned} \tag{3.1.5}$$

Thus, if $n \in \mathbb{N}$ and $\mathsf{h}_{\phi^n}(x) < \infty$ for all $x \in X$, then $\mathsf{h}_{\phi^j}(x) < \infty$ for all $j \in \{1, \ldots, n\}$ and $x \in X$. This hereditary property is no longer true for a single point $x \in X$ (see [40, Example 4.2]). In fact, given any nonempty subset $\varXi$ of $\mathbb{N}$, we can construct a discrete measure μ on a set X and a self-map ϕ of X such that $\varXi = \{n \in \mathbb{N}\colon \mathsf{h}_{\phi^n}(x_0) = \infty\}$ for some $x_0 \in X$. This is shown below.

Example 3.1.1. Fix $k \in \mathbb{N} \cup \{\infty\}$. Let X, $\{x_i\}_{i=0}^\infty$, $\{x_{i,j}\}_{i=1}^{\eta+1}{}_{j=1}^{l_i}$ and ϕ be as in Theorem 3.3.2(ii-b*) with $\eta = \infty$ and $l_i = k$ for all $i \in \mathbb{N}$. Suppose that $\varXi$ is a nonempty subset of J_k (see Section 1.2 for the definition of J_k). Define the discrete measure μ on X by

$$\mu(\{x\}) = \begin{cases} 2^{-i} & \text{if } x \in \{x_{i,j}\colon i \in \mathbb{N},\, j \in J_k \setminus \varXi\},\\ 1 & \text{otherwise},\end{cases} \qquad x \in X.$$

It is a matter of routine to show that

$$\mathsf{h}_{\phi^n}(x_0) = \begin{cases} \infty & \text{if } n \in \varXi, \\ 1 & \text{if } n \in J_k \setminus \varXi, \\ 0 & \text{if } k < \infty \text{ and } n > k, \end{cases} \qquad n \in \mathbb{N}.$$

Assume now that C_ϕ is densely defined (as before, μ is a discrete measure on a set X and ϕ is a self-map of X), or equivalently, by (3.1.2) and (3.1.5), that $\mu(\phi^{-1}(\{x\})) < \infty$ for all $x \in \phi(X)$. Then for every function $f\colon X \to \overline{\mathbb{R}}_+$, there exists a unique $\phi^{-1}(2^X)$-measurable function $\mathsf{E}_\phi(f)\colon X \to \overline{\mathbb{R}}_+$ such that

$$\int_{\phi^{-1}(\varDelta)} f \mathrm{d}\mu = \int_{\phi^{-1}(\varDelta)} \mathsf{E}_\phi(f) \mathrm{d}\mu, \quad \varDelta \subseteq X,$$

or equivalently, such that for every $x \in \phi(X)$,

$$\mathsf{E}_\phi(f)(z) = \frac{\sum_{y \in \phi^{-1}(\{x\})} \mu(y) f(y)}{\mu(\phi^{-1}(\{x\}))}, \quad z \in \phi^{-1}(\{x\}). \tag{3.1.6}$$

The above definition is correct because $X = \bigsqcup_{x\in\phi(X)} \phi^{-1}(\{x\})$. The function $\mathsf{E}_\phi(f)$ is called the *conditional expectation* of a function $f\colon X \to \overline{\mathbb{R}}_+$ with respect to the σ-algebra $\phi^{-1}(2^X)$ (see [15]). Following [16], we say that a family $\{P(x,\cdot)\}_{x\in X}$ of Borel probability measures on $\mathbb{R}_+$ satisfies the *consistency condition* if

$$\mathsf{h}_\phi(\phi(z)) \cdot \mathsf{E}_\phi(P(\cdot,\sigma))(z) = \int_\sigma t P(\phi(z), \mathrm{d}t), \quad z \in X, \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

In view of (3.1.2) and (3.1.6), we see that $\{P(x,\cdot)\}_{x\in X}$ satisfies the consistency condition if and only if

$$\frac{1}{\mu(x)} \sum_{y\in\phi^{-1}(\{x\})} \mu(y) P(y,\sigma) = \int_\sigma t P(x, \mathrm{d}t), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\, x \in \phi(X). \tag{CC}$$

The following characterization of hyponormality of C_ϕ can be deduced from (3.1.2), (3.1.6) and [19, Corollary 6.7] (see also [11, Lemma 2.1]).

Proposition 3.1.2. *Let μ be a discrete measure on a set X and ϕ be a self-map of X. Assume that C_ϕ is densely defined. Then C_ϕ is hyponormal if and only if for every $x \in X$, $\mathsf{h}_\phi(x) > 0$ and*

$$\frac{1}{\mu(x)} \sum_{y\in\phi^{-1}(\{x\})} \frac{\mu(y)^2}{\mu(\phi^{-1}(\{y\}))} \leqslant 1. \tag{3.1.7}$$

A criterion for subnormality of unbounded composition operators given in [16, Theorems 9 and 17] takes in the present situation the following form (recall that if C_ϕ is subnormal, then $\mathsf{h}_\phi(x) > 0$ for every $x \in X$, or equivalently C_ϕ is injective, see [15, Section 6]).

Theorem 3.1.3. *Let μ be a discrete measure on a set X and ϕ be a self-map of X. Assume that C_ϕ is densely defined and $\mathsf{h}_\phi(x) > 0$ for every $x \in X$. Suppose that there exists a family $\{P(x,\cdot)\}_{x\in X}$ of Borel probability measures on $\mathbb{R}_+$ that satisfies (CC). Then C_ϕ is subnormal and*

$$\mathsf{h}_{\phi^n}(x) = \int_0^\infty t^n P(x, \mathrm{d}t), \quad n \in \mathbb{Z}_+,\, x \in X. \tag{3.1.8}$$

Let $\mathscr{G} = (V, E)$ be a directed graph (i.e., V is a nonempty set called the *vertex set* of $\mathscr{G}$ and E is a subset of $V \times V$ called *the edge set* of $\mathscr{G}$). We say that a vertex u is a *parent* of a vertex v, and write $\mathsf{par}(v) = u$, if $(u, v) \in E$ and $u = w$ whenever $(w, v) \in E$. The directed graph $\mathscr{G}$ is said to be *connected* if for every pair (u, v) of distinct vertices there exists an *undirected path* joining u and v, i.e., a finite sequence $\{u_i\}_{i=1}^k$ of vertices with $k \geq 2$ such that $u_1 = u$, $u_k = v$ and for every $i \in J_{k-1}$, either $(u_i, u_{i+1}) \in E$ or $(u_{i+1}, u_i) \in E$. We say that a finite sequence $\{u_j\}_{j=1}^n$ of distinct vertices is a *circuit* if $n \geq 2$, $(u_j, u_{j+1}) \in E$ for all $j \in J_{n-1}$ and $(u_n, u_1) \in E$. By a *rootless directed tree* we mean a directed graph $\mathscr{T} = (V, E)$ which is connected, has no circuits, each vertex of $\mathscr{T}$ has a parent and $(u, u) \notin E$ for every $u \in V$. In this case, obviously, the partial function par is a self-map of V.

Given a rootless directed tree $\mathscr{T} = (V, E)$ and a family $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V}$ of complex numbers, we define a *weighted shift* $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}$ with weights $\boldsymbol{\lambda}$ via

$$\mathcal{D}(S_{\boldsymbol{\lambda}}) = \{f \in \ell^2(V) \colon \boldsymbol{\lambda} \cdot f \circ \mathsf{par} \in \ell^2(V)\} \text{ and } S_{\boldsymbol{\lambda}} f = \boldsymbol{\lambda} \cdot f \circ \mathsf{par} \text{ for } f \in \mathcal{D}(S_{\boldsymbol{\lambda}}),$$

where $(\boldsymbol{\lambda} \cdot f \circ \mathsf{par})(v) = \lambda_v f(\mathsf{par}(v))$ for $v \in V$. We refer the reader to [41] for more information on directed trees and weighted shifts on them.

It follows from [42, Lemma 4.3.1] and [41, Theorem 3.2.1] that each weighted shift $S_{\boldsymbol{\lambda}}$ on a countable rootless directed tree $\mathscr{T} = (V, E)$ with nonzero weights is unitarily equivalent to a composition operator C_{par} in $L^2(V, 2^V, \mu)$ for some discrete measure μ on V (the assumption that V is infinite made in [42, Lemma 4.3.1] is redundant). As explicated below, the proof of [42, Lemma 4.3.1] contains more information.

Lemma 3.1.4. *Let $\mathscr{T} = (V, E)$ be a rootless directed tree. Suppose that μ is a discrete measure on V. Then the composition operator C_{par} in $L^2(\mu)$ is unitarily equivalent to a weighted shift on $\mathscr{T}$ with positive weights.*

Proof. Since μ is a discrete measure on V, we see that $\operatorname{card}(V) \leqslant \aleph_0$. Consider the weighted shift $S_{\boldsymbol{\lambda}}$ on the directed tree $\mathscr{T}$ with weights $\Big\{\sqrt{\frac{\mu(v)}{\mu(\mathsf{par}(v))}}\,\Big\}_{v\in V}$. A careful inspection of the proof of [42, Lemma 4.3.1] reveals that the composition operator C_{par} is unitarily equivalent to $S_{\boldsymbol{\lambda}}$ via the unitary isomorphism $U\colon \ell^2(V) \to L^2(\mu)$ defined in [42, (4.3.4)]. $\square$

3.2. A class of directed graphs with one circuit

In this section we classify connected directed graphs induced by self-maps whose vertices, all but one, have valency one and the valency of the remaining vertex is nonzero (see Theorem 3.2.1 below; see also Figs. 2 and 3 which illustrate this theorem).

Let X be a nonempty set and ϕ be a self-map of X. Set

$$E^\phi = \{(x, y) \in X \times X\colon x = \phi(y)\}. \tag{3.2.1}$$

Then (X, E^ϕ) is a directed graph which we call the *directed graph induced by* ϕ. Note that the valency of a vertex $x \in X$ is equal to $\mathrm{card}(\phi^{-1}(\{x\}))$ and that $\phi(x)$ is the parent of x. We will write $\phi^{-n}(A) = (\phi^n)^{-1}(A)$ whenever $n \in \mathbb{Z}_+$ and $A \subseteq X$.

Theorem 3.2.1. *Let X and ϕ be as above and let $\eta \in \mathbb{N} \cup \{\infty\}$. Then the following two conditions are equivalent:*

(i) *the directed graph (X, E^ϕ) is connected and there exists $\omega \in X$ such that $\mathrm{card}(\phi^{-1}(\{\omega\})) = \eta + 1$ and $\mathrm{card}(\phi^{-1}(\{x\})) = 1$ for every $x \in X \setminus \{\omega\}$,*

(ii) *one of the following two conditions is satisfied:*

(ii-a) *there exist $\kappa \in \mathbb{Z}_+$ and two disjoint systems $\{x_i\}_{i=0}^{\kappa}$ and $\{x_{i,j}\}_{i=1}^{\eta}{}_{j=1}^{\infty}$ of distinct points of X such that*

$$X = \{x_0, \ldots, x_\kappa\} \cup \{x_{i,j}\colon i \in J_\eta, j \in \mathbb{N}\}, \tag{3.2.2}$$

$$\phi(x) = \begin{cases} x_{i,j-1} & \text{if } x = x_{i,j} \text{ with } i \in J_\eta \text{ and } j \geqslant 2, \\ x_\kappa & \text{if } x = x_{i,1} \text{ with } i \in J_\eta \text{ or } x = x_0, \\ x_{i-1} & \text{if } x = x_i \text{ with } i \in J_\kappa, \end{cases} \tag{3.2.3}$$

(ii-b) *there exist two disjoint systems $\{x_i\}_{i=0}^{\infty}$ and $\{x_{i,j}\}_{i=1}^{\eta+1}{}_{j=1}^{\infty}$ of distinct points of X such that*

$$X = \{x_i\colon i \in \mathbb{Z}_+\} \cup \{x_{i,j}\colon i \in J_{\eta+1}, j \in \mathbb{N}\},$$

$$\phi(x) = \begin{cases} x_{i,j-1} & \text{if } x = x_{i,j} \text{ with } i \in J_{\eta+1} \text{ and } j \geqslant 2, \\ x_0 & \text{if } x = x_{i,1} \text{ with } i \in J_{\eta+1}, \\ x_{i+1} & \text{if } x = x_i \text{ with } i \in \mathbb{Z}_+. \end{cases} \tag{3.2.4}$$

Proof. Clearly, we need only to prove the implication (i)⇒(ii). We do it in five steps. The proof of Step 1 being simple is omitted.

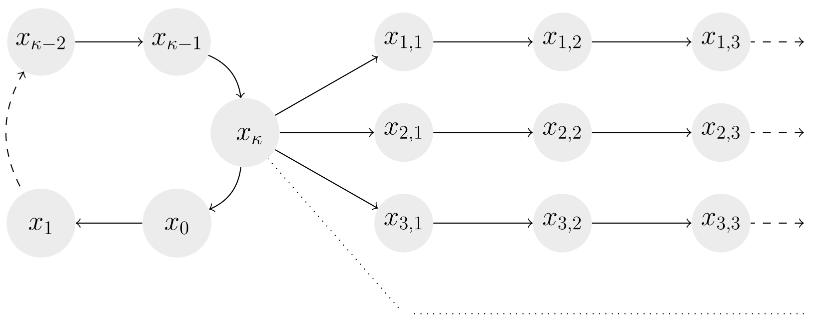

Fig. 2. The directed graph (X, E^ϕ) in the case of (ii-a) in Theorem 3.2.1 (the self-map ϕ acts in accordance with the reverted arrows).

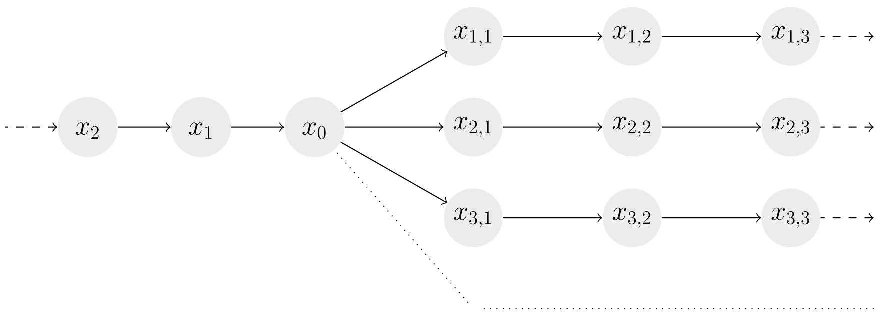

Fig. 3. The directed graph (X, E^ϕ) in the case of (ii-b) in Theorem 3.2.1.

Step 1. If $\omega \in X$ is such that $\operatorname{card}(\phi^{-1}(x)) \leqslant 1$ for all $x \in X \setminus \{\omega\}$, then the restriction of ϕ to $\phi^{-1}(X \setminus \{\omega\})$ is injective.

Step 2. If $\omega \in X$ and $\Omega \subseteq X$ are such that $\omega \in \Omega$, $\phi(\Omega) \subseteq \Omega$, $\operatorname{card}(\phi^{-1}(\{\omega\})) \geqslant 2$, $\operatorname{card}(\phi^{-1}(\{x\})) = 1$ for every $x \in X \setminus \{\omega\}$ and $\Delta := \phi^{-1}(\{\omega\}) \setminus \Omega \neq \emptyset$, then

(s1) $\operatorname{card}(\phi^{-n}(\{x\})) = 1$ provided $n \in \mathbb{Z}_+$ and $x \in X \setminus \Omega$,
(s2) $\phi^{-n}(x) \in X \setminus \Omega$ provided $n \in \mathbb{Z}_+$ and $x \in X \setminus \Omega$, where $\phi^{-n}(x)$ is a unique element of X such that $\{\phi^{-n}(x)\} = \phi^{-n}(\{x\})$ (see (s1)),
(s3) $x = \phi^n(\phi^{-n}(x))$ and $\phi^{-(m+n)}(x) = \phi^{-m}(\phi^{-n}(x))$ provided $m, n \in \mathbb{Z}_+$ and $x \in X \setminus \Omega$,
(s4) $\{\phi^{-n}(x) \colon n \in \mathbb{Z}_+\}$ are distinct points of X provided $x \in \Delta$,
(s5) $\{\phi^{-m}(x) \colon m \in \mathbb{Z}_+\} \cap \{\phi^{-n}(y) \colon n \in \mathbb{Z}_+\} = \emptyset$ provided $x, y \in \Delta$ and $x \neq y$.

For this, take $x \in X \setminus \Omega$. By assumption, $\operatorname{card}(\phi^{-1}(\{x\})) = 1$ and thus there exists a unique $\phi^{-1}(x) \in X$ such that $\{\phi^{-1}(x)\} = \phi^{-1}(\{x\})$. Since $\phi(\Omega) \subseteq \Omega$ and $x = \phi(\phi^{-1}(x))$, we deduce that $\phi^{-1}(x) \in X \setminus \Omega$. An induction argument yields (s1), (s2) and (s3). To prove (s4) and (s5), suppose that $x, y \in \Delta$ and $\phi^{-i}(x) = \phi^{-j}(y)$ for some integers $0 \leqslant i \leqslant j$. If $i < j$, then by (s2) and (s3), we have

$$x = \phi^i(\phi^{-i}(x)) = \phi^i(\phi^{-j}(y)) = \phi^i(\phi^{-i}(\phi^{-(j-i)}(y))) = \phi^{-(j-i)}(y)$$

and so

$$\omega = \phi(x) = \phi(\phi^{-(j-i)}(y)) = \phi(\phi^{-1}(\phi^{-(j-i-1)}(y))) = \phi^{-(j-i-1)}(y),$$

which yields $y = \phi^{j-i-1}(\omega) \in \Omega$, a contradiction. Finally, if $i = j$, then

$$x = \phi^i(\phi^{-i}(x)) = \phi^j(\phi^{-j}(y)) = y,$$

which completes the proof of (s4) and (s5).

Step 3. If (X, E^ϕ) is connected and Y is a nonempty subset of X such that $\phi(Y) \subseteq Y$ and $\phi(X \setminus Y) \subseteq X \setminus Y$, then $X = Y$.

Indeed, otherwise there exists $x \in X \setminus Y$. Take $y \in Y$. Since the graph (X, E^ϕ) is connected, there exists a finite sequence $\{u_i\}_{i=1}^k$ of elements of X with $k \geq 2$ such that $u_1 = x$, $u_k = y$ and for every $i \in J_{k-1}$, either $(u_i, u_{i+1}) \in E^\phi$ or $(u_{i+1}, u_i) \in E^\phi$. Then there exists $j \in J_{k-1}$ such that $u_j \in X \setminus Y$ and $u_{j+1} \in Y$. Thus either $u_j = \phi(u_{j+1}) \in Y$ or $u_{j+1} = \phi(u_j) \in X \setminus Y$, a contradiction.

Step 4. If (X, E^ϕ) is connected and $\omega \in X$ is such that $\operatorname{card}(\phi^{-1}(\{\omega\})) = \eta + 1$, $\operatorname{card}(\phi^{-1}(\{x\})) = 1$ for every $x \in X \setminus \{\omega\}$ and $\omega \in \phi^{-n}(\{\omega\})$ for some $n \in \mathbb{N}$, then (ii-a) holds.

For this, we set

$$\kappa = \min\{n \in \mathbb{N} \colon \omega \in \phi^{-n}(\{\omega\})\} - 1. \tag{3.2.5}$$

First we show that $\{\phi^i(\omega)\}_{i=0}^\kappa$ is a sequence of distinct points of X. Indeed, if $\phi^i(\omega) = \phi^j(\omega)$ for some integers $0 \leqslant i < j \leqslant \kappa$, then $1 \leqslant \kappa + 1 - (j - i) < \kappa + 1$ and

$$\omega \overset{(3.2.5)}{=} \phi^{\kappa+1}(\omega) = \phi^{\kappa+1-j}(\phi^j(\omega)) = \phi^{\kappa+1-j}(\phi^i(\omega)) = \phi^{\kappa+1-(j-i)}(\omega),$$

which contradicts (3.2.5). Set $\Omega = \big\{x_i \colon i \in \{0, \ldots, \kappa\}\big\}$ with $x_i = \phi^{\kappa-i}(\omega)$ for $i \in \{0, \ldots, \kappa\}$, and $\Delta = \phi^{-1}(\{\omega\}) \setminus \Omega$. Then clearly $\phi(x_0) = x_\kappa = \omega$ and $\phi(x_i) = x_{i-1}$ for $i \in J_\kappa$, and $\phi(\Omega) \subset \Omega$. Since $\phi|_\Omega$ is injective, we see that $\phi^{-1}(\{\omega\}) \cap \Omega = \{x_0\}$. This and $\operatorname{card}(\phi^{-1}(\{\omega\})) = \eta + 1$ imply that $\operatorname{card}(\Delta) = \eta$. Thus, by Step 2, the conditions (s1)–(s5) hold. Let $\{x_{i,1}\}_{i=1}^\eta$ be a sequence of distinct points of Δ. Setting $x_{i,j} = \phi^{-(j-1)}(x_{i,1})$ for $i \in J_\eta$ and $j \in \mathbb{N}$, we verify that $\{x_i\}_{i=0}^\kappa$ and $\{x_{i,j}\}_{i=1}^\eta{}_{j=1}^\infty$ are disjoint systems of distinct points of X which satisfy (3.2.3). Hence, $\phi(Y) = Y$ and $\phi^{-1}(\{x\}) \cap Y \neq \emptyset$ for every $x \in Y$, where

$$Y = \{x_0, \ldots, x_\kappa\} \cup \{x_{i,j} \colon i \in J_\eta, j \in \mathbb{N}\}. \tag{3.2.6}$$

Since $\operatorname{card}(\phi^{-1}(\{x\})) = 1$ for every $x \in X \setminus \{\omega\}$ and $\operatorname{card}(\phi^{-1}(\{\omega\})) = \eta + 1$, we deduce that $\phi^{-1}(Y) \subseteq Y$, or equivalently that $\phi(X \setminus Y) \subseteq X \setminus Y$. This together with Step 3 completes the proof of Step 4.

Step 5. If (X, E^ϕ) is connected and $\omega \in X$ is such that $\mathrm{card}(\phi^{-1}(\{\omega\})) = \eta + 1$, $\mathrm{card}(\phi^{-1}(\{x\})) = 1$ for every $x \in X \setminus \{\omega\}$ and $\omega \notin \phi^{-n}(\{\omega\})$ for every $n \in \mathbb{N}$, then (ii-b) holds.

We begin by proving that

$$\{\phi^i(\omega)\}_{i=0}^\infty \text{ is a sequence of distinct points of } X. \tag{3.2.7}$$

For this, suppose that $\phi^i(\omega) = \phi^j(\omega)$ for some integers $0 \leqslant i < j$. Since, by assumption $\phi^k(\omega) \neq \omega$ for all $k \in \mathbb{N}$, we can assume that $i \geqslant 1$. If $i \geqslant 2$, then $\phi(\phi^{j-1}(\omega)) = \phi(\phi^{i-1}(\omega))$, which in view of Step 1 implies that $\phi^{j-1}(\omega) = \phi^{i-1}(\omega)$. An induction argument shows that $\phi^{j-i+1}(\omega) = \phi(\omega)$. Clearly, the last equality remains valid if $i = 1$. Hence, in both cases, we see that $\omega, \phi^{j-i}(\omega) \in \phi^{-1}(\{\phi(\omega)\})$. As $\phi(\omega) \neq \omega$ and consequently $\mathrm{card}(\phi^{-1}(\{\phi(\omega)\})) = 1$, we deduce that $\omega = \phi^{j-i}(\omega)$, a contradiction. This proves (3.2.7). Set $\varOmega = \{x_i : i \in \mathbb{Z}_+\}$ with $x_i = \phi^i(\omega)$, and $\varDelta = \phi^{-1}(\{\omega\}) \setminus \varOmega$. Clearly $\omega \in \varOmega$, $\phi(\varOmega) \subseteq \varOmega$ and $\phi^{-1}(\{\omega\}) \cap \varOmega = \emptyset$. This yields $\mathrm{card}(\varDelta) = \eta + 1$. Therefore, by Step 2, the conditions (s1)–(s5) hold. Let $\{x_{i,1}\}_{i=1}^{\eta+1}$ be a sequence of distinct points of $\varDelta$. Setting $x_{i,j} = \phi^{-(j-1)}(x_{i,1})$ for $i \in J_{\eta+1}$ and $j \in \mathbb{N}$, we verify that $\{x_i\}_{i=0}^\infty$ and $\{x_{i,j}\}_{i=1}^{\eta+1}{}_{j=1}^{\infty}$ are disjoint systems of distinct points of X which satisfy (3.2.4). Now, arguing as in the final stage of the proof of Step 4 with

$$Y := \{x_i : i \in \mathbb{Z}_+\} \cup \{x_{i,j} : i \in J_{\eta+1}, j \in \mathbb{N}\}, \tag{3.2.8}$$

we complete the proof of Step 5 and Theorem 3.2.1. $\square$

Now we make some comments on Theorem 3.2.1 that support the idea of studying composition operators with symbols of type (ii-a). We also shed more light on the question of simplicity of directed graphs discussed in Section 1.1.

Remark 3.2.2. 1) Suppose that the directed graph (X, E^ϕ) is connected and C_ϕ is a composition operator in $L^2(X, 2^X, \mu)$, where μ is a discrete measure on X. By (3.1.2) and [15, Proposition 6.2], C_ϕ is injective if and only if

$$\mathrm{card}(\phi^{-1}(\{x\})) \geqslant 1 \text{ for all } x \in X. \tag{3.2.9}$$

To guarantee injectivity of C_ϕ, we assume that (3.2.9) holds. We also exclude the (more complex) case when the directed graph (X, E^ϕ) has more than one vertex of valency greater than 1. The case when (X, E^ϕ) has exactly one vertex of valency greater than 1 has been described in Theorem 3.2.1. If $\mathrm{card}(\phi^{-1}(\{x\})) = 1$ for every $x \in X$ (the *flat* case), then, by [59, Proposition 2.4] and Step 3 of the proof of Theorem 3.2.1, ϕ is bijectively isomorphic to the mapping $i + k\mathbb{Z} \mapsto (i+1) + k\mathbb{Z}$ acting on $\mathbb{Z}_k := \mathbb{Z}/k\mathbb{Z}$ for some $k \in \mathbb{Z}_+$. Hence, the composition operator C_ϕ is unitarily equivalent to an injective bilateral weighted shift (the case of $\mathbb{Z}$) or to a bijective finite dimensional weighted shift (the case of $\mathbb{Z}_k$ for $k \in \mathbb{N}$).

2) Now we assume that the directed graph (X, E^ϕ) is not connected. Note that if there exists $\omega \in X$ such that $\mathrm{card}(\phi^{-1}(\{\omega\})) = \eta + 1$ for some $\eta \in \mathbb{N} \cup \{\infty\}$ and $\mathrm{card}(\phi^{-1}(\{x\})) = 1$ for every $x \in X \setminus \{\omega\}$ (hence (3.2.9) holds), then $X \setminus Y \neq \emptyset$, where Y is given either by (3.2.6) or by (3.2.8) depending on whether $\omega \in \phi^{-n}(\{\omega\})$ for some $n \in \mathbb{N}$ or not (see the proofs of Steps 4 and 5 of Theorem 3.2.1). Indeed, this is a consequence of the easily verifiable fact that $(Y, E^{\phi|_Y})$ is a connected subgraph of (X, E^ϕ). Next observe that $\phi|_{X \setminus Y}$ is a bijective self-map of $X \setminus Y$ and thus it is bijectively isomorphic to a disjoint sum of a number of self-maps $i + k\mathbb{Z} \mapsto (i+1) + k\mathbb{Z}$ of $\mathbb{Z}_k$, where $k \in \mathbb{Z}_+$ (see [59, Proposition 2.4]). Clearly, the directed graph $(Y, E^{\phi|_Y})$ satisfies the condition (i) of Theorem 3.2.1. Hence, the composition operator C_ϕ is unitarily equivalent to an orthogonal sum of composition operators whose symbols are described above (see [16, Appendix C]). One can draw a similar conclusion for symbols ϕ discussed in the flat case in 1).

3) The directed graph (X, E^ϕ) appearing in (ii-b) (see Fig. 3) is isomorphic to the directed tree $\mathscr{T}_{\eta+1,\infty}$ defined in [41, (6.2.10)]. By Lemma 3.1.4, the corresponding composition operator C_ϕ is unitarily equivalent to a weighted shift on $\mathscr{T}_{\eta+1,\infty}$ with nonzero weights. Subnormality of such operators has been studied in [41,13].

3.3. Injectivity problem

If we allow the directed graph (X, E^ϕ) to have vertices of valency 0, then the question is how many such vertices can there be. As in Remark 3.2.2, we exclude from our considerations the case when (X, E^ϕ) has more than one vertex of valency greater than 1. The answer is given in Proposition 3.3.1 and in the proof of Theorem 3.3.2 (see (3.3.2)). The question becomes especially interesting when the composition operator C_ϕ generates Stieltjes moment sequences. In this version, the question is a particular case of a more general problem, called here the *injectivity problem* (see Problem 3.3.6). In the present section we investigate the injectivity problem in the context of directed graphs (X, E^ϕ) having at most one vertex of valency greater than 1.

The following assumption will be used many times in this section.

$$\text{Let } X \text{ be a nonempty set, } \phi \text{ be a self-map of } X,\ E^\phi \text{ be as in (3.2.1) and } C_\phi \text{ be a composition operator in } L^2(X, 2^X, \mu) \text{ with symbol } \phi, \text{ where } \mu \text{ is a discrete measure on } X. \tag{3.3.1}$$

If (3.3.1) holds, then we set

$$Z_\phi = \{x \in X : \mathrm{card}(\phi^{-1}(\{x\})) = 0\}.$$

Recall that the case of $Z_\phi = \emptyset$ has been discussed in Remark 3.2.2.

We begin by considering the situation in which the valency of each vertex of the directed graph (X, E^ϕ) does not exceed 1.

Proposition 3.3.1. *Assume that (3.3.1) holds. If the directed graph (X, E^{ϕ}) is connected and $\mathrm{card}(\phi^{-1}(\{x\})) \leqslant 1$ for every $x \in X$, then*

(i) $\mathrm{card}(Z_\phi) \leqslant 1$,
(ii) *C_ϕ is injective whenever C_ϕ generates Stieltjes moment sequences.*

Proof. (i) For if not, there are two distinct vertices $u, v \in Z_\phi$. By the connectivity of (X, E^{ϕ}), there exists an undirected path $\{u_i\}_{i=1}^{k} \subseteq X$ joining u and v of smallest possible length $k \geqslant 2$ (with $u_1 = u$ and $u_k = v$). It is easily seen that the sequence $\{u_i\}_{i=1}^{k}$ is injective. Hence, since $u, v \in Z_\phi$, we see that $k \geqslant 3$, $u_2 = \phi(u)$ and $u_{k-1} = \phi(v)$. By induction, there exists $j \in \{1, \ldots, k-2\}$ such that $u_{j+1} = \phi(u_j)$ and $u_{j+1} = \phi(u_{j+2})$. As a consequence, $u_j, u_{j+2} \in \phi^{-1}(\{u_{j+1}\})$, which contradicts the inequality $\mathrm{card}(\phi^{-1}(\{u_{j+1}\})) \leqslant 1$. This proves (i).

(ii) Suppose, on the contrary, that C_ϕ is not injective, or equivalently that $Z_\phi \neq \emptyset$. By (i), $Z_\phi = \{\omega\}$ for some $\omega \in X$, and thus, by [59, Proposition 2.4] and Steps 1 and 3 of the proof of Theorem 3.2.1, ϕ is bijectively isomorphic to the mapping $i \mapsto i + 1$ acting on $\mathbb{Z}_+$. Hence, without loss of generality, we can assume that $X = \mathbb{Z}_+$ and $\phi(i) = i + 1$ for all $i \in \mathbb{Z}_+$. Then $\chi_{\{1\}} \in \mathcal{D}^\infty(C_\phi)$, $C_\phi \chi_{\{1\}} \neq 0$ and $C_\phi^2 \chi_{\{1\}} = 0$. This contradicts [61, Lemma 1.1(ii)] because C_ϕ generates Stieltjes moment sequences. □

It remains to consider the case when the directed graph (X, E^{ϕ}) has exactly one vertex of valency greater than 1. The following theorem which describes such graphs (see Figs. 4 and 5) will be deduced from Theorem 3.2.1.

Theorem 3.3.2. *Assume that (3.3.1) holds and $\eta \in \mathbb{N} \cup \{\infty\}$. Then the following two conditions are equivalent:*

(i) *the directed graph (X, E^{ϕ}) is connected and there exists $\omega \in X$ such that $\mathrm{card}(\phi^{-1}(\{\omega\})) = \eta + 1$ and $\mathrm{card}(\phi^{-1}(\{x\})) \leqslant 1$ for every $x \in X \setminus \{\omega\}$,*
(ii) *one of the following two conditions is satisfied:*
  (ii-a*) *there exist $\kappa \in \mathbb{Z}_+$, a sequence $\{l_i\}_{i=1}^{\eta} \subseteq \mathbb{N} \cup \{\infty\}$ and two disjoint systems $\{x_i\}_{i=0}^{\kappa}$ and $\{x_{i,j}\}_{i=1}^{\eta}{}_{j=1}^{l_i}$ of distinct points of X such that*

$$X = \{x_0, \ldots, x_\kappa\} \cup \bigcup_{i=1}^{\eta} \Big\{x_{i,j} \colon j \in J_{l_i}\Big\},$$

$$\phi(x) = \begin{cases} x_{i,j-1} & \text{if } x = x_{i,j} \text{ with } i \in J_\eta \text{ and } j \in J_{l_i} \setminus \{1\}, \\ x_\kappa & \text{if } x = x_{i,1} \text{ with } i \in J_\eta \text{ or } x = x_0, \\ x_{i-1} & \text{if } x = x_i \text{ with } i \in J_\kappa, \end{cases}$$

  (ii-b*) *there exist a sequence $\{l_i\}_{i=1}^{\eta+1} \subseteq \mathbb{N} \cup \{\infty\}$ and two disjoint systems $\{x_i\}_{i=0}^{\infty}$ and $\{x_{i,j}\}_{i=1}^{\eta+1}{}_{j=1}^{l_i}$ of distinct points of X such that*

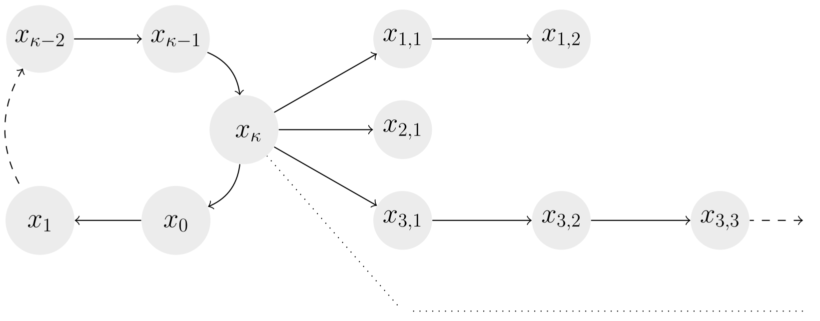

Fig. 4. The directed graph (X, E^ϕ) in the case of (ii-a*) in Theorem 3.3.2 with $l_1 = 2$, $l_2 = 1$, $l_3 = \infty$, ...

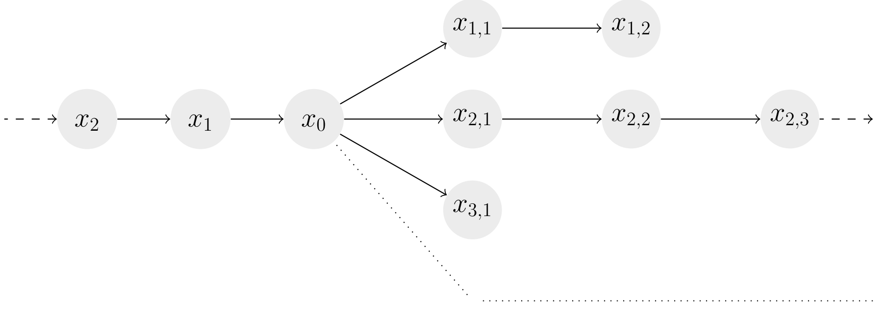

Fig. 5. The directed graph (X, E^ϕ) in the case of (ii-b*) in Theorem 3.3.2 with $l_1 = 2$, $l_2 = \infty$, $l_3 = 1$, ...

$$X = \{x_i \colon i \in \mathbb{Z}_+\} \cup \bigcup_{i=1}^{\eta+1} \Big\{x_{i,j} \colon j \in J_{l_i}\Big\},$$

$$\phi(x) = \begin{cases} x_{i,j-1} & \textit{if } x = x_{i,j} \textit{ with } i \in J_{\eta+1} \textit{ and } j \in J_{l_i} \setminus \{1\}, \\ x_0 & \textit{if } x = x_{i,1} \textit{ with } i \in J_{\eta+1}, \\ x_{i+1} & \textit{if } x = x_i \textit{ with } i \in \mathbb{Z}_+. \end{cases}$$

Proof. (ii)⇒(i) Obvious.

(i)⇒(ii) In view of Theorem 3.2.1, we may assume that $Z_\phi \neq \emptyset$. Let $\{Y_z\}_{z \in Z_\phi}$ be a family of pairwise disjoint countably infinite sets such that $X \cap \bigsqcup_{z \in Z_\phi} Y_z = \emptyset$. Set $\widehat{X} = X \sqcup \bigsqcup_{z \in Z_\phi} Y_z$. For every $z \in Z_\phi$, let $\{y_{z,i}\}_{i=1}^\infty$ be a sequence of distinct points of Y_z such that $Y_z = \{y_{z,i} \colon i \in \mathbb{N}\}$. Define the self-map $\hat{\phi}$ of $\widehat{X}$ by

$$\hat{\phi}(x) = \begin{cases} \phi(x) & \text{if } x \in X, \\ z & \text{if } x = y_{z,1} \text{ for some } z \in Z_\phi, \\ y_{z,i-1} & \text{if } x = y_{z,i} \text{ for some } z \in Z_\phi \text{ and } i \geqslant 2. \end{cases}$$

It is a matter of routine to verify that $\phi \subseteq \hat{\phi}$ (i.e., $\hat{\phi}$ extends ϕ), the directed graph $(\widehat{X}, E^{\hat{\phi}})$ is connected, $\hat{\phi}^{-1}(\{\omega\}) = \phi^{-1}(\{\omega\})$ and $\mathrm{card}(\hat{\phi}^{-1}(\{x\})) = 1$ for every $x \in \widehat{X} \setminus \{\omega\}$.

Hence, by Theorem 3.2.1, $(\widehat{X}, \hat{\phi})$ takes the form (ii-a) or (ii-b) (with $(\widehat{X}, \hat{\phi})$ in place of (X, ϕ)). Set $\tilde{\eta} = \eta$ if $(\widehat{X}, \hat{\phi})$ takes the form (ii-a) and $\tilde{\eta} = \eta + 1$ otherwise. Since $\hat{\phi}^{-1}(\{\omega\}) = \phi^{-1}(\{\omega\})$, we deduce that $x_{i,1} \in X$ for every $i \in J_{\tilde{\eta}}$. This, the explicit description of $(\widehat{X}, \hat{\phi})$ and an induction argument combined with $\phi \subseteq \hat{\phi}$ imply that there exists a sequence $\{l_i\}_{i=1}^{\tilde{\eta}} \subseteq \mathbb{N} \cup \{\infty\}$ such that (ii) holds. In particular, we have

$$Z_\phi = \{x_{i,l_i} \colon i \in J_{\tilde{\eta}},\, l_i < \infty\}. \tag{3.3.2}$$

This completes the proof. $\square$

Now we take a closer look at those directed graphs described by parts (ii-a*) and (ii-b*) of Theorem 3.3.2 which admit composition operators generating Stieltjes moment sequences.

Proposition 3.3.3. *Assume that (3.3.1) holds. If (X, E^ϕ) is as in Theorem 3.3.2*(ii-a*) *with $\eta \in \mathbb{N} \cup \{\infty\}$ and C_ϕ generates Stieltjes moment sequences, then*

(i) *$l_i \in \{1\} \cup \{\infty\}$ for every $i \in J_\eta$,*
(ii) $\operatorname{card}(Z_\phi) \leqslant \eta - 1$,
(iii) *C_ϕ is injective whenever the Stieltjes moment sequence $\{\mathsf{h}_{\phi^{n+1}}(x_\kappa)\}_{n=0}^\infty$ is S-determinate.*

Proof. (i) Indeed, otherwise $l_i \in \mathbb{N}_2$ for some $i \in J_\eta$, which implies that $\chi_{\{x_{i,l_i-1}\}} \in \mathcal{D}^\infty(C_\phi)$, $C_\phi \chi_{\{x_{i,l_i-1}\}} \neq 0$ and $C_\phi^2 \chi_{\{x_{i,l_i-1}\}} = 0$. This contradicts [61, Lemma 1.1(ii)] because C_ϕ generates Stieltjes moment sequences.

(ii) Suppose, on the contrary, that $\operatorname{card}(Z_\phi) > \eta - 1$. Then, by (3.3.2), $\operatorname{card}(Z_\phi) = \eta \in \mathbb{N}$ and consequently $C_\phi \in \boldsymbol{B}(L^2(\mu))$ and C_ϕ is not injective. In view of Lambert's theorem (we use [64, Theorem 7]), C_ϕ is subnormal and, as such, is injective (see [34, Theorem 9d]). This gives a contradiction and proves (ii).

(iii) By [15, Theorem 10.4], the sequence $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $x \in X$. Suppose, on the contrary, that C_ϕ is not injective. Using (i), we see that there exists $i \in J_\eta$ such that $l_i = 1$. We easily verify that δ_0 is an S-representing measure of the Stieltjes moment sequence $\{\mathsf{h}_{\phi^n}(x_{i,l_i})\}_{n=0}^\infty$ (see (3.1.2)). Applying [16, Lemma 38] to $x = x_\kappa$, we are led to a contradiction. $\square$

The next result is a (ii-b*)-analog of Proposition 3.3.3.

Proposition 3.3.4. *Assume that (3.3.1) holds. If (X, E^ϕ) is as in Theorem 3.3.2*(ii-b*) *with $\eta \in \mathbb{N} \cup \{\infty\}$ and C_ϕ generates Stieltjes moment sequences, then*

(i) *$l_i \in \{1\} \cup \{\infty\}$ for every $i \in J_{\eta+1}$,*
(ii) $\operatorname{card}(Z_\phi) \leqslant \eta$,

(iii) C_ϕ *is injective whenever the Stieltjes moment sequence* $\{\mathsf{h}_{\phi^{n+1}}(x_0)\}_{n=0}^{\infty}$ *is S-determinate.*

Proof. Arguing as in the proof of Proposition 3.3.3 (but now applying [16, Lemma 38] to $x = x_0$), we get (i) and (iii).

(ii) Suppose, on the contrary, that $\operatorname{card}(Z_\phi) > \eta$. Then, by (3.3.2), $\eta \in \mathbb{N}$ and $\operatorname{card}(Z_\phi) = \eta + 1$. It follows from (i) that $l_i = 1$ for every $i \in J_{\eta+1}$. As a consequence, $\chi_{\{x_0\}} \in \mathcal{D}^\infty(C_\phi)$, $C_\phi\chi_{\{x_0\}} \neq 0$ and $C_\phi^2\chi_{\{x_0\}} = 0$, which contradicts [61, Lemma 1.1(ii)] because C_ϕ generates Stieltjes moment sequences. □

Similar reasoning as in the proof of part (iii) of Proposition 3.3.3 gives a criterion for injectivity of C_ϕ in a more general situation.

Proposition 3.3.5. *Suppose that* (3.3.1) *holds. If* C_ϕ *generates Stieltjes moment sequences and the Stieltjes moment sequence* $\{\mathsf{h}_{\phi^{n+1}}(x)\}_{n=0}^{\infty}$ *is S-determinate for every* $x \in X$*, then* C_ϕ *is injective.*

The above considerations lead us to the following injectivity problem (see [15] for the necessary definitions).

Problem 3.3.6. Suppose that $(X, \mathscr{A}, \mu)$ is a σ-finite measure space, ϕ is a nonsingular self-map of X and C_ϕ is a composition operator in $L^2(\mu)$ with symbol ϕ generating Stieltjes moment sequences. Is it true that C_ϕ is injective?

Problem 3.3.6 seems to be hard to solve. To shed more light on this we make the following remark.

Remark 3.3.7. First, we note that Problem 3.3.6 has an affirmative solution for bounded composition operators. Indeed, by Lambert's theorem (we use [64, Theorem 7] again), a bounded composition operator in an L^2-space generating Stieltjes moment sequences is subnormal and consequently, by [34, Theorem 9d], it is injective. If the composition operator in question is over a rootless directed tree and it has sufficiently many quasi-analytic vectors, then the property of generating Stieltjes moment sequences is equivalent to subnormality (use Lemma 3.1.4, its proof and [12, Theorem 5.3.1]). Hence, in view of [15, Corollary 6.3], in this particular case, Problem 3.3.6 has an affirmative solution as well (this can be also deduced from Proposition 3.3.5 by applying (3.1.4) and Proposition 2.5.1). Propositions 3.3.1, 3.3.3, 3.3.4 and 3.3.5 provide yet another examples for which Problem 3.3.6 has an affirmative solution.

3.4. The Radon–Nikodym derivatives

The following assumption will be used frequently through this paper.

$$\begin{aligned}&\text{Let } \eta \in \mathbb{N}\cup\{\infty\} \text{ and } \kappa\in\mathbb{Z}_+, \text{ and let } X = X_{\eta,\kappa} \text{ be a set satisfying}\\ &\text{(3.2.2), where } \{x_i\}_{i=0}^{\kappa} \text{ and } \{x_{i,j}\}_{i=1}^{\eta}{}_{j=1}^{\infty} \text{ are two disjoint systems}\\ &\text{of distinct points of } X,\ \phi=\phi_{\eta,\kappa} \text{ be a self-map of } X \text{ satisfying}\\ &\text{(3.2.3) and } \mu \text{ be a discrete measure on } X. \text{ We adhere to the}\\ &\text{convention that } x_{-1}=x_\kappa \text{ and } x_{i,0}=x_\kappa \text{ for } i\in J_\eta.\end{aligned} \tag{3.4.1}$$

It is easily seen that

$$\textit{if (3.4.1) holds, then the composition operator } C_\phi \textit{ is injective.} \tag{3.4.2}$$

We begin by deriving a formula for iterated inverses of ϕ at the point x_κ.

Lemma 3.4.1. *Suppose (3.4.1) holds. If $n = j(\kappa+1)+r$ for some $j\in\mathbb{Z}_+$ and $r\in\{0,\dots,\kappa\}$, then*

$$\phi^{-n}(\{x_\kappa\}) = \{x_{r-1}\}\cup\Big\{x_{i,l(\kappa+1)+r}\colon i\in J_\eta,\, l\in\{0,\dots,j\}\Big\}. \tag{3.4.3}$$

Proof. We proceed by induction on j. The case of $j=0$ is easily verified. This and (3.2.3) imply that

$$\begin{aligned}\phi^{-(\kappa+1)}(\{x_\kappa\}) = \phi^{-1}(\phi^{-\kappa}(\{x_\kappa\})) &= \phi^{-1}(\{x_{\kappa-1}\}\cup\{x_{i,\kappa}\colon i\in J_\eta\})\\ &= \{x_\kappa\}\cup\{x_{i,\kappa+1}\colon i\in J_\eta\}.\end{aligned} \tag{3.4.4}$$

Suppose that (3.4.3) holds for a fixed $j\in\mathbb{Z}_+$. Then, by (3.4.4) and induction hypothesis, we have

$$\begin{aligned}\phi^{-((j+1)(\kappa+1)+r)}&(\{x_\kappa\}) = \phi^{-(j(\kappa+1)+r)}(\{x_\kappa\}\cup\{x_{i,\kappa+1}\colon i\in J_\eta\})\\ &= \{x_{r-1}\}\cup\Big\{x_{i,l(\kappa+1)+r}\colon i\in J_\eta,\, l\in\{0,\dots,j\}\Big\}\cup\Big\{x_{i,(j+1)(\kappa+1)+r}\colon i\in J_\eta\Big\},\end{aligned}$$

which completes the proof. □

Applying (3.1.2) to ϕ and ϕ^n and using Lemma 3.4.1, we can easily calculate the Radon–Nikodym derivatives $\{\mathsf{h}_\phi(x)\colon x\in X\}$ and $\{\mathsf{h}_{\phi^n}(x_\kappa)\colon n\in\mathbb{Z}_+\}$.

Proposition 3.4.2. *Suppose (3.4.1) holds. Then for every $x\in X$,*

$$\mathsf{h}_\phi(x) = \begin{cases}\frac{\mu(x_{j+1})}{\mu(x_j)} & \textit{if } x=x_j \textit{ with } j\in\{0,\dots,\kappa-1\},\\ \frac{\mu(x_0)+\sum_{i=1}^{\eta}\mu(x_{i,1})}{\mu(x_\kappa)} & \textit{if } x=x_\kappa,\\ \frac{\mu(x_{i,j+1})}{\mu(x_{i,j})} & \textit{if } x=x_{i,j} \textit{ with } i\in J_\eta \textit{ and } j\in\mathbb{N}.\end{cases} \tag{3.4.5}$$

If $n=j(\kappa+1)+r$ for some $j\in\mathbb{Z}_+$ and $r\in\{0,\dots,\kappa\}$, then

$$\mathsf{h}_{\phi^n}(x_\kappa) = \begin{cases} 1 & \text{if } n = 0, \\ 1 + \sum_{i=1}^{\eta} \sum_{l=1}^{j} \frac{\mu(x_{i,l(\kappa+1)})}{\mu(x_\kappa)} & \text{if } j \geqslant 1 \text{ and } r = 0, \\ \frac{\mu(x_{r-1})}{\mu(x_\kappa)} + \sum_{i=1}^{\eta} \sum_{l=0}^{j} \frac{\mu(x_{i,l(\kappa+1)+r})}{\mu(x_\kappa)} & \text{if } r \in J_\kappa. \end{cases} \tag{3.4.6}$$

Now we calculate the Radon–Nikodym derivatives h_{ϕ^n}, $n \geqslant 0$, at the vertices lying on the circuit.

Lemma 3.4.3. *Suppose (3.4.1) holds. Then*

$$\mathsf{h}_{\phi^{n+1}}(x_{r-1}) = \frac{\mu(x_r)}{\mu(x_{r-1})} \mathsf{h}_{\phi^n}(x_r), \qquad r \in J_\kappa,\ n \in \mathbb{Z}_+, \tag{3.4.7}$$

$$\mathsf{h}_{\phi^{n+r}}(x_0) = \frac{\mu(x_r)}{\mu(x_0)} \mathsf{h}_{\phi^n}(x_r), \qquad r \in \{0, \ldots, \kappa\},\ n \in \mathbb{Z}_+. \tag{3.4.8}$$

Proof. If $r \in J_\kappa$ and $n \in \mathbb{Z}_+$, then by (3.1.2) and (3.2.3) we have

$$\begin{aligned} \mathsf{h}_{\phi^{n+1}}(x_{r-1}) &= \frac{\mu(\phi^{-n}(\phi^{-1}(\{x_{r-1}\})))}{\mu(x_{r-1})} \\ &= \frac{\mu(\phi^{-n}(\{x_r\}))}{\mu(x_r)} \frac{\mu(x_r)}{\mu(x_{r-1})} \\ &= \frac{\mu(x_r)}{\mu(x_{r-1})} \mathsf{h}_{\phi^n}(x_r), \end{aligned}$$

which gives (3.4.7). Applying induction on r and (3.4.7), we obtain (3.4.8). □

The subsequent lemma plays an essential role in the present paper.

Lemma 3.4.4. *If (3.4.1) holds, then*

$$\mathsf{h}_{\phi^{n+\kappa+1}}(x_0) = \mathsf{h}_{\phi^n}(x_0) + \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_0)} \mathsf{h}_{\phi^n}(x_{i,1}), \quad n \in \mathbb{Z}_+. \tag{3.4.9}$$

Proof. Observe that by (3.1.2) and (3.2.3) we have

$$\begin{aligned} \mathsf{h}_{\phi^{n+\kappa+1}}(x_0) &= \frac{\mu(\phi^{-n}(\phi^{-1}(\{x_\kappa\})))}{\mu(x_0)} \\ &= \frac{\mu(\phi^{-n}(\{x_0\} \sqcup \{x_{i,1} : i \in J_\eta\}))}{\mu(x_0)} \end{aligned}$$

$$= \mathsf{h}_{\phi^n}(x_0) + \sum_{i=1}^{\eta} \frac{\mu(\phi^{-n}(\{x_{i,1}\}))}{\mu(x_0)}$$

$$= \mathsf{h}_{\phi^n}(x_0) + \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_0)} \mathsf{h}_{\phi^n}(x_{i,1}), \quad n \in \mathbb{Z}_+,$$

which completes the proof of (3.4.9). □

The question of density of domains of powers of C_ϕ can be answered in terms of the Radon–Nikodym derivatives h_{ϕ^n}, $n \geqslant 0$, calculated at x_0.

Proposition 3.4.5. *Suppose (3.4.1) holds and $n \in \mathbb{N}$. Then the following conditions are equivalent*:

(i) $\mathcal{D}(C_\phi^n)$ *is dense in* $L^2(\mu)$,
(ii) $\mathsf{h}_{\phi^{n+r}}(x_0) < \infty$ *for every* $r \in \{0, \ldots, \kappa\}$.

Moreover, if $r \in \{0, \ldots, \kappa\}$, then the following conditions are equivalent:

(iii) $\mathcal{D}^\infty(C_\phi)$ *is dense in* $L^2(\mu)$,
(iv) $\mathcal{D}(C_\phi^j)$ *is dense in* $L^2(\mu)$ *for all* $j \in \mathbb{N}$,
(v) $\mathsf{h}_{\phi^j}(x) < \infty$ *for all* $j \in \mathbb{N}$ *and* $x \in X$,
(vi) $\mathsf{h}_{\phi^j}(x_r) < \infty$ *for all* $j \in \mathbb{N}$.

Proof. (i)⇒(ii) By (3.1.5), $\mathsf{h}_{\phi^l}(x_\kappa) < \infty$ for all $l \in \{0, \ldots, n\}$, and thus, by (3.4.8) with $r = \kappa$, $\mathsf{h}_{\phi^{l+\kappa}}(x_0) < \infty$ for all $l \in \{0, \ldots, n\}$. Since, by (3.1.2), $\mathsf{h}_{\phi^l}(x_0) < \infty$ for every $l \in \{0, \ldots, \kappa - 1\}$, we see that $\mathsf{h}_{\phi^l}(x_0) < \infty$ for every $l \in \{0, \ldots, n + \kappa\}$.

(ii)⇒(i) Applying (3.4.8), we deduce that $\mathsf{h}_{\phi^n}(x_r) < \infty$ for every $r \in \{0, \ldots, \kappa\}$. It follows from (3.1.2) that $\mathsf{h}_{\phi^n}(x_{i,j}) < \infty$ for all $i \in J_\eta$ and $j \in \mathbb{N}$. This, (3.1.5) and (3.2.2) yield (i).

Now we prove the "moreover" part. By (3.1.5) and [15, Theorem 4.7], it suffices to prove that (vi) implies (iv). It follows from (3.4.8) that $\mathsf{h}_{\phi^j}(x_0) < \infty$ for all integers $j \geqslant \kappa$. Applying (3.1.2), we deduce that $\mathsf{h}_{\phi^j}(x_0) < \infty$ for all $j \in \mathbb{Z}_+$. Hence, by the implication (ii)⇒(i), $\mathcal{D}(C_\phi^j)$ is dense in $L^2(\mu)$ for all $j \in \mathbb{N}$. □

Corollary 3.4.6. *Suppose (3.4.1) holds and $n \in \mathbb{N}$. Then the following conditions are equivalent*:

(i) $\overline{\mathcal{D}(C_\phi^n)} = L^2(\mu)$ *and* $\overline{\mathcal{D}(C_\phi^{n+1})} \subsetneq L^2(\mu)$,
(ii) $\mathsf{h}_{\phi^{n+r}}(x_0) < \infty$ *for every* $r \in \{0, \ldots, \kappa\}$ *and* $\mathsf{h}_{\phi^{n+\kappa+1}}(x_0) = \infty$.

4. Subnormality of $C_{\phi_{\eta,\kappa}}$ via the consistency condition (CC)

4.1. Characterizations of (CC)

This section deals with the consistency condition which, according to Theorem 3.1.3, automatically implies subnormality of $C_{\phi_{\eta,\kappa}}$ (because, under our standing assumption (3.4.1), $\mathsf{h}_{\phi_{\eta,\kappa}}(x) > 0$ for all $x \in X$).

Theorem 4.1.1. *Suppose (3.4.1) holds, C_ϕ is densely defined and $\{P(x,\cdot)\}_{x\in X}$ is a family of Borel probability measures on $\mathbb{R}_+$. Then $\{P(x,\cdot)\}_{x\in X}$ satisfies (CC) if and only if the following three conditions are satisfied:*

(i) $P(x_r,\sigma) = \frac{\mu(x_0)}{\mu(x_r)} \int_\sigma t^r P(x_0,\mathrm{d}t)$ *for all* $r \in \{0,\ldots,\kappa\}$ *and* $\sigma \in \mathfrak{B}(\mathbb{R}_+)$,
(ii) $\sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_0)} P(x_{i,1},\sigma) = \int_\sigma (t^{\kappa+1}-1) P(x_0,\mathrm{d}t)$ *for all* $\sigma \in \mathfrak{B}(\mathbb{R}_+)$,
(iii) $P(x_{i,j},\sigma) = \frac{\mu(x_{i,1})}{\mu(x_{i,j})} \int_\sigma t^{j-1} P(x_{i,1},\mathrm{d}t)$ *for all* $i \in J_\eta$, $j \in \mathbb{N}_2$ *and* $\sigma \in \mathfrak{B}(\mathbb{R}_+)$.

Moreover, if $\{P(x,\cdot)\}_{x\in X}$ satisfies (CC), then

(iv) $P(x_r,[0,1)) = P(x_{i,j},[0,1]) = 0$ *for all* $r \in \{0,\ldots,\kappa\}$, $i \in J_\eta$ *and* $j \in \mathbb{N}$,
(v) $P(x_0,\sigma \cap (1,\infty)) = \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_0)} \int_\sigma \frac{1}{t^{\kappa+1}-1} P(x_{i,1},\mathrm{d}t)$ *for all* $\sigma \in \mathfrak{B}(\mathbb{R}_+)$,
(vi) $\sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_0)} \int_0^\infty \frac{1}{t^{\kappa+1}-1} P(x_{i,1},\mathrm{d}t) \leqslant 1$,
(vii) $P(x_0,\{1\}) = \vartheta := 1 - \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_0)} \int_0^\infty \frac{1}{t^{\kappa+1}-1} P(x_{i,1},\mathrm{d}t)$,
(viii) $P(x_0,\sigma) = \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_0)} \int_\sigma \frac{1}{t^{\kappa+1}-1} P(x_{i,1},\mathrm{d}t) + \vartheta\delta_1(\sigma)$ *for all* $\sigma \in \mathfrak{B}(\mathbb{R}_+)$,
(ix) $\sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_0)} \int_0^\infty \frac{t^{\kappa+1}}{t^{\kappa+1}-1} P(x_{i,1},\mathrm{d}t) < \infty$.

Proof. Since, by (3.1.5), $\mathsf{h}_\phi(x_\kappa) < \infty$, we infer from Proposition 3.4.2 that

$$\sum_{i=1}^{\eta} \mu(x_{i,1}) < \infty. \tag{4.1.1}$$

Assume now that $\{P(x,\cdot)\}_{x\in X}$ satisfies (CC). Substituting $x = x_r$ with $r \in \{0,\ldots,\kappa-1\}$ into (CC), we get

$$P(x_{r+1},\sigma) = \frac{\mu(x_r)}{\mu(x_{r+1})} \int_\sigma t P(x_r,\mathrm{d}t), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\ r \in \{0,\ldots,\kappa-1\}.$$

An induction argument shows that (i) holds. Substituting $x = x_\kappa$ into (CC) and using (i), we obtain

$$\mu(x_0)P(x_0,\sigma) + \sum_{i=1}^{\eta} \mu(x_{i,1})P(x_{i,1},\sigma) = \mu(x_\kappa) \int_\sigma t P(x_\kappa,\mathrm{d}t)$$

$$= \mu(x_0) \int_\sigma t^{\kappa+1} P(x_0, \mathrm{d}t), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

This implies (ii). Substituting $x = x_{i,j}$ into (CC) yields

$$P(x_{i,j+1}, \sigma) = \frac{\mu(x_{i,j})}{\mu(x_{i,j+1})} \int_\sigma t P(x_{i,j}, \mathrm{d}t), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\ i \in J_\eta,\ j \in \mathbb{N}.$$

An induction argument leads to (iii).

Similar reasoning shows that the conditions (i)–(iii) imply that $\{P(x, \cdot)\}_{x\in X}$ satisfies (CC).

To prove the "moreover" part, we assume that $\{P(x, \cdot)\}_{x\in X}$ satisfies the condition (CC). Since $P(x, \mathbb{R}_+) = 1$ for all $x \in X$, we deduce from (ii) and (4.1.1) that $\int_0^\infty |t^{\kappa+1} - 1| P(x_0, \mathrm{d}t) < \infty$. Hence, by (ii) again, $P(x_{i,1}, [0, 1]) = 0$ for all $i \in J_\eta$ and $P(x_0, [0, 1)) = 0$. Applying (i) and (iii) gives (iv). It follows from (ii) that

$$\sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_0)} P(x_{i,1}, \sigma) = \int_{\sigma\cap(1,\infty)} (t^{\kappa+1} - 1) P(x_0, \mathrm{d}t), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{4.1.2}$$

Using (iv) and (4.1.2) and integrating the function $t \mapsto \frac{\chi_\sigma(t)}{t^{\kappa+1}-1}$ with respect to the measure $\sum_{i=1}^\eta \frac{\mu(x_{i,1})}{\mu(x_0)} P(x_{i,1}, \cdot)$, we obtain (v). Since $P(x_0, \mathbb{R}_+) = 1$, the conditions (vi) and (vii) follow from (v). The equality (viii) is a direct consequence of (iv), (v) and (vii). Finally, integrating the function $t \mapsto t^{\kappa+1}$ with respect to the (positive) measure $\sigma \mapsto \sum_{i=1}^\eta \frac{\mu(x_{i,1})}{\mu(x_0)} \int_\sigma \frac{1}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t)$, we deduce from (v) and (3.1.8) that

$$\sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_0)} \int_0^\infty \frac{t^{\kappa+1}}{t^{\kappa+1} - 1} P(x_{i,1}, \mathrm{d}t) \leqslant \mathsf{h}_{\phi^{\kappa+1}}(x_0).$$

Applying Proposition 3.4.5 with $n = 1$ and $r = \kappa$, we get (ix). □

The following proposition provides new criteria for $C_{\phi_{\eta,\kappa}}$ to have densely defined nth power (cf. Proposition 3.4.5).

Proposition 4.1.2. *Suppose (3.4.1) holds, the composition operator C_ϕ is densely defined and $\{P(x, \cdot)\}_{x\in X}$ is a family of Borel probability measures on $\mathbb{R}_+$ that satisfies (CC). Then for every $n \in \mathbb{N}$, the following conditions are equivalent*:

(i) *C_ϕ^n is densely defined,*
(ii) *$\mathsf{h}_{\phi^{n+\kappa}}(x_0) < \infty$,*
(iii) *$\sum_{i=1}^\eta \mu(x_{i,1}) \int_0^\infty \frac{t^{n+\kappa}}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) < \infty$.*

Proof. By (3.1.8) and Theorem 4.1.1(iv), the sequence $\{\mathsf{h}_{\phi^j}(x_0)\}_{n=0}^\infty$ is monotonically increasing. Hence, by Proposition 3.4.5, the conditions (i) and (ii) are equivalent. Integrating the function $t \mapsto t^{n+\kappa}$ with respect to the measure $P(x_0, \cdot)$ and using (3.1.8) and Theorem 4.1.1(viii), we deduce that the conditions (ii) and (iii) are equivalent. □

As a consequence of Theorem 4.1.1(iv) and Proposition 4.1.2, we have the following corollary (cf. Corollary 3.4.6).

Corollary 4.1.3. *Suppose (3.4.1) holds, C_ϕ is densely defined and $\{P(x,\cdot)\}_{x\in X}$ is a family of Borel probability measures on $\mathbb{R}_+$ that satisfies (CC). Let $n \in \mathbb{N}$. Then $\overline{\mathcal{D}(C_\phi^{n+1})} \subsetneq \overline{\mathcal{D}(C_\phi^n)} = L^2(\mu)$ if and only if the following two conditions hold:*

(i) $\sum_{i=1}^\eta \mu(x_{i,1}) \int_0^\infty \frac{t^{n+\kappa}}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) < \infty$,
(ii) $\sum_{i=1}^\eta \mu(x_{i,1}) \int_0^\infty \frac{t^{n+\kappa+1}}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) = \infty$.

4.2. Modelling subnormality via (CC)

In Procedure 4.2.1 below, we propose a method of constructing all possible subnormal composition operators $C_{\phi_{\eta,\kappa}}$ in $L^2(X_{\eta,\kappa}, \mu)$ that satisfy (CC) in the meaning that they admit families of probability measures satisfying (CC). The starting point of our procedure is a family $\{P(x_{i,1},\cdot)\}_{i\in J_\eta}$ of Borel probability measures on $\mathbb{R}_+$ that satisfies the conditions (4.2.1)–(4.2.3) below. Let us point out that if a densely defined $C_{\phi_{\eta,\kappa}}$ admits a family $\{P(x,\cdot)\}_{x\in X_{\eta,\kappa}}$ of Borel probability measures on $\mathbb{R}_+$ that satisfies (CC), then, by Theorem 4.1.1, the measures $P(x_{i,1},\cdot)$, $i \in J_\eta$, satisfy the conditions (4.2.1)–(4.2.3).

Procedure 4.2.1. Let $\eta \in \mathbb{N} \cup \{\infty\}$, $\kappa \in \mathbb{Z}_+$, ϕ be a self-map of a set X, $\{x_i\}_{i=0}^\kappa$ and $\{x_{i,j}\}_{i=1}^\eta{}_{j=1}^\infty$ be two disjoint systems of distinct points of X that satisfy (3.2.2) and (3.2.3). Let $\{P(x_{i,1},\cdot)\}_{i\in J_\eta}$ be a family of Borel probability measures on $\mathbb{R}_+$ that satisfies the following three conditions:

$$P(x_{i,1}, [0,1]) = 0, \quad i \in J_\eta, \tag{4.2.1}$$

$$\int_0^\infty t^j P(x_{i,1}, \mathrm{d}t) < \infty, \quad j \in \mathbb{N},\ i \in J_\eta, \tag{4.2.2}$$

$$\int_0^\infty \frac{t^{\kappa+1}}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) < \infty, \quad i \in J_\eta. \tag{4.2.3}$$

Let $\{\mu(x_{i,1})\}_{i\in J_\eta}$ be a family of positive real numbers such that

$$\sum_{i=1}^\eta \mu(x_{i,1}) \int_0^\infty \frac{t^{\kappa+1}}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) < \infty. \tag{4.2.4}$$

It follows from (4.2.1) and (4.2.4) that

$$0 \leqslant \sum_{i=1}^{\eta} \mu(x_{i,1}) \int_0^\infty \frac{t^r}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) < \infty, \quad r \in \{0, \ldots, \kappa+1\}. \tag{4.2.5}$$

Using (4.2.1) and (4.2.5), we get

$$\begin{aligned} \sum_{i=1}^{\eta} \mu(x_{i,1}) &= \sum_{i=1}^{\eta} \mu(x_{i,1}) \int_0^\infty \frac{t^{\kappa+1}}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) \\ &\quad - \sum_{i=1}^{\eta} \mu(x_{i,1}) \int_0^\infty \frac{1}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) < \infty. \end{aligned} \tag{4.2.6}$$

Now, by (4.2.5), we can take $\mu(x_0) \in (0, \infty)$ such that

$$0 \leqslant \varTheta := \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_0)} \int_0^\infty \frac{1}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) \leqslant 1. \tag{4.2.7}$$

Then we define the Borel measure $P(x_0, \cdot)$ on $\mathbb{R}_+$ by

$$P(x_0, \sigma) = \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_0)} \int_\sigma \frac{1}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) + (1-\varTheta)\delta_1(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{4.2.8}$$

By (4.2.1) and (4.2.7), $P(x_0, \cdot)$ is a probability measure such that $P(x_0, [0,1)) = 0$. Moreover, $P(x_0, \cdot)$ satisfies the condition (ii) of Theorem 4.1.1. Since $P(x_{i,1}, \cdot)$, $i \in J_\eta$, are probability measures, we infer from (4.2.1) and (4.2.2) that $0 < \int_0^\infty t^j P(x_{i,1}, \mathrm{d}t) < \infty$ for all $j \in \mathbb{N}$ and $i \in J_\eta$. This enables us to define the family $\{\mu(x_{i,j})\}_{i=1}^{\eta}{}_{j=2}^{\infty}$ of positive real numbers by

$$\mu(x_{i,j}) = \mu(x_{i,1}) \int_0^\infty t^{j-1} P(x_{i,1}, \mathrm{d}t), \quad i \in J_\eta,\ j \in \mathbb{N}_2, \tag{4.2.9}$$

and the family $\{P(x_{i,j}, \cdot)\}_{i=1}^{\eta}{}_{j=2}^{\infty}$ of Borel measures on $\mathbb{R}_+$ by

$$P(x_{i,j}, \sigma) = \frac{\mu(x_{i,1})}{\mu(x_{i,j})} \int_\sigma t^{j-1} P(x_{i,1}, \mathrm{d}t), \quad i \in J_\eta,\ j \in \mathbb{N}_2,\ \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

In view of (4.2.9), the family $\{P(x_{i,j}, \cdot)\}_{i=1}^{\eta}{}_{j=2}^{\infty}$ consists of probability measures. According to (4.2.1), (4.2.5) and (4.2.8), $0 < \int_0^\infty t^r P(x_0, \mathrm{d}t) < \infty$ for every $r \in \{0, \ldots, \kappa\}$. Hence, we can define positive real numbers $\mu(x_r)$, $r \in J_\kappa$, via

$$\mu(x_r) = \mu(x_0) \int_0^\infty t^r P(x_0, \mathrm{d}t), \quad r \in J_\kappa.$$

As a consequence, the measures $P(x_r, \cdot)$, $r \in J_\kappa$, defined by

$$P(x_r, \sigma) = \frac{\mu(x_0)}{\mu(x_r)} \int_\sigma t^r P(x_0, \mathrm{d}t), \quad r \in J_\kappa, \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

are Borel probability measures on $\mathbb{R}_+$. Let μ be the discrete measure on X such that $\mu(\{x_r\}) = \mu(x_r)$ and $\mu(\{x_{i,j}\}) = \mu(x_{i,j})$ for all $r \in \{0, \ldots, \kappa\}$, $i \in J_\eta$ and $j \in \mathbb{N}$, and let C_ϕ be the corresponding composition operator in $L^2(\mu)$ with $\phi = \phi_{\eta,\kappa}$. By (3.1.5), (3.4.5) and (4.2.6), C_ϕ is densely defined. Applying Theorem 4.1.1, we see that the family $\{P(x, \cdot)\}_{x \in X}$ satisfies (CC). Hence, by Theorem 3.1.3, C_ϕ is subnormal.

Our procedure enables us to model all subnormal composition operators $C_{\phi_{\eta,\kappa}}$ that satisfy (CC) and have densely defined nth power (n is a fixed positive integer). It suffices to replace (4.2.3) by the condition

$$\int_0^\infty \frac{t^{\kappa+n}}{t^{\kappa+1} - 1} P(x_{i,1}, \mathrm{d}t) < \infty, \quad i \in J_\eta \tag{4.2.10}$$

(leaving the assumptions (4.2.1) and (4.2.2) unchanged) and to choose a family $\{\mu(x_{i,1})\}_{i \in J_\eta} \subseteq (0, \infty)$ that satisfies, in place of (4.2.4), the following inequality

$$\sum_{i=1}^\eta \mu(x_{i,1}) \int_0^\infty \frac{t^{\kappa+n}}{t^{\kappa+1} - 1} P(x_{i,1}, \mathrm{d}t) < \infty. \tag{4.2.11}$$

Indeed, by (4.2.1), the conditions (4.2.10) and (4.2.11) imply (4.2.3) and (4.2.4), respectively. On the other hand, under the assumptions (4.2.1)–(4.2.3), C_ϕ^n is densely defined if and only if (4.2.11) holds (see Proposition 4.1.2). As a consequence (see also Corollary 4.1.3), $\overline{\mathcal{D}(C_\phi^{n+1})} \subsetneq \overline{\mathcal{D}(C_\phi^n)} = L^2(\mu)$ if and only if both (4.2.11) and (4.2.12) hold, where

$$\sum_{i=1}^\eta \mu(x_{i,1}) \int_0^\infty \frac{t^{\kappa+n+1}}{t^{\kappa+1} - 1} P(x_{i,1}, \mathrm{d}t) = \infty. \tag{4.2.12}$$

Using Procedure 4.2.1, we will show that for every $n \in \mathbb{N}$, there exists a subnormal composition operator C_ϕ such that C_ϕ^n is densely defined, while C_ϕ^{n+1} is not. Examples of this kind have been given in [14] by using weighted shifts on directed trees (see also a recent paper [17] for more subtle examples).

Example 4.2.2. Fix $n \in \mathbb{N}$. Consider a sequence $\{P(x_{i,1}, \cdot)\}_{i=1}^\infty$ of Borel probability measures on $\mathbb{R}_+$ given by $P(x_{i,1}, \sigma) = \delta_{i+1}(\sigma)$ for all $\sigma \in \mathfrak{B}(\mathbb{R}_+)$ and $i \in \mathbb{N}$. Set

$\mu(x_{i,1}) = \frac{1}{(i+1)^{n+1}}$ for $i \in \mathbb{N}$. It is now a routine matter to verify that the conditions (4.2.1), (4.2.2), (4.2.11) and (4.2.12) hold for $\eta = \infty$ and for arbitrary $\kappa \in \mathbb{Z}_+$. Hence, applying Procedure 4.2.1, we get a composition operator C_ϕ with the required properties, i.e., $\overline{\mathcal{D}(C_\phi^{n+1})} \subsetneq \overline{\mathcal{D}(C_\phi^n)} = L^2(\mu)$. Note that by (3.4.6) and Proposition 3.4.5, C_ϕ^j is densely defined for every $j \in \mathbb{N}$ whenever $\eta < \infty$.

4.3. Criteria for subnormality related to x_0

In this section we give criteria for subnormality of composition operators C_ϕ in $L^2(X, \mu)$ with $X = X_{\eta,\kappa}$ and $\phi = \phi_{\eta,\kappa}$ written in terms of the Radon–Nikodym derivatives $\{\mathsf{h}_{\phi^n}\}_{n=0}^\infty$ calculated at the points x_0 and $x_{i,1}$, $i \in J_\eta$ (see Theorem 4.3.3). Surprisingly, in the case of $\eta = 1$ the subnormality of C_ϕ can be inferred from the behaviour of $\{\mathsf{h}_{\phi^n}\}_{n=0}^\infty$ only at the point x_0 (see Proposition 4.3.4). We begin by stating two necessary lemmata.

Lemma 4.3.1. *Let $\{\gamma_n\}_{n=0}^\infty$ be a Stieltjes moment sequence and let $p \in \mathbb{N}$. Then $\{\gamma_{jp}\}_{j=0}^\infty$ is a Stieltjes moment sequence and the following assertions hold*:

(i) *if $\{\gamma_{jp}\}_{j=0}^\infty$ is S-determinate, then so is $\{\gamma_n\}_{n=0}^\infty$,*
(ii) *if $\{\gamma_{jp}\}_{j=0}^\infty$ is S-determinate and $\{\gamma_{(j+1)p} - \gamma_{jp}\}_{j=0}^\infty$ is a Stieltjes moment sequence, then $\{\gamma_n\}_{n=0}^\infty$ is S-determinate and its unique S-representing measure vanishes on $[0, 1)$,*
(iii) *if $\{\gamma_{n+1} - \gamma_n\}_{n=0}^\infty$ is a Stieltjes moment sequence, then $\{\gamma_n\}_{n=0}^\infty$ satisfies the Carleman condition if and only if $\{\gamma_{n+1} - \gamma_n\}_{n=0}^\infty$ satisfies the Carleman condition.*

Proof. (i) Let ρ be an S-representing measure of $\{\gamma_n\}_{n=0}^\infty$, $W\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a function given by $W(t) = t^p$ for $t \in \mathbb{R}_+$, and $\rho \circ W^{-1}$ be a Borel measure on $\mathbb{R}_+$ given by $\rho \circ W^{-1}(\sigma) = \rho(W^{-1}(\sigma))$ for $\sigma \in \mathfrak{B}(\mathbb{R}_+)$. Using the measure transport theorem, we see that $\{\gamma_{jp}\}_{j=0}^\infty$ is a Stieltjes moment sequence with the S-representing measure $\rho \circ W^{-1}$. If ρ' is another S-representing measure of $\{\gamma_n\}_{n=0}^\infty$, then the measure $\rho' \circ W^{-1}$, being an S-representing measure of $\{\gamma_{jp}\}_{j=0}^\infty$, coincides with $\rho \circ W^{-1}$, and consequently $\rho = \rho'$.

(ii) In view of (i), $\{\gamma_n\}_{n=0}^\infty$ is S-determinate. Denote by ρ its unique S-representing measure. Let ν be an S-representing measure of $\{\gamma_{(j+1)p} - \gamma_{jp}\}_{j=0}^\infty$. Then

$$\sum_{i,j=0}^{n} \big(\gamma_{(i+j+1)p} - \gamma_{(i+j)p}\big)\lambda_i\bar{\lambda}_j = \int_0^\infty \Big|\sum_{j=0}^{n} \lambda_j t^j\Big|^2 \mathrm{d}\nu(t) \geqslant 0$$

for all finite sequences $\{\lambda_j\}_{j=0}^n \subseteq \mathbb{C}$. By Lemma 2.1.6 and the S-determinacy of $\{\gamma_{jp}\}_{j=0}^\infty$, we deduce that $\rho \circ W^{-1}([0, 1)) = 0$. Hence $\rho([0, 1)) = 0$.

(iii) Set $\varDelta_n = \gamma_{n+1} - \gamma_n$ for $n \in \mathbb{Z}_+$. Since $\varDelta_n \leqslant \gamma_{n+1}$ for all $n \in \mathbb{Z}_+$, we infer from Proposition 2.5.1(ii) that if $\{\gamma_n\}_{n=0}^\infty$ satisfies the Carleman condition, then so does

$\{\varDelta_n\}_{n=0}^\infty$. To prove the converse implication assume that $\{\varDelta_n\}_{n=0}^\infty$ satisfies the Carleman condition. Note that

$$\gamma_n = \sum_{l=0}^{n-1} \varDelta_l + \gamma_0, \quad n \in \mathbb{N}. \tag{4.3.1}$$

Let τ be an S-representing measure of $\{\varDelta_n\}_{n=0}^\infty$. Set $\varDelta_n' = \int_{(1,\infty)} t^n \mathrm{d}\tau(t)$ for $n \in \mathbb{Z}_+$. If $\tau((1,\infty)) = 0$, then, by (4.3.1),

$$\gamma_n \leqslant n\big(\tau([0,1]) + \gamma_0\big), \quad n \in \mathbb{N}. \tag{4.3.2}$$

If $\tau((1,\infty)) > 0$, then using (4.3.1) and the fact that the sequence $\{\varDelta_n'\}_{n=0}^\infty$ is monotonically increasing, we obtain

$$\gamma_n \leqslant n\big(\tau([0,1]) + \gamma_0 + \varDelta_n'\big) \leqslant n\bigg(\frac{\tau([0,1]) + \gamma_0}{\tau((1,\infty))} + 1\bigg)\varDelta_n, \quad n \in \mathbb{N}. \tag{4.3.3}$$

Combining (4.3.2) and (4.3.3) completes the proof. □

The next lemma is a direct consequence of Lemma 3.4.4.

Lemma 4.3.2. *Assume that (3.4.1) holds, $\mathsf{h}_{\phi^n}(x_0) < \infty$ for every $n \in \mathbb{N}$ and $\{\mathsf{h}_{\phi^n}(x_{i,1})\}_{n=0}^\infty$ is a Stieltjes moment sequence with an S-representing measure $P(x_{i,1}, \cdot)$ for every $i \in J_\eta$. Then $\{\mathsf{h}_{\phi^{n+\kappa+1}}(x_0) - \mathsf{h}_{\phi^n}(x_0)\}_{n=0}^\infty$ is a Stieltjes moment sequence with an S-representing measure ν given by*

$$\nu(\sigma) = \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_0)} P(x_{i,1}, \sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

The above enables us to prove the aforementioned criteria for subnormality.

Theorem 4.3.3. *Assume that (3.4.1) holds and $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $x \in \{x_0\} \cup \{x_{i,1}\colon i \in J_\eta\}$. Then $\{\mathsf{h}_{\phi^{n+\kappa+1}}(x_0) - \mathsf{h}_{\phi^n}(x_0)\}_{n=0}^\infty$ is a Stieltjes moment sequence. Moreover, if one of the following four conditions is satisfied:*

(i) *$\{\mathsf{h}_{\phi^{n+\kappa+1}}(x_0) - \mathsf{h}_{\phi^n}(x_0)\}_{n=0}^\infty$ is S-determinate and there exists an S-representing measure $P(x_0, \cdot)$ of $\{\mathsf{h}_{\phi^n}(x_0)\}_{n=0}^\infty$ such that $P(x_0, [0,1)) = 0$,*
(ii) *$\{\mathsf{h}_{\phi^{n+\kappa+1}}(x_0) - \mathsf{h}_{\phi^n}(x_0)\}_{n=0}^\infty$ and $\{\mathsf{h}_{\phi^{j(\kappa+1)}}(x_0)\}_{j=0}^\infty$ are S-determinate,*
(iii) *$\{\mathsf{h}_{\phi^{j(\kappa+1)}}(x_0)\}_{j=0}^\infty$ satisfies the Carleman condition,*
(iv) *$\{\mathsf{h}_{\phi^{(j+1)(\kappa+1)}}(x_0) - \mathsf{h}_{\phi^{j(\kappa+1)}}(x_0)\}_{j=0}^\infty$ satisfies the Carleman condition,*

then C_ϕ is subnormal.

Proof. According to Lemma 4.3.2, the sequence $\{\mathsf{h}_{\phi^{n+\kappa+1}}(x_0)-\mathsf{h}_{\phi^n}(x_0)\}_{n=0}^\infty$ is a Stieltjes moment sequence. By Proposition 3.4.5, C_ϕ is densely defined. It follows from Lemmata 4.3.1 and 4.3.2 that the conditions (iii) and (iv) are equivalent. If (iv) holds, then by Proposition 2.5.1(i) and Lemma 4.3.1(i), $\{\mathsf{h}_{\phi^{n+\kappa+1}}(x_0)-\mathsf{h}_{\phi^n}(x_0)\}_{n=0}^\infty$ is S-determinate, and thus, because of (iv)$\Rightarrow$(iii), the condition (ii) holds. Applying Lemma 4.3.1(ii), we see that (ii) implies (i). All this means that it suffices to prove that (i) implies the subnormality of C_ϕ.

To this end, assume that (i) holds. Let $P(x_{i,1},\cdot)$ be an S-representing measure of $\{\mathsf{h}_{\phi^n}(x_{i,1})\}_{n=0}^\infty$ for $i \in J_\eta$. Note that

$$\mathsf{h}_{\phi^{n+\kappa+1}}(x_0)-\mathsf{h}_{\phi^n}(x_0)=\int_0^\infty t^n(t^{\kappa+1}-1)P(x_0,\mathrm{d}t), \quad n\in\mathbb{Z}_+.$$

As $P(x_0,[0,1))=0$, the set-function $\sigma\mapsto\int_\sigma(t^{\kappa+1}-1)P(x_0,\mathrm{d}t)$ is a (positive) measure. Applying Lemma 4.3.2 and using the S-determinacy assumption, we deduce that the condition (ii) of Theorem 4.1.1 holds. Now we define the measures $\{P(x_r,\cdot)\colon r\in J_\kappa\}$ and $\{P(x_{i,j},\cdot)\colon i\in J_\eta,\ j\in\mathbb{N}_2\}$ by the conditions (i) and (iii) of Theorem 4.1.1, respectively. Using (3.1.2) and the fact that $P(x,\cdot)$ is an S-representing measure of $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ for every $x\in\{x_0\}\cup\{x_{i,1}\colon i\in J_\eta\}$, we verify that $\{P(x,\cdot)\}_{x\in X}$ is a family of Borel probability measures on $\mathbb{R}_+$ that satisfies (CC). Applying Theorem 3.1.3, we conclude that C_ϕ is subnormal. $\square$

The situation changes drastically if η equals 1.

Proposition 4.3.4. *Suppose (3.4.1) holds and $\eta=1$. Then $\overline{\mathcal{D}^\infty(C_\phi)}=L^2(\mu)$, $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty\subseteq(0,\infty)$ for every $x\in X$ and the following conditions are equivalent:*

(i) *there exists a family $\{P(x,\cdot)\}_{x\in X}$ of Borel probability measures on $\mathbb{R}_+$ that satisfies (CC),*
(ii) *$\{\mathsf{h}_{\phi^n}(x_0)\}_{n=0}^\infty$ is a Stieltjes moment sequence which has an S-representing measure ρ vanishing on $[0,1)$,*
(iii) *$0\leqslant\sum_{i,j=0}^n\mathsf{h}_{\phi^{i+j}}(x_0)\lambda_i\bar{\lambda}_j\leqslant\sum_{i,j=0}^n\mathsf{h}_{\phi^{i+j+1}}(x_0)\lambda_i\bar{\lambda}_j$ for all finite sequences $\{\lambda_i\}_{i=0}^n$ of complex numbers.*

Moreover, if any of the above conditions holds, then C_ϕ is subnormal.

Proof. It follows from (3.1.2), (3.4.6) and Proposition 3.4.5 that $\overline{\mathcal{D}^\infty(C_\phi)}=L^2(\mu)$ and $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty\subseteq(0,\infty)$ for every $x\in X$.

(i)$\Rightarrow$(ii) Apply Theorem 3.1.3 and Theorem 4.1.1(iv).

(ii)$\Rightarrow$(i) Set $P(x_0,\cdot)=\rho(\cdot)$. By Lemma 3.4.4, $\{\mathsf{h}_{\phi^n}(x_{1,1})\}_{n=0}^\infty$ is a Stieltjes moment sequence with an S-representing probability measure $P(x_{1,1},\cdot)$ given by

$$P(x_{1,1}, \sigma) = \frac{\mu(x_0)}{\mu(x_{1,1})} \int_\sigma (t^{\kappa+1} - 1) P(x_0, \mathrm{d}t), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{4.3.4}$$

Clearly, the condition (ii) of Theorem 4.1.1 holds. Next, we define $\{P(x_r, \cdot)\}_{r=1}^{\kappa}$, the Borel measures on $\mathbb{R}_+$, using the condition (i) of Theorem 4.1.1. Since $P(x_0, \cdot)$ is an S-representing measure of $\{\mathsf{h}_{\phi^n}(x_0)\}_{n=0}^\infty$, we deduce from (3.1.2) that the so-defined measures are probabilistic. Finally, we define $\{P(x_{1,j}, \cdot)\}_{j=2}^\infty$, the Borel measures on $\mathbb{R}_+$, using the condition (iii) of Theorem 4.1.1. Noting that

$$\begin{aligned}
\int_0^\infty t^{j-1} P(x_{1,1}, \mathrm{d}t) &\overset{(4.3.4)}{=} \frac{\mu(x_0)}{\mu(x_{1,1})} \int_0^\infty t^{j-1}(t^{\kappa+1} - 1) P(x_0, \mathrm{d}t) \\
&= \frac{\mu(x_0)}{\mu(x_{1,1})} \left(\mathsf{h}_{\phi^{j-1+(\kappa+1)}}(x_0) - \mathsf{h}_{\phi^{j-1}}(x_0)\right) \\
&\overset{(3.4.9)}{=} \mathsf{h}_{\phi^{j-1}}(x_{1,1}) \\
&\overset{(3.1.2)}{=} \frac{\mu(x_{1,j})}{\mu(x_{1,1})}, \quad j \in \mathbb{N}_2,
\end{aligned}$$

we see that the measures $\{P(x_{1,j}, \cdot)\}_{j=2}^\infty$ are probabilistic. Now, applying Theorem 4.1.1, we conclude that $\{P(x, \cdot)\}_{x \in X}$ satisfies (CC).

(ii)⇔(iii) Apply Lemma 2.1.6.

The "moreover" part is a direct consequence of Theorem 3.1.3. □

Regarding the implication (ii)⇒(i) of Proposition 4.3.4, we note that the assumption that $\{\mathsf{h}_{\phi^n}(x_0)\}_{n=0}^\infty$ is a Stieltjes moment sequence is not sufficient for C_ϕ to be subnormal, even if $C_\phi \in \boldsymbol{B}(L^2(\mu))$.

Example 4.3.5. First, we show that if

$$\{\gamma_n\}_{n=0}^\infty \subseteq (0, \infty), \ \gamma_0 = 1 \text{ and } \gamma_{n+\kappa+1} - \gamma_n > 0 \text{ for every } n \in \mathbb{Z}_+, \tag{4.3.5}$$

then there exists a discrete measure μ on $X = X_{1,\kappa}$ such that $\mathsf{h}_{\phi^n}(x_0) = \gamma_n$ for every $n \in \mathbb{Z}_+$ with $\phi = \phi_{1,\kappa}$. For this, take any $\mu(x_0) \in (0, \infty)$ and set $\mu(x_r) = \mu(x_0)\gamma_r$ for every $r \in J_\kappa$. Next we put

$$\mu(x_{1,n+1}) = \mu(x_0)(\gamma_{n+\kappa+1} - \gamma_n), \quad n \in \mathbb{Z}_+. \tag{4.3.6}$$

Since $\gamma_{n+\kappa+1} - \gamma_n > 0$ for every $n \in \mathbb{Z}_+$, we see that $\mu(x_{1,j}) \in (0, \infty)$ for every $j \in \mathbb{N}$. Clearly, by (3.1.2), $\mathsf{h}_{\phi^r}(x_0) = \gamma_r$ for every $r \in \{0, \ldots, \kappa\}$. Using induction, Lemma 3.4.4, (3.1.2) and (4.3.6), we verify that $\mathsf{h}_{\phi^n}(x_0) = \gamma_n$ for every $n \in \mathbb{Z}_+$.

It is worth mentioning that if (3.4.1) holds and $\eta = 1$, then the sequence $\{\gamma_n\}_{n=0}^\infty$ defined by $\gamma_n = \mathsf{h}_{\phi^n}(x_0)$ for every $n \in \mathbb{Z}_+$ satisfies (4.3.5). Indeed, this is a direct consequence of Proposition 4.3.4, Lemma 3.4.4 and (3.1.2).

Now, we can apply the above procedure as follows. Take $a \in (0,1)$ and set $\gamma_n = \frac{1}{2}(a^n + 2^n)$ for every $n \in \mathbb{Z}_+$. Clearly, $\{\gamma_n\}_{n=0}^\infty$ is an S-determinate Stieltjes moment sequence with the S-representing measure $\frac{1}{2}(\delta_a + \delta_2)$. Since

$$a^{n+\kappa+1} + 2^{n+\kappa+1} > 2^{n+\kappa+1} \geqslant 1 + 2^n \geqslant a^n + 2^n, \quad n \in \mathbb{Z}_+,$$

we see that $\gamma_{n+\kappa+1} - \gamma_n > 0$ for every $n \in \mathbb{Z}_+$. Applying our procedure, we get a discrete measure μ on X such that $\mathsf{h}_{\phi^n}(x_0) = \gamma_n$ for every $n \in \mathbb{Z}_+$. Note that $C_\phi \in \boldsymbol{B}(L^2(\mu))$. Indeed, by (3.1.2) and (4.3.6), we have

$$\lim_{n\to\infty} \mathsf{h}_\phi(x_{1,n}) = \lim_{n\to\infty} \frac{\gamma_{n+\kappa+1} - \gamma_n}{\gamma_{n+\kappa} - \gamma_{n-1}} = \frac{2^{\kappa+2} - 2}{2^{\kappa+1} - 1} < \infty.$$

This, combined with (3.1.2) and (3.1.3), yields $C_\phi \in \boldsymbol{B}(L^2(\mu))$. It is worth mentioning that C_ϕ is not subnormal (hence, by Proposition 4.3.6 below, $\{\mathsf{h}_{\phi^n}(x_{1,1})\}_{n=0}^\infty$ is not a Stieltjes moment sequence). Indeed, otherwise, by [16, Theorem 13], C_ϕ admits a family $\{P(x,\cdot)\}_{x\in X}$ of Borel probability measures on $\mathbb{R}_+$ that satisfies (CC). This, together with Proposition 4.3.4, contradicts the S-determinacy of $\{\gamma_n\}_{n=0}^\infty$.

The question of characterizing subnormality of bounded composition operators of the form $C_{\phi_{\eta,\kappa}}$ has a simple solution.

Proposition 4.3.6. *Suppose (3.4.1) holds and $C_\phi \in \boldsymbol{B}(L^2(\mu))$. Then C_ϕ is subnormal if and only if $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $x \in \{x_0\} \cup \{x_{i,1} \colon i \in J_\eta\}$.*

Proof. Since, by (3.1.3), $\mathsf{h}_{\phi^n}(x_0) \leqslant \|C_\phi\|^{2n}$ for all $n \in \mathbb{Z}_+$, the sufficiency follows from Theorem 4.3.3(iii). The sufficiency can also be deduced from Lambert's theorem (see [49]) by using (3.1.2) and (3.4.8). The necessity is a direct consequence of the corresponding part of Lambert's theorem. $\square$

Now we discuss the question of "optimality" of the assumptions of Proposition 4.3.6. As shown in Example 4.3.5, for $\eta = 1$ and for every $\kappa \in \mathbb{Z}_+$, there exists a non-subnormal $C_\phi \in \boldsymbol{B}(L^2(\mu))$ such that $\{\mathsf{h}_{\phi^n}(x_0)\}_{n=0}^\infty$ is a Stieltjes moment sequence. In Example 4.3.7 below, we will show that for $\eta = 1$ and $\kappa = 0$ the assumption that $\{\mathsf{h}_{\phi^n}(x_{1,1})\}_{n=0}^\infty$ is a Stieltjes moment sequence is not sufficient for C_ϕ to be subnormal, even if $C_\phi \in \boldsymbol{B}(L^2(\mu))$.

Example 4.3.7. Assume that $\eta = 1$ and $\kappa = 0$. Take any positive real numbers $\mu(x_0)$ and $\mu(x_{1,1})$. Let $\{\gamma_n\}_{n=0}^\infty$ be a Stieltjes moment sequence with an S-representing measure ν such that $\gamma_0 = 1$, $\nu((1,\infty)) = 0$ and $\nu(\{1\}) > 0$. Then

$$0 < \gamma_n \leqslant 1, \quad n \in \mathbb{Z}_+, \tag{4.3.7}$$

which implies that $\{\gamma_n\}_{n=0}^\infty$ is S-determinate (see e.g., Proposition 2.5.1(i)). Set $\mu(x_{1,j}) = \mu(x_{1,1})\gamma_{j-1}$ for every $j \in \mathbb{N}_2$. Clearly, $\mathsf{h}_{\phi^n}(x_{1,1}) = \gamma_n$ for all $n \in \mathbb{Z}_+$, which means that $\{\mathsf{h}_{\phi^n}(x_{1,1})\}_{n=0}^\infty$ is a Stieltjes moment sequence. Since $\{\gamma_n\}_{n=0}^\infty$ is monotonically decreasing, we see that $\mathsf{h}_\phi(x_{1,j}) = \frac{\gamma_j}{\gamma_{j-1}} \leqslant 1$ for every $j \in \mathbb{N}$. This, together with (3.1.3), implies that $C_\phi \in \boldsymbol{B}(L^2(\mu))$. Note that

$$\begin{aligned} \mathsf{h}_{\phi^n}(x_0) &= \mathsf{h}_{\phi^0}(x_0) + \sum_{l=0}^{n-1}(\mathsf{h}_{\phi^{l+1}}(x_0) - \mathsf{h}_{\phi^l}(x_0)) \\ &\overset{(3.4.9)}{=} 1 + \frac{\mu(x_{1,1})}{\mu(x_0)}\sum_{l=0}^{n-1}\gamma_l, \quad n \in \mathbb{N}. \end{aligned} \tag{4.3.8}$$

This and (4.3.7) imply that

$$\lim_{n\to\infty} \mathsf{h}_{\phi^n}(x_0)^{1/n} = 1. \tag{4.3.9}$$

Now we show that $\{\mathsf{h}_{\phi^n}(x_0)\}_{n=0}^\infty$ is not a Stieltjes moment sequence. Indeed, otherwise by (4.3.9) and [55, Exercise 4(e) on p. 71], we see that $\rho((1,\infty)) = 0$, where ρ is an S-representing measure of $\{\mathsf{h}_{\phi^n}(x_0)\}_{n=0}^\infty$. Hence

$$1 = \rho(\mathbb{R}_+) \geqslant \mathsf{h}_{\phi^n}(x_0) \overset{(4.3.8)}{\geqslant} 1 + \frac{\mu(x_{1,1})\nu(\{1\})}{\mu(x_0)} n, \quad n \in \mathbb{N},$$

which contradicts $\nu(\{1\}) > 0$. By Proposition 4.3.6, C_ϕ is not subnormal.

4.4. *Extending to families satisfying* (CC)

We begin this section by providing necessary and sufficient conditions for the extendibility of a given family of Borel probability measures on $\mathbb{R}_+$ indexed by $\{x_{i,1}\}_{i\in J_\eta}$ to a family of Borel probability measures on $\mathbb{R}_+$ satisfying (CC).

Theorem 4.4.1. *Suppose* (3.4.1) *holds and C_ϕ is densely defined. Then the following assertions hold.*

(i) *If $\{P(x,\cdot)\}_{x\in X}$ is a family of Borel probability measures on $\mathbb{R}_+$ that satisfies* (CC), *then*
  - (i-a) *$\{\mathsf{h}_{\phi^n}(x_{i,1})\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $i \in J_\eta$,*
  - (i-b) *for every $i \in J_\eta$, the measure $P(x_{i,1},\cdot)$ is an S-representing measure of $\{\mathsf{h}_{\phi^n}(x_{i,1})\}_{n=0}^\infty$ vanishing on $[0,1]$,*
  - (i-c) $\sum_{i=1}^\eta \frac{\mu(x_{i,1})}{\mu(x_r)}\int_0^\infty \frac{t^r-1}{t^{\kappa+1}-1}P(x_{i,1},\mathrm{d}t) + \frac{\mu(x_0)}{\mu(x_r)} = 1$ *for every $r \in J_\kappa$,*
  - (i-d) $\sum_{i=1}^\eta \frac{\mu(x_{i,1})}{\mu(x_0)}\int_0^\infty \frac{1}{t^{\kappa+1}-1}P(x_{i,1},\mathrm{d}t) \leqslant 1.$

(ii) *If* (i-a) *holds and* $\{P(x_{i,1},\cdot)\}_{i\in J_\eta}$ *is a family of Borel probability measures on* $\mathbb{R}_+$ *satisfying* (i-b), (i-c) *and* (i-d), *then there exists a family* $\{P(x,\cdot)\}_{x\in X\setminus\{x_{i,1}\colon i\in J_\eta\}}$ *of Borel probability measures on* $\mathbb{R}_+$ *such that* $\{P(x,\cdot)\}_{x\in X}$ *satisfies* (CC).

Proof. (i) The conditions (i-a) and (i-b) follow from (3.1.2), Theorem 3.1.3 and Theorem 4.1.1(iv). The condition (i-d) is a direct consequence of Theorem 4.1.1(vi). Using the conditions (i), (vii) and (viii) of Theorem 4.1.1 and integrating the function $t\mapsto t^r$ with respect to the measure $P(x_0,\cdot)$, we obtain

$$\begin{aligned}\frac{\mu(x_r)}{\mu(x_0)} &= \int_0^\infty t^r P(x_0,\mathrm{d}t)\\ &= \sum_{i=1}^{\eta}\frac{\mu(x_{i,1})}{\mu(x_0)}\int_0^\infty \frac{t^r}{t^{\kappa+1}-1}P(x_{i,1},\mathrm{d}t)\\ &\qquad + 1-\sum_{i=1}^{\eta}\frac{\mu(x_{i,1})}{\mu(x_0)}\int_0^\infty \frac{1}{t^{\kappa+1}-1}P(x_{i,1},\mathrm{d}t)\\ &= 1+\sum_{i=1}^{\eta}\frac{\mu(x_{i,1})}{\mu(x_0)}\int_0^\infty \frac{t^r-1}{t^{\kappa+1}-1}P(x_{i,1},\mathrm{d}t),\quad r\in J_\kappa,\end{aligned}$$

which implies (i-c).

(ii) For $i\in J_\eta$ and $j\in\mathbb{N}_2$, we define the measure $P(x_{i,j},\cdot)$ by the condition (iii) of Theorem 4.1.1. Note that $P(x_{i,j},\mathbb{R}_+)=1$ because

$$P(x_{i,j},\mathbb{R}_+)=\frac{\mu(x_{i,1})}{\mu(x_{i,j})}\int_0^\infty t^{j-1}P(x_{i,1},\mathrm{d}t)\overset{\text{(i-b)}}{=}\frac{\mu(x_{i,1})}{\mu(x_{i,j})}\mathsf{h}_{\phi^{j-1}}(x_{i,1})\overset{(3.1.2)}{=}1.$$

Next, we define the measures $P(x_0,\cdot),\ldots,P(x_\kappa,\cdot)$ by

$$P(x_r,\sigma)=\sum_{i=1}^{\eta}\frac{\mu(x_{i,1})}{\mu(x_r)}\int_\sigma \frac{t^r}{t^{\kappa+1}-1}P(x_{i,1},\mathrm{d}t)+\vartheta\frac{\mu(x_0)}{\mu(x_r)}\delta_1(\sigma)\tag{4.4.1}$$

for $\sigma\in\mathfrak{B}(\mathbb{R}_+)$ and $r\in\{0,\ldots,\kappa\}$ with $\vartheta=1-\sum_{i=1}^{\eta}\frac{\mu(x_{i,1})}{\mu(x_0)}\int_0^\infty\frac{1}{t^{\kappa+1}-1}P(x_{i,1},\mathrm{d}t)$. By (i-b), (i-c) and (i-d), the quantity ϑ is well-defined, $\vartheta\in[0,1]$ and $P(x_r,\cdot)$ is a well-defined finite measure. That it is probabilistic follows from the equalities

$$\begin{aligned}P(x_r,\mathbb{R}_+)&\overset{(4.4.1)}{=}\sum_{i=1}^{\eta}\frac{\mu(x_{i,1})}{\mu(x_r)}\int_0^\infty\frac{t^r-1}{t^{\kappa+1}-1}P(x_{i,1},\mathrm{d}t)+\frac{\mu(x_0)}{\mu(x_r)}\\ &\overset{\text{(i-c)}}{=}1,\quad r\in\{0,\ldots,\kappa\}.\end{aligned}$$

Now, integrating the function $t \mapsto t^r$ with respect to $P(x_0, \cdot)$, we see that the condition (i) of Theorem 4.1.1 is satisfied. Finally, integrating the function $t \mapsto t^{\kappa+1} - 1$ (which is positive on $(1, \infty)$) with respect to $P(x_0, \cdot)$, we deduce that the condition (ii) of Theorem 4.1.1 is satisfied. This combined with Theorem 4.1.1 completes the proof. □

Below we introduce the condition (i-d′), a weaker version of the condition (i-d) of Theorem 4.4.1 that will lead us to constructing exotic examples (see Section 5). For more information concerning the conditions (e1) and (e2) below, the reader is referred to Lemma 2.1.4 and Remark 2.1.5.

Theorem 4.4.2. *Suppose* (3.4.1) *holds, the condition* (i-a) *of* Theorem 4.4.1 *is satisfied and* $\{P(x_{i,1}, \cdot)\}_{i \in J_\eta}$ *is a family of Borel probability measures on* $\mathbb{R}_+$ *satisfying the conditions* (i-b) *and* (i-c) *of* Theorem 4.4.1 *and the condition below*:

(i-d′) $\sum_{i=1}^{\eta} \mu(x_{i,1}) \int_0^\infty \frac{1}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) < \infty.$

Consider the following three conditions:

(e1) $\{\mathsf{h}_{\phi^n}(x_\kappa) + c\}_{n=0}^\infty$ *is an S-determinate Stieltjes moment sequence for every* $c \in (0, \infty)$,

(e2) $\{\mathsf{h}_{\phi^{n+1}}(x_\kappa) + c\}_{n=0}^\infty$ *is an S-determinate Stieltjes moment sequence for every* $c \in (0, \infty)$,

(e3) $\{\mathsf{h}_{\phi^n}(x_\kappa)\}_{n=0}^\infty$ *is a Stieltjes moment sequence that satisfies the Carleman condition.*

Then (e3)⇒(e2)⇒(e1). *In turn, if* (e1) *holds, then the composition operator* C_ϕ *is densely defined and there exists a family* $\{P(x, \cdot)\}_{x \in X \setminus \{x_{i,1} \colon i \in J_\eta\}}$ *of Borel probability measures on* $\mathbb{R}_+$ *such that* $\{P(x, \cdot)\}_{x \in X}$ *satisfies* (CC). *Moreover, if* (e3) *holds, then* $\{\mathsf{h}_{\phi^n}(x_r)\}_{n=0}^\infty$ *satisfies the Carleman condition for every* $r \in \{0, \ldots, \kappa\}$.

Proof. It follows from (i-b) and (i-d′) that the quantity ξ defined below

$$\xi := \frac{\mu(x_0)}{\mu(x_\kappa)} - \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \int_0^\infty \frac{1}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) \tag{4.4.2}$$

is a real number. By (i-b), (i-c) and (i-d′), we have

$$\xi = \frac{\mu(x_r)}{\mu(x_\kappa)} - \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \int_0^\infty \frac{t^r}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t), \quad r \in \{0, \ldots, \kappa\}. \tag{4.4.3}$$

In particular, the above series are convergent in $\mathbb{R}_+$.

First, we show that

$$\mathsf{h}_{\phi^{n+1}}(x_\kappa) = \xi + \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \int_0^\infty t^n \frac{t^{\kappa+1}}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t), \quad n \in \mathbb{Z}_+. \tag{4.4.4}$$

To this end, we fix $n \in \mathbb{Z}_+$. Then $n = j(\kappa+1) + r$ for some $j \in \mathbb{Z}_+$ and $r \in \{0, \ldots, \kappa\}$. We begin by considering the case of $r < \kappa$. Since then $r + 1$ is the remainder of the division of $n + 1$ by $\kappa + 1$, we infer from (i-b) that

$$\begin{aligned}
\mathsf{h}_{\phi^{n+1}}(x_\kappa) &\overset{(3.4.6)}{=} \frac{\mu(x_r)}{\mu(x_\kappa)} + \sum_{i=1}^{\eta} \sum_{l=0}^{j} \frac{\mu(x_{i,l(\kappa+1)+r+1})}{\mu(x_\kappa)} \\
&\overset{(3.1.2)}{=} \frac{\mu(x_r)}{\mu(x_\kappa)} + \sum_{i=1}^{\eta} \sum_{l=0}^{j} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \mathsf{h}_{\phi^{l(\kappa+1)+r}}(x_{i,1}) \\
&= \frac{\mu(x_r)}{\mu(x_\kappa)} + \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \int_{(1,\infty)} \sum_{l=0}^{j} (t^{\kappa+1})^l t^r P(x_{i,1}, \mathrm{d}t) \\
&= \frac{\mu(x_r)}{\mu(x_\kappa)} + \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \int_{(1,\infty)} \frac{t^{(j+1)(\kappa+1)}-1}{t^{\kappa+1}-1} t^r P(x_{i,1}, \mathrm{d}t) \\
&\overset{(4.4.3)}{=} \xi + \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \int_0^\infty t^n \frac{t^{\kappa+1}}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t).
\end{aligned}$$

If $r = \kappa$, then mimicking the above proof we get

$$\begin{aligned}
\mathsf{h}_{\phi^{n+1}}(x_\kappa) &\overset{(3.4.6)}{=} 1 + \sum_{i=1}^{\eta} \sum_{l=1}^{j+1} \frac{\mu(x_{i,l(\kappa+1)})}{\mu(x_\kappa)} \\
&= 1 + \sum_{i=1}^{\eta} \sum_{l=1}^{j+1} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \mathsf{h}_{\phi^{l(\kappa+1)-1}}(x_{i,1}) \\
&\overset{(4.4.3)}{=} \xi + \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \int_0^\infty t^n \frac{t^{\kappa+1}}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t),
\end{aligned}$$

which proves (4.4.4).

It follows from (4.4.4) that

$$\mathsf{h}_{\phi^{n}}(x_\kappa) = \xi + \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \int_0^\infty t^n \frac{t^{\kappa}}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t), \quad n \in \mathbb{Z}_+. \tag{4.4.5}$$

(The case of $n = 0$ can be deduced from (4.4.3) with $r = \kappa$.)

(e3)⇒(e2) Apply the assertions (i)–(iii) of Proposition 2.5.1.

(e2)⇒(e1) Since the class of Stieltjes moment sequences is closed under the operation of taking pointwise limits (which follows from the Stieltjes theorem, see [4, Theorem 6.2.5]) and $\mathsf{h}_{\phi^{n+1}}(x_\kappa) = \lim_{c\to 0+}(\mathsf{h}_{\phi^{n+1}}(x_\kappa) + c)$ for all $n \in \mathbb{Z}_+$, we see that $\{\mathsf{h}_{\phi^{n+1}}(x_\kappa)\}_{n=0}^\infty$ is a Stieltjes moment sequence. Let ρ be an S-representing measure of $\{\mathsf{h}_{\phi^{n+1}}(x_\kappa)\}_{n=0}^\infty$. Then, by (4.4.4), we have

$$\int_0^\infty t^n \mathrm{d}\rho(t) = \mathsf{h}_{\phi^{n+1}}(x_\kappa) = \xi + \int_0^\infty t^n \mathrm{d}\nu(t), \quad n \in \mathbb{Z}_+, \tag{4.4.6}$$

where ν is the Borel measure on $\mathbb{R}_+$ given by

$$\nu(\sigma) = \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \int_\sigma \frac{t^{\kappa+1}}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{4.4.7}$$

Now we prove that the condition (i-d) of Theorem 4.4.1 is satisfied. Indeed, otherwise $\xi \in (-\infty, 0)$. By (e2), the Stieltjes moment sequence $\{\mathsf{h}_{\phi^{n+1}}(x_\kappa) + |\xi|\}_{n=0}^\infty$ is S-determinate. This together with (4.4.6) implies that

$$\rho(\sigma) + |\xi|\delta_1(\sigma) = \nu(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{4.4.8}$$

Using (4.4.7) and (i-b), and substituting $\sigma = \{1\}$ into (4.4.8), we deduce that $\xi = 0$, a contradiction. This shows that the condition (i-d) of Theorem 4.4.1 is satisfied.

Since $\mathsf{h}_{\phi^n}(x_\kappa) < \infty$ for all $n \in \mathbb{Z}_+$, we infer from Proposition 3.4.5 that $\mathcal{D}^\infty(C_\phi)$ is dense in $L^2(\mu)$. Hence, by Theorem 4.4.1(ii), there exists $\{P(x,\cdot)\}_{x\in X\setminus\{x_{i,1}\colon i\in J_\eta\}}$, a family of Borel probability measures on $\mathbb{R}_+$, such that $\{P(x,\cdot)\}_{x\in X}$ satisfies (CC). By Theorem 3.1.3, $\mathsf{h}_{\phi^n}(x_\kappa) = \int_0^\infty t^n P(x_\kappa, \mathrm{d}t)$ for all $n \in \mathbb{Z}_+$. This means that for every $c \in (0,\infty)$, $\{\mathsf{h}_{\phi^n}(x_\kappa)+c\}_{n=0}^\infty$ is a Stieltjes moment sequence and, by (e2), $\{\mathsf{h}_{\phi^{n+1}}(x_\kappa)+c\}_{n=0}^\infty$ is an S-determinate Stieltjes moment sequence, which implies that $\{\mathsf{h}_{\phi^n}(x_\kappa) + c\}_{n=0}^\infty$ is S-determinate for every $c \in (0,\infty)$ (see [57, Proposition 5.12]; see also [12, Lemma 2.4.1]).

Assume now that (e1) holds. Passing to the limit, as in the proof of (e2)⇒(e1), we see that $\{\mathsf{h}_{\phi^n}(x_\kappa)\}_{n=0}^\infty$ is a Stieltjes moment sequence. Let ρ' be an S-representing measure of $\{\mathsf{h}_{\phi^n}(x_\kappa)\}_{n=0}^\infty$. Then, by (4.4.5), we have

$$\int_0^\infty t^n \mathrm{d}\rho'(t) = \mathsf{h}_{\phi^n}(x_\kappa) = \xi + \int_0^\infty t^n \mathrm{d}\nu'(t), \quad n \in \mathbb{Z}_+,$$

where ν' is the Borel measure on $\mathbb{R}_+$ given by

$$\nu'(\sigma) = \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \int_\sigma \frac{t^{\kappa}}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

Arguing as in the paragraph containing (4.4.7) (using (e1) in place of (e2)), we deduce that the condition (i-d) of Theorem 4.4.1 is satisfied. According to Proposition 3.4.5, C_ϕ is densely defined. Applying Theorem 4.4.1(ii), we get the required family of Borel probability measures on $\mathbb{R}_+$ that satisfies (CC).

Now we prove the "moreover" part. Assume (e3) holds. First note that, by what has been proved above, the assumptions of Theorem 3.1.3 are satisfied. Hence, by Proposition 3.4.5, $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $x \in X$. Since $\{\mathsf{h}_{\phi^n}(x_\kappa)\}_{n=0}^\infty$ satisfies the Carleman condition, we deduce from (3.4.7) and Proposition 2.5.1(ii) that $\{\mathsf{h}_{\phi^n}(x_{\kappa-1})\}_{n=0}^\infty$ satisfies the Carleman condition as well. An induction argument completes the proof. $\square$

Remark 4.4.3. It follows from the proof of (4.4.4) that, under the assumptions of Theorem 4.4.2, $\int_0^\infty \frac{t^n}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) < \infty$ for all $n \in \mathbb{Z}_+$ and $i \in J_\eta$, and in the case of $\eta = \infty$,

$$\sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \int_0^\infty \frac{t^n}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) < \infty \tag{4.4.9}$$

for any integer n such that $0 \leqslant n \leqslant \kappa$. However, the series in (4.4.9) may be divergent to infinity for some integers $n \geqslant \kappa + 1$.

The following is related to the "moreover" part of the conclusion of Theorem 4.4.2.

Proposition 4.4.4. *Suppose (3.4.1) holds, $\{\mathsf{h}_{\phi^n}(x_r)\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $r \in \{0, \ldots, \kappa\}$, and $\{\mathsf{h}_{\phi^n}(x_\kappa)\}_{n=0}^\infty$ is S-determinate. Then for every $r \in \{0, \ldots, \kappa\}$, $\{\mathsf{h}_{\phi^n}(x_r)\}_{n=0}^\infty$ is S-determinate.*

Proof. It follows from (3.4.7) applied to $r = \kappa$ that the Stieltjes moment sequence $\{\mathsf{h}_{\phi^{n+1}}(x_{\kappa-1})\}_{n=0}^\infty$ is S-determinate. This combined with [57, Proposition 5.12] (see also [42, Lemma 2.1.1]) implies that $\{\mathsf{h}_{\phi^n}(x_{\kappa-1})\}_{n=0}^\infty$ is S-determinate as well. Thus, an induction argument completes the proof. $\square$

5. Examples of exotic non-hyponormal operators

5.1. Outline

In the last part of the paper we construct non-hyponormal injective composition operators generating Stieltjes moment sequences. The construction relies on the key observation that there is a gap between the conditions (i-d) and (i-d′) of Theorems 4.4.1 and 4.4.2. The indices of H-determinacy of the sequences $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$, $x \in X$, are discussed as well.

We begin by introducing for any $\eta \in \mathbb{N} \cup \{\infty\}$ and $\kappa \in \mathbb{Z}_+$ a class of composition operators over the directed graph $\mathscr{G}_{\eta,\kappa} := (X_{\eta,\kappa}, E^{\phi_{\eta,\kappa}})$ which admit families $\{P(x_{i,1}, \cdot)\}_{i \in J_\eta}$

of Borel probability measures on $\mathbb{R}_+$ satisfying the conditions (i-a), (i-b), (i-c) and (i-d′) of Theorems 4.4.1 and 4.4.2, but not (i-d) of Theorem 4.4.1 (see Procedure 5.2.1 and Lemma 5.3.1). The construction of these operators essentially depends on a choice of specific N-extremal measures ν and τ, and a partition $\{\varDelta_i\}_{i=1}^{\eta}$ of $\operatorname{supp}(\tau)$. The fundamental properties of the so-constructed operators, including the characterization of their hyponormality for $\kappa = 0$, are proven in Lemma 5.3.1. Section 5.4 shows that the gap between the conditions (i-d) and (i-d′) does exist. Theorem 5.5.2 is the culminating result of the present paper. It shows that there exists a non-hyponormal composition operator generating Stieltjes moment sequences over the locally finite directed graph $\mathscr{G}_{2,0}$. Its proof heavily depends on the existence of N-extremal probability measures ζ and ρ satisfying a restrictive condition which is not easy to deal with. Fortunately, there are N-extremal probability measures coming from shifted Al-Salam–Carlitz q-polynomials or from a quartic birth and death process that satisfy this condition. It is worth pointing out that, without the use of these special N-extremal measures, Step 1 of the proof of Theorem 5.5.2 combined with (5.4.1) implies that for every sufficiently large integer η, there exists a non-hyponormal composition operator generating Stieltjes moment sequences over the locally finite directed graph $\mathscr{G}_{\eta,0}$.

5.2. General scheme

In this section we introduce the aforementioned class of composition operators. We do it according to the following procedure.

Procedure 5.2.1. Fix $\eta \in \mathbb{N} \cup \{\infty\}$ and $\kappa \in \mathbb{Z}_+$. Suppose that

$$\nu \text{ and } \tau \text{ are N-extremal measures of the same Stieltjes moment sequence,} \tag{5.2.1}$$

$$1 = \inf \operatorname{supp}(\nu) < \inf \operatorname{supp}(\tau), \tag{5.2.2}$$

$$\nu(\mathbb{R}_+) = 1 + \nu(\{1\}), \tag{5.2.3}$$

$$1 + \int_0^\infty \frac{1}{t^\kappa} \mathrm{d}\tau(t) > \tau(\mathbb{R}_+), \tag{5.2.4}$$

and

$$\{\varDelta_i\}_{i=1}^{\eta} \text{ is a partition of } \operatorname{supp}(\tau). \tag{5.2.5}$$

(A partition of a nonempty set is always assumed to consist of nonempty sets.) Since, by Lemma 2.1.1, $\operatorname{card}(\operatorname{supp}(\tau)) = \aleph_0$, such partition always exists.

It follows from Lemma 2.1.1, (5.2.1) and (5.2.2) that $\int_{\varDelta_i} \frac{t^{\kappa+1}-1}{t^\kappa} \mathrm{d}\tau(t) \in (0,\infty)$ for every $i \in J_\eta$ (see Section 1.2 for the definition of J_η). Hence,

$$c_i := \Big(\int\limits_{\varDelta_i} \frac{t^{\kappa+1}-1}{t^\kappa} \mathrm{d}\tau(t)\Big)^{-1} \in (0,\infty), \quad i \in J_\eta. \tag{5.2.6}$$

Set

$$P(x_{i,1},\sigma) = c_i \int\limits_{\varDelta_i\cap\sigma} \frac{t^{\kappa+1}-1}{t^\kappa} \mathrm{d}\tau(t), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\, i \in J_\eta. \tag{5.2.7}$$

Clearly, $P(x_{i,1},\cdot)$ is a Borel probability measure on $\mathbb{R}_+$ such that $\int_0^\infty t^n P(x_{i,1},\mathrm{d}t) \in (0,\infty)$ for all $n \in \mathbb{Z}_+$ and $i \in J_\eta$ (use (5.2.1) and (5.2.2)). Take any $\mu(x_\kappa) \in (0,\infty)$ and define the family $\{\mu(x_{i,1})\}_{i\in J_\eta}$ of positive real numbers by

$$\mu(x_{i,1}) = \frac{1}{c_i}\,\mu(x_\kappa), \quad i \in J_\eta. \tag{5.2.8}$$

Next, we define the family $\{\mu(x_{i,j})\}_{(i,j)\in J_\eta\times\mathbb{N}_2}$ of positive real numbers and the family $\{P(x_{i,j},\cdot)\}_{(i,j)\in J_\eta\times\mathbb{N}_2}$ of Borel probability measures on $\mathbb{R}_+$ by

$$\mu(x_{i,j}) = \mu(x_{i,1}) \int\limits_0^\infty t^{j-1} P(x_{i,1},\mathrm{d}t), \quad i \in J_\eta,\, j \in \mathbb{N}_2, \tag{5.2.9}$$

$$P(x_{i,j},\sigma) = \frac{\mu(x_{i,1})}{\mu(x_{i,j})} \int\limits_\sigma t^{j-1} P(x_{i,1},\mathrm{d}t), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\, i \in J_\eta,\, j \in \mathbb{N}_2. \tag{5.2.10}$$

Let $P(x_\kappa,\cdot)$ be the Borel measure on $\mathbb{R}_+$ given by

$$P(x_\kappa,\sigma) = \nu(\sigma) - \nu(\{1\})\delta_1(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+). \tag{5.2.11}$$

By (5.2.1) and (5.2.3), $P(x_\kappa,\cdot)$ is a probability measure. It is also clear that $0 < \int_0^\infty t^n \mathrm{d}P(x_\kappa,\mathrm{d}t) < \infty$ for every $n \in \mathbb{Z}_+$. In view of (5.2.1) and (5.2.2), we have

$$0 < \int\limits_0^\infty \frac{1}{t^n} \mathrm{d}\tau(t) < \infty, \quad n \in \mathbb{Z}_+, \tag{5.2.12}$$

and

$$\begin{aligned} \int\limits_0^\infty \frac{1}{t^{\kappa-r}} \mathrm{d}\tau(t) - \nu(\{1\}) \;&\geqslant\; \int\limits_0^\infty \frac{1}{t^\kappa} \mathrm{d}\tau(t) - \nu(\{1\}) \\ &\overset{(5.2.4)}{>} \tau(\mathbb{R}_+) - 1 - \nu(\{1\}) \\ &\overset{(5.2.1)}{=} \nu(\mathbb{R}_+) - 1 - \nu(\{1\}) \\ &\overset{(5.2.3)}{=} 0, \quad r \in \{0,\ldots,\kappa\}. \end{aligned} \tag{5.2.13}$$

If $\kappa \geqslant 1$, then we set

$$\mu(x_r) = \mu(x_\kappa)\bigg(\int_0^\infty \frac{1}{t^{\kappa-r}}\mathrm{d}\tau(t) - \nu(\{1\})\bigg), \quad r \in \{0, \ldots, \kappa-1\}. \tag{5.2.14}$$

It follows from (5.2.12) and (5.2.13) that $\mu(x_r) \in (0,\infty)$ for every $r \in \{0, \ldots, \kappa-1\}$.

Finally, let μ be the (unique) discrete measure on $X = X_{\eta,\kappa}$ such that $\mu(\{x\}) = \mu(x)$ for every $x \in X$ (we follow the convention (3.1.1)), and let C_ϕ be the corresponding composition operator in $L^2(\mu)$ with the symbol $\phi = \phi_{\eta,\kappa}$. Since $\phi(X) = X$, we infer from (3.1.2) that $\mathsf{h}_\phi(x) > 0$ for every $x \in X$.

5.3. Three key lemmata

We begin by listing the most fundamental properties of the composition operator C_ϕ constructed in Procedure 5.2.1.

Lemma 5.3.1. *Let κ, η, ν, τ, $\{\varDelta_i\}_{i=1}^\eta$, X, μ, $P(x_\kappa,\cdot)$, $\{P(x_{i,j},\cdot)\}_{i\in J_\eta, j\in\mathbb{N}}$ and C_ϕ be as in Procedure 5.2.1. Then $\overline{\mathcal{D}^\infty(C_\phi)} = L^2(\mu)$ and the following holds:*

- (i) *$\{\mathsf{h}_{\phi^n}(x_\kappa)\}_{n=0}^\infty$ is an H-determinate Stieltjes moment sequence with the S-representing measure $P(x_\kappa,\cdot)$ whose index of H-determinacy at 0 is 0,*
- (ii) *for all $i \in J_\eta$ and $j \in \mathbb{N}$, $\{\mathsf{h}_{\phi^n}(x_{i,j})\}_{n=0}^\infty$ is a Stieltjes moment sequence with the S-representing measure $P(x_{i,j},\cdot)$,*
- (iii) *the condition* (i-a) *of Theorem 4.4.1 holds and the family $\{P(x_{i,1},\cdot)\}_{i\in J_\eta}$ satisfies the conditions* (i-b), (i-c) *and* (i-d′) *of Theorems 4.4.1 and 4.4.2,*
- (iv) *$\{P(x_{i,1},\cdot)\}_{i\in J_\eta}$ does not satisfy the condition* (i-d) *of Theorem 4.4.1,*
- (v) *if $\kappa = 0$, then C_ϕ is hyponormal if and only if*

$$\sum_{i=1}^\eta \frac{\left(\int_{\varDelta_i}(t-1)\mathrm{d}\tau(t)\right)^2}{\int_{\varDelta_i} t(t-1)\mathrm{d}\tau(t)} \leqslant \frac{\int_0^\infty (t-1)\mathrm{d}\tau(t)}{1+\int_0^\infty (t-1)\mathrm{d}\tau(t)}, \tag{5.3.1}$$

- (vi) *if $\kappa = 0$, then C_ϕ generates Stieltjes moment sequences,*
- (vii) *if $\kappa = 0$ and $\eta = 1$, then there exists a family $\{P'(x,\cdot)\}_{x\in X}$ of Borel probability measures on $\mathbb{R}_+$ that satisfies (CC) and thus C_ϕ is subnormal.*

Proof. (ii) It follows from (3.1.2) and (5.2.9) that

$$\mathsf{h}_{\phi^n}(x_{i,1}) = \int_0^\infty t^n P(x_{i,1}, \mathrm{d}t), \quad n \in \mathbb{Z}_+,\ i \in J_\eta. \tag{5.3.2}$$

Moreover, we have

$$
\begin{aligned}
\int_0^\infty t^n P(x_{i,j}, \mathrm{d}t) &\overset{(5.2.10)}{=} \frac{\mu(x_{i,1})}{\mu(x_{i,j})} \int_0^\infty t^{j+n-1} P(x_{i,1}, \mathrm{d}t) \\
&\overset{(5.2.9)}{=} \frac{\mu(x_{i,1})}{\mu(x_{i,j})} \frac{\mu(x_{i,j+n})}{\mu(x_{i,1})} \\
&\overset{(3.1.2)}{=} \mathsf{h}_{\phi^n}(x_{i,j}), \quad n \in \mathbb{Z}_+,\ i \in J_\eta,\ j \in \mathbb{N}_2.
\end{aligned}
$$

Altogether this implies that (ii) holds. In particular, the condition (i-a) of Theorem 4.4.1 holds. By (5.2.2), (5.2.5), (5.2.7) and (5.3.2), the family $\{P(x_{i,1}, \cdot)\}_{i \in J_\eta}$ satisfies the condition (i-b) of Theorem 4.4.1.

(iv) The condition (i-b) of Theorem 4.4.1, (5.2.5), (5.2.7) and (5.2.8) yield

$$
\begin{aligned}
0 < \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \int_0^\infty \frac{1}{t^{\kappa+1} - 1} P(x_{i,1}, \mathrm{d}t) &= \sum_{i=1}^{\eta} \int_{\Delta_i} \frac{1}{t^\kappa} \mathrm{d}\tau(t) \\
&= \int_0^\infty \frac{1}{t^\kappa} \mathrm{d}\tau(t) \overset{(5.2.12)}{<} \infty. \qquad (5.3.3)
\end{aligned}
$$

Hence, the family $\{P(x_{i,1}, \cdot)\}_{i \in J_\eta}$ satisfies the condition (i-d′) of Theorem 4.4.2 and the quantity ξ defined by (4.4.2) is a real number. Since the closed support of an N-extremal measure has no accumulation point in $\mathbb{R}$ (see Lemma 2.1.1), we infer from (5.2.1) and (5.2.2) that

$$
\nu(\{1\}) > 0. \qquad (5.3.4)
$$

Thus, noting that

$$
\begin{aligned}
\xi &\overset{(4.4.2)}{=} \frac{\mu(x_0)}{\mu(x_\kappa)} - \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \int_0^\infty \frac{1}{t^{\kappa+1} - 1} P(x_{i,1}, \mathrm{d}t) \\
&\overset{(5.3.3)}{=} \frac{\mu(x_0)}{\mu(x_\kappa)} - \int_0^\infty \frac{1}{t^\kappa} \mathrm{d}\tau(t) \\
&\overset{(*)}{=} \begin{cases} -\nu(\{1\}) \text{ if } \kappa \geqslant 1, \\ 1 - \nu(\mathbb{R}_+) \overset{(5.2.3)}{=} -\nu(\{1\}) \text{ if } \kappa = 0, \end{cases}
\end{aligned}
$$

where $(*)$ refers to (5.2.14) if $\kappa \geqslant 1$ and to (5.2.1) if $\kappa = 0$, we obtain

$$
\xi = -\nu(\{1\}) < 0. \qquad (5.3.5)
$$

It follows from (5.3.5) that the assertion (iv) holds.

(iii) In view of what has been done, it remains to show that the condition (i-c) of Theorem 4.4.1 holds for $\kappa \in \mathbb{N}$. Suppose $\kappa \in \mathbb{N}$ and $r \in J_\kappa$. Using (5.2.5), (5.2.7) and (5.2.8), we get

$$\begin{aligned}\frac{\mu(x_r)}{\mu(x_\kappa)} - \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \int_0^\infty \frac{t^r}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) & \\ &= \frac{\mu(x_r)}{\mu(x_\kappa)} - \int_0^\infty \frac{1}{t^{\kappa-r}} \mathrm{d}\tau(t) \\ &\overset{(\dagger)}{=} \begin{cases} -\nu(\{1\}) \text{ if } r \in J_{\kappa-1}, \\ 1-\nu(\mathbb{R}_+) \overset{(5.2.3)}{=} -\nu(\{1\}) \text{ if } r = \kappa, \end{cases}\end{aligned}$$

where $(\dagger)$ refers to (5.2.1) and (5.2.14). This and (5.3.5) yield

$$\frac{\mu(x_r)}{\mu(x_\kappa)} - \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \int_0^\infty \frac{t^r}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) = \xi, \quad r \in J_\kappa. \tag{5.3.6}$$

It follows from (4.4.2) and (5.3.6) that the family $\{P(x_{i,1}, \cdot)\}_{i \in J_\eta}$ satisfies the condition (i-c) of Theorem 4.4.1. Therefore (iii) holds.

(i) Arguing as in the proof of Theorem 4.4.2, we verify that (4.4.5) is satisfied. Hence, applying (5.2.5), (5.2.7) and (5.2.8) we get

$$\begin{aligned}\mathsf{h}_{\phi^n}(x_\kappa) &\overset{(4.4.5)}{=} \xi + \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_\kappa)} \int_0^\infty t^n \frac{t^\kappa}{t^{\kappa+1}-1} P(x_{i,1}, \mathrm{d}t) \\ &= \xi + \int_0^\infty t^n \mathrm{d}\tau(t) \\ &\overset{(5.2.1)}{=} \xi + \int_0^\infty t^n \mathrm{d}\nu(t) \\ &\overset{(5.2.11)\&(5.3.5)}{=} \int_0^\infty t^n \mathrm{d}P(x_\kappa, \mathrm{d}t), \quad n \in \mathbb{Z}_+.\end{aligned}$$

This means that $\{\mathsf{h}_{\phi^n}(x_\kappa)\}_{n=0}^\infty$ is a Stieltjes moment sequence with the S-representing measure $P(x_\kappa, \cdot)$. Employing (5.2.1), (5.3.4), (5.2.11) and [5, Theorem 3.6], we deduce that $\{\mathsf{h}_{\phi^n}(x_\kappa)\}_{n=0}^\infty$ is H-determinate, $\mathrm{ind}_z(P(x_\kappa, \cdot)) = 0$ if $z \in \mathbb{C} \setminus \mathrm{supp}(P(x_\kappa, \cdot))$ and $\mathrm{ind}_z(P(x_\kappa, \cdot)) = 1$ if $z \in \mathrm{supp}(P(x_\kappa, \cdot))$. Hence, by (5.2.2), $\mathrm{ind}_0(P(x_\kappa, \cdot)) = 0$.

It follows from (i) and Proposition 3.4.5 that $\mathcal{D}^\infty(C_\phi)$ is dense in $L^2(\mu)$. In particular, C_ϕ is densely defined.

Assume now that $\kappa = 0$.

(v) It follows from the Cauchy–Schwarz inequality that

$$\mu(x_{i,j+1})^2 \overset{(5.2.9)}{\leqslant} \mu(x_{i,j})\mu(x_{i,j+2}), \quad i \in J_\eta,\ j \in \mathbb{N}.$$

Hence, the inequality (3.1.7) holds for every $x \in X \setminus \{x_0\}$. Note that

$$\begin{aligned}
\frac{1}{\mu(x_0)} \sum_{y \in \phi^{-1}(\{x_0\})} \frac{\mu(y)^2}{\mu(\phi^{-1}(\{y\}))} &= \frac{1}{1 + \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_0)}} + \sum_{i=1}^{\eta} \frac{\mu(x_{i,1})}{\mu(x_0)} \frac{\mu(x_{i,1})}{\mu(x_{i,2})} \\
&\overset{(5.2.8)\&(5.2.9)}{=} \frac{1}{1 + \sum_{i=1}^{\eta} \frac{1}{c_i}} + \sum_{i=1}^{\eta} \frac{1}{c_i \int_0^\infty t P(x_{i,1}, \mathrm{d}t)} \\
&\overset{(5.2.5)\&(5.2.6)}{=} \frac{1}{1 + \int_0^\infty (t-1)\mathrm{d}\tau(t)} + \sum_{i=1}^{\eta} \frac{\int_{\varDelta_i} (t-1)\mathrm{d}\tau(t)}{\int_0^\infty t P(x_{i,1}, \mathrm{d}t)} \\
&\overset{(5.2.6)\&(5.2.7)}{=} \frac{1}{1 + \int_0^\infty (t-1)\mathrm{d}\tau(t)} + \sum_{i=1}^{\eta} \frac{(\int_{\varDelta_i} (t-1)\mathrm{d}\tau(t))^2}{\int_{\varDelta_i} t(t-1)\mathrm{d}\tau(t)}.
\end{aligned}$$

Therefore, the inequality (3.1.7) holds for $x = x_0$ if and only if (5.3.1) is satisfied. This combined with Proposition 3.1.2 yields (v).

(vi) This follows from (i), (ii) and [15, Theorem 10.4].

(vii) Apply (i), (5.2.2), (5.2.11), Proposition 4.3.4 and Theorem 3.1.3. □

The next lemma, which is of technical nature, will play a key role in constructing examples of exotic composition operators in Sections 5.4 and 5.5.

Lemma 5.3.2. *Let $\beta \in \mathscr{M}^+$ be such that $\beta(\mathbb{R}_+) > 1$, $0 < \inf \operatorname{supp}(\beta)$ and $\operatorname{supp}(\beta) = \{\theta_1, \theta_2, \ldots\}$, where $\{\theta_i\}_{i=1}^\infty$ is an injective sequence. Set $\theta_i^{(a)} = \psi_{a^{-1},a}(\theta_i)$ and $\beta^{(a)} = \beta \circ \psi_{a^{-1},a}^{-1}$ for $i \in \mathbb{N}$ and $a \in (0, \infty)$ (see (2.2.1) and (2.2.5) for the necessary definitions). Then the following holds:*

(i) *$\beta^{(a)} \in \mathscr{M}^+$ and $\operatorname{supp}(\beta^{(a)}) = \{\theta_1^{(a)}, \theta_2^{(a)}, \ldots\} \subseteq (1, \infty)$ for all $a \in (0, \infty)$,*

(ii) *there exists $m \in \mathbb{N}$ such that $\beta(\{\theta_1, \ldots, \theta_j\}) > 1$ for every integer $j \geqslant m$,*

(iii) *if $m \in \mathbb{N}$ is such that $\beta(\{\theta_1, \ldots, \theta_m\}) > 1$, then there exists $a_1 \in (0, \infty)$ such that for every $a \in (0, a_1)$,*

$$\sum_{i=1}^{m} \frac{\theta_i^{(a)} - 1}{\theta_i^{(a)}} \beta^{(a)}(\{\theta_i^{(a)}\}) > \frac{\int_0^\infty (t-1)\mathrm{d}\beta^{(a)}(t)}{1 + \int_0^\infty (t-1)\mathrm{d}\beta^{(a)}(t)}, \tag{5.3.7}$$

(iv) *there exists $a_2 \in (0, \infty)$ such that for every $a \in (0, a_2)$,*

$$\sum_{i=1}^{\infty} \frac{\theta_i^{(a)} - 1}{\theta_i^{(a)}} \beta^{(a)}(\{\theta_i^{(a)}\}) > \frac{\int_0^\infty (t-1)\mathrm{d}\beta^{(a)}(t)}{1 + \int_0^\infty (t-1)\mathrm{d}\beta^{(a)}(t)}, \tag{5.3.8}$$

(v) *if $\kappa \in \mathbb{N}$, then there exists $a_3 \in (0, \infty)$ such that for every $a \in (a_3, \infty)$,*

$$1 + \int_0^\infty \frac{1}{t^\kappa} \mathrm{d}\beta^{(a)}(t) > \beta^{(a)}(\mathbb{R}_+). \tag{5.3.9}$$

Proof. (i) Apply Lemma 2.2.2.

(ii) Use the fact that $\lim_{j\to\infty} \beta(\{\theta_1, \ldots, \theta_j\}) = \beta(\mathbb{R}_+) > 1$.

(iii) It follows from our assumptions that

$$\lim_{a\to 0+} \sum_{i=1}^m \frac{\theta_i}{a + \theta_i} \beta(\{\theta_i\}) = \beta(\{\theta_1, \ldots, \theta_m\}) > 1$$

and $\lim_{a\to 0+} \frac{\int_0^\infty t\mathrm{d}\beta(t)}{a+\int_0^\infty t\mathrm{d}\beta(t)} = 1$. Hence, there exists $a_1 \in (0, \infty)$ such that

$$\sum_{i=1}^m \frac{\theta_i}{a + \theta_i} \beta\big(\{\theta_i\}\big) > \frac{\int_0^\infty t\mathrm{d}\beta(t)}{a + \int_0^\infty t\mathrm{d}\beta(t)}, \quad a \in (0, a_1). \tag{5.3.10}$$

Using (2.2.1), we easily verify that

$$\sum_{i=1}^m \frac{\theta_i^{(a)} - 1}{\theta_i^{(a)}} \beta^{(a)}\big(\{\theta_i^{(a)}\}\big) = \sum_{i=1}^m \frac{\theta_i}{a + \theta_i} \beta\big(\{\theta_i\}\big), \quad a \in (0, \infty). \tag{5.3.11}$$

Applying the measure transport theorem, we obtain

$$\frac{\int_0^\infty (t-1)\mathrm{d}\beta^{(a)}(t)}{1 + \int_0^\infty (t-1)\mathrm{d}\beta^{(a)}(t)} = \frac{\int_0^\infty t\mathrm{d}\beta(t)}{a + \int_0^\infty t\mathrm{d}\beta(t)}, \quad a \in (0, \infty). \tag{5.3.12}$$

Combining (5.3.10), (5.3.11) and (5.3.12) yields (iii).

(iv) This follows from (i), (ii) and (iii).

(v) Note that

$$\lim_{a\to\infty} \sum_{i=1}^\infty \Big(\frac{a}{\theta_i + a}\Big)^\kappa \beta(\{\theta_i\}) = \beta(\mathbb{R}_+).$$

Hence, there exists $a_3 \in (0, \infty)$ such that

$$1 + \sum_{i=1}^\infty \Big(\frac{a}{\theta_i + a}\Big)^\kappa \beta(\{\theta_i\}) > \beta(\mathbb{R}_+), \quad a \in (a_3, \infty). \tag{5.3.13}$$

Using (i) and the measure transport theorem, we get

$$
\begin{aligned}
1+\int_0^\infty \frac{1}{t^\kappa}\mathrm{d}\beta^{(a)}(t) &= 1+\int_0^\infty \frac{1}{\big(\psi_{a^{-1},a}(t)\big)^\kappa}\mathrm{d}\beta(t) \\
&= 1+\sum_{i=1}^\infty \Big(\frac{a}{\theta_i+a}\Big)^\kappa \beta(\{\theta_i\}) \\
&\overset{(5.3.13)}{>} \beta(\mathbb{R}_+) = \beta^{(a)}(\mathbb{R}_+), \quad a\in(a_3,\infty).
\end{aligned}
$$

This completes the proof. □

The third lemma is related to q-Pochhammer symbol $(z;q)_\infty$ (see (2.3.1) for its definition). The function $(0,1)\ni q\mapsto (q;q)_\infty\in(0,1)$ is called the *Euler function*.

Lemma 5.3.3. *Let $a\in(1,\infty)$. Then there exists $q_0\in\big(0,\frac{1}{a}\big)$ such that*

$$(q/a;q)_\infty + (aq;q)_\infty > 1, \quad q\in(0,q_0).$$

Proof. Since the function $n\mapsto \frac{(3n-1)n}{2}$ maps $\mathbb{Z}$ injectively into $\mathbb{Z}_+$, we deduce from Euler's pentagonal-number theorem (see [2, Theorem 14.3]) that

$$
\begin{aligned}
(q;q)_\infty &= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{(3n-1)n}{2}} \\
&= \sum_{k=-\infty}^{\infty} q^{(6k-1)k} - \sum_{k=-\infty}^{\infty} q^{(3k+1)(2k+1)} \\
&> 1-\sum_{k=-\infty}^{\infty} q^{(3k+1)(2k+1)} \\
&> 1-\sum_{k=1}^{\infty} q^k \\
&= 1-\frac{q}{1-q}, \quad q\in(0,1).
\end{aligned} \tag{5.3.14}
$$

It is easily seen that there exists $q_0\in(0,\frac{1}{a})$ such that

$$1-\frac{q}{1-q} > \frac{1}{2(1-aq)}, \quad q\in(0,q_0).$$

Hence, by (5.3.14), we have

$$2(1-aq)(q;q)_\infty > 1, \quad q\in(0,q_0).$$

This combined with $\frac{1}{a}<a$ and $aq_0<1$ imply that

$$\prod_{j=1}^{\infty}\Big(1-\frac{1}{a}q^j\Big)+\prod_{j=1}^{\infty}(1-aq^j) > 2(1-aq)\prod_{j=1}^{\infty}(1-aqq^j)$$
$$> 2(1-aq)(q;q)_\infty > 1, \quad q\in(0,q_0),$$

which completes the proof. □

5.4. The gap between the conditions (i-d) and (i-d′)

In this section, we show that the assertion (ii) of Theorem 4.4.1 is no longer true if the condition (i-d) of Theorem 4.4.1 is replaced by the condition (i-d′) of Theorem 4.4.2. As shown in Theorem 5.4.2 below, this can happen even for subnormal composition operators. In fact, this phenomenon is independent of whether the operator in question is subnormal or not (see Remark 5.5.5).

We begin with the case of $\eta \geqslant 2$.

Theorem 5.4.1. *Let $\kappa \in \mathbb{Z}_+$ and $\eta \in \mathbb{N}_2 \cup \{\infty\}$. Then there exists a discrete measure μ on $X = X_{\eta,\kappa}$ such that the composition operator C_ϕ in $L^2(\mu)$ with $\phi = \phi_{\eta,\kappa}$ has the property that $\overline{\mathcal{D}^\infty(C_\phi)} = L^2(\mu)$ and the following conditions hold:*

(i) *$\{\mathsf{h}_{\phi^n}(x_\kappa)\}_{n=0}^\infty$ is an H-determinate Stieltjes moment sequence with index of H-determinacy at 0 equal to 0,*
(ii) *for all $i \in J_\eta$ and $j \in \mathbb{N}$, $\{\mathsf{h}_{\phi^n}(x_{i,j})\}_{n=0}^\infty$ is an H-determinate Stieltjes moment sequence with infinite index of H-determinacy; in particular, the condition* (i-a) *of Theorem 4.4.1 holds,*
(iii) *there exists a unique family $\{P(x_{i,1},\cdot)\}_{i\in J_\eta}$ of Borel probability measures on $\mathbb{R}_+$ that satisfies the conditions* (i-b), (i-c) *and* (i-d′) *of Theorems 4.4.1 and 4.4.2; this family does not satisfy the condition* (i-d) *of Theorem 4.4.1,*
(iv) *there is no family $\{P'(x,\cdot)\}_{x\in X}$ of Borel probability measures on $\mathbb{R}_+$ that satisfies (CC),*
(v) *if $\kappa = 0$, then C_ϕ generates Stieltjes moment sequences.*

Proof. First, we note that if (i) holds, then by Proposition 3.4.5, $\mathcal{D}^\infty(C_\phi)$ is dense in $L^2(\mu)$. We begin by proving that the condition (iv) follows from (i), (ii) and (iii). Suppose, on the contrary, that there exists a family $\{P'(x,\cdot)\}_{x\in X}$ of Borel probability measures on $\mathbb{R}_+$ that satisfies (CC). It follows from Theorem 3.1.3 that $P'(x_{i,1},\cdot)$ is an S-representing measure of $\{\mathsf{h}_{\phi^n}(x_{i,1})\}_{n=0}^\infty$ for every $i \in J_\eta$. Hence, by (ii) and (iii), $P'(x_{i,1},\cdot) = P(x_{i,1},\cdot)$ for every $i \in J_\eta$. This together with Theorem 4.4.1(i) implies that the family $\{P(x_{i,1},\cdot)\}_{i\in J_\eta}$ satisfies the condition (i-d) of Theorem 4.4.1, which contradicts (iii).

Take any S-indeterminate Stieltjes moment sequence $\boldsymbol{\gamma} = \{\gamma_n\}_{n=0}^\infty$ (see e.g., the classical example due to Stieltjes [58]; see also Section 2.3). Let α and β be the Krein and the Friedrichs measures of $\boldsymbol{\gamma}$ (see Section 2.1). Then by Lemma 2.1.1 and (2.1.1),

$0 < \alpha((0,\infty)) < \alpha(\mathbb{R}_+)$. Hence, replacing (α,β) by $(r\alpha, r\beta)$ with $r := \alpha((0,\infty))^{-1}$ if necessary, we may assume that

$$\begin{aligned}&\alpha \text{ and } \beta \text{ are N-extremal measures of the same Stieltjes}\\ &\text{moment sequence such that } 0 = \inf\operatorname{supp}(\alpha) < \inf\operatorname{supp}(\beta)\\ &\text{and } \alpha(\mathbb{R}_+) = 1 + \alpha(\{0\}) > 1.\end{aligned} \tag{5.4.1}$$

It follows from Lemma 5.3.2(v) that there exists $a \in (0,\infty)$ such that (5.3.9) holds. This combined with (5.4.1) and Lemma 2.2.2 shows that the measures $\nu := \alpha^{(a)}$ and $\tau := \beta^{(a)}$ satisfy the conditions (5.2.1)–(5.2.4) of Procedure 5.2.1. Since the measure τ is N-extremal and $\eta \geqslant 2$, we infer from Lemma 2.1.1 that there exists a partition $\{\varDelta_i\}_{i=1}^{\eta}$ of $\operatorname{supp}(\tau)$ such that

$$\operatorname{card}(\operatorname{supp}(\tau) \setminus \varDelta_i) = \aleph_0, \quad i \in J_\eta. \tag{5.4.2}$$

Let X, μ, $P(x_\kappa,\cdot)$, $\{P(x_{i,j},\cdot)\}_{i\in J_\eta, j\in\mathbb{N}}$ and C_ϕ be as in Procedure 5.2.1. We will show that the operator C_ϕ has all the required properties. Indeed, set $\xi_i = \sum_{\lambda\in\varDelta_i} \tau(\{\lambda\})\delta_\lambda$ for every $i \in J_\eta$. We deduce from the equality (5.4.2) and Theorem 2.4.1 that each ξ_i is H-determinate and

$$\operatorname{ind}_z(\xi_i) = \infty \text{ for all } z \in \mathbb{C} \text{ and } i \in J_\eta. \tag{5.4.3}$$

Fix $(i,j) \in J_\eta \times \mathbb{N}$ and $z \in \mathbb{C}$. It follows from (5.2.2) that $\operatorname{supp}(\xi_i) \subseteq (1,\infty)$. Hence, by (5.2.7) and (5.2.10), we see that for every $k \in \mathbb{Z}_+$,

$$\begin{aligned}\int_\sigma |t-z|^{2k} P(x_{i,j},\mathrm{d}t) &= c_i \frac{\mu(x_{i,1})}{\mu(x_{i,j})} \int_\sigma |t-z|^{2k} t^{j-1} \frac{t^{\kappa+1}-1}{t^\kappa} \chi_{\varDelta_i}(t)\mathrm{d}\tau(t)\\ &\leqslant c_i \frac{\mu(x_{i,1})}{\mu(x_{i,j})} (1+|z|)^{2k} \int_\sigma t^{j+\kappa+2k} \mathrm{d}\xi_i(t)\\ &\leqslant c_i \frac{\mu(x_{i,1})}{\mu(x_{i,j})} (1+|z|)^{2k} \int_\sigma t^{2(j+\kappa+k)} \mathrm{d}\xi_i(t), \quad \sigma \in \mathfrak{B}(\mathbb{R}).\end{aligned}$$

This, (5.4.3), (2.4.1) and Proposition 2.1.3 imply that the measure $P(x_{i,j},\cdot)$ is H-determinate and $\operatorname{ind}_z(P(x_{i,j},\cdot)) = \infty$. Applying Lemma 5.3.1, we conclude that C_ϕ satisfies the conditions (i)–(v). This completes the proof. $\square$

The case of $\eta = 1$ turns out to be surprisingly different from the previous one (compare Theorem 5.4.1(iv) with Theorem 5.4.2(iv)).

Theorem 5.4.2. *Let $\kappa \in \mathbb{Z}_+$. Then there exists a discrete measure μ on $X = X_{1,\kappa}$ such that the composition operator C_ϕ in $L^2(\mu)$ with $\phi = \phi_{1,\kappa}$ has the property that $\overline{\mathcal{D}^\infty(C_\phi)} = L^2(\mu)$ and the following conditions hold:*

(i) $\{\mathsf{h}_{\phi^n}(x_\kappa)\}_{n=0}^\infty$ *is an H-determinate Stieltjes moment sequence with index of H-determinacy at* 0 *equal to* 0*,*

(ii) *for every* $j \in \mathbb{N}$*,* $\{\mathsf{h}_{\phi^n}(x_{1,j})\}_{n=0}^\infty$ *is an S-indeterminate Stieltjes moment sequence; in particular, the condition* (i-a) *of Theorem* 4.4.1 *holds,*

(iii) *there exists a Borel probability measure* $P(x_{1,1}, \cdot)$ *on* $\mathbb{R}_+$ *that satisfies the conditions* (i-b)*,* (i-c) *and* (i-d′) *of Theorems* 4.4.1 *and* 4.4.2*, and does not satisfy the condition* (i-d) *of Theorem* 4.4.1*,*

(iv) *if* $\kappa = 0$*, then there exists a family* $\{P'(x, \cdot)\}_{x\in X}$ *of Borel probability measures on* $\mathbb{R}_+$ *that satisfies (CC) and, consequently,* C_ϕ *is subnormal.*

Proof. As in the proof of Theorem 5.4.1, we see that there exist measures ν and τ satisfying the conditions (5.2.1)–(5.2.4) of Procedure 5.2.1. The only possible partition $\{\varDelta_i\}_{i=1}^\eta$ of $\operatorname{supp}(\tau)$ with $\eta = 1$ is $\varDelta_1 = \operatorname{supp}(\tau)$. Let X, μ, $P(x_\kappa, \cdot)$, $\{P(x_{1,j}, \cdot)\}_{j=1}^\infty$ and C_ϕ be as in Procedure 5.2.1. In view of Lemma 5.3.1, it remains to show that for every $j \in \mathbb{N}$, the Stieltjes moment sequence $\{\mathsf{h}_{\phi^n}(x_{1,j})\}_{n=0}^\infty$ is S-indeterminate.

Suppose, on the contrary, that there exists $k \in \mathbb{N}$ such that $\{\mathsf{h}_{\phi^n}(x_{1,k})\}_{n=0}^\infty$ is S-determinate. We show that then $\{\mathsf{h}_{\phi^n}(x_{1,1})\}_{n=0}^\infty$ is S-determinate. For this we may assume that $k \geqslant 2$. It follows from (3.1.2) that

$$\mathsf{h}_{\phi^n}(x_{1,k}) = \frac{\mu(x_{1,k-1})}{\mu(x_{1,k})}\mathsf{h}_{\phi^{n+1}}(x_{1,k-1}), \quad n \in \mathbb{Z}_+.$$

Hence, by [57, Proposition 5.12], $\{\mathsf{h}_{\phi^n}(x_{1,k-1})\}_{n=0}^\infty$ is S-determinate. Applying backward induction, we deduce that $\{\mathsf{h}_{\phi^n}(x_{1,1})\}_{n=0}^\infty$ is S-determinate. Since, by Lemma 5.3.1(ii), $P(x_{1,1}, \cdot)$ is an S-representing measure of $\{\mathsf{h}_{\phi^n}(x_{1,1})\}_{n=0}^\infty$ and $P(x_{1,1}, \{0\}) = 0$ (see (5.2.2) and (5.2.7)), we infer from [21, Corollary on p. 481] (see also [42, Lemma 2.2.5]) that $P(x_{1,1}, \cdot)$ is H-determinate. By (5.2.2), there exists $M \in (0, \infty)$ such that

$$\frac{t^{\kappa+1} - 1}{t^\kappa} \geqslant M, \quad t \in [\inf \operatorname{supp}(\tau), \infty).$$

This combined with (5.2.7) and Proposition 2.1.3 (applied to τ and $P(x_{1,1}, \cdot)$) implies that τ is H-determinate, which contradicts (5.2.1). □

Regarding Theorem 5.4.2, note that if $\kappa = 0$, then (ii) can also be deduced from (iii) and (iv) by arguing as in the first paragraph of the proof of Theorem 5.4.1.

5.5. Exotic non-hyponormality

The main purpose of this section is to construct non-hyponormal composition operators C_ϕ in $L^2(X, 2^X, \mu)$ that generate Stieltjes moment sequences with $X = X_{\eta,0}$ and $\phi = \phi_{\eta,0}$. Recall that $C_{\phi_{\eta,\kappa}}$ is always injective (see (3.4.2)). We begin with the case of $\eta = \infty$.

Theorem 5.5.1. *There exists a discrete measure μ on $X = X_{\infty,0}$ such that the composition operator C_ϕ in $L^2(\mu)$ with $\phi = \phi_{\infty,0}$ has the following properties:*

(i) *C_ϕ is injective and generates Stieltjes moment sequences,*
(ii) *C_ϕ is not hyponormal,*
(iii) *$\{\mathsf{h}_{\phi^n}(x_0)\}_{n=0}^\infty$ is an H-determinate Stieltjes moment sequence with index of H-determinacy at 0 equal to 0,*
(iv) *$\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ is an H-determinate Stieltjes moment sequence with infinite index of H-determinacy for all $x \in X \setminus \{x_0\}$.*

Proof. Arguing as in the proof of Theorem 5.4.1, we see that there exist measures α and β satisfying (5.4.1). Let $\{\theta_i\}_{i=1}^\infty$ be an injective sequence such that $\operatorname{supp}(\beta) = \{\theta_1, \theta_2, \ldots\}$ (see Lemma 2.1.1). It follows from Lemma 5.3.2(iv) that there exists $a \in (0, \infty)$ such that (5.3.8) holds. This and Lemma 2.2.2 show that the measures $\nu := \alpha^{(a)}$ and $\tau := \beta^{(a)}$ satisfy the conditions (5.2.1)–(5.2.4) of Procedure 5.2.1 for $\kappa = 0$ and $\eta = \infty$. Note also that in view of Lemma 5.3.2(i), $\operatorname{supp}(\tau) = \{\theta_1^{(a)}, \theta_2^{(a)}, \ldots\}$. Set $\varDelta_i = \{\theta_i^{(a)}\}$ for $i \in \mathbb{N}$. Clearly, the sequence $\{\varDelta_i\}_{i=1}^\infty$ satisfies the condition (5.2.5) for $\eta = \infty$.

Let X, μ, $\{P(x, \cdot)\}_{x \in X}$ and C_ϕ be as in Procedure 5.2.1 for $\kappa = 0$ and $\eta = \infty$. It follows from (5.2.7), (5.2.9) and (5.2.10) that $P(x_{i,j}, \cdot) = \delta_{\theta_i^{(a)}}(\cdot)$ for all $i, j \in \mathbb{N}$. This is easily seen to imply (iv). Since (5.3.8) yields

$$\sum_{i=1}^\infty \frac{(\int_{\varDelta_i} (t-1) \mathrm{d}\tau(t))^2}{\int_{\varDelta_i} t(t-1) \mathrm{d}\tau(t)} > \frac{\int_0^\infty (t-1) \mathrm{d}\tau(t)}{1 + \int_0^\infty (t-1) \mathrm{d}\tau(t)},$$

an application of Lemma 5.3.1 completes the proof. □

Now we concern ourselves with the case when η is an arbitrary integer greater than or equal to 2. The most striking case is that of $\eta = 2$. Below, given $x \in \mathbb{R}$, we write $\lfloor x \rfloor$ for the largest integer not greater than x, and $\lceil x \rceil$ for the smallest integer not less than x.

Theorem 5.5.2. *Let $\eta \in \mathbb{N}_2$. Then there exists a discrete measure μ on $X = X_{\eta,0}$ such that the composition operator C_ϕ in $L^2(\mu)$ with $\phi = \phi_{\eta,0}$ has the following properties:*

(i) *C_ϕ is injective and generates Stieltjes moment sequences,*
(ii) *C_ϕ is not hyponormal,*
(iii) *$\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $x \in X$,*
(iv) *$\{\mathsf{h}_{\phi^n}(x_0)\}_{n=0}^\infty$ is H-determinate with index of H-determinacy at 0 equal to 0,*
(v) *for all $i \in J_{\eta-1}$ and $j \in \mathbb{N}$, the sequence $\{\mathsf{h}_{\phi^n}(x_{i,j})\}_{n=0}^\infty$ is H-determinate with infinite index of H-determinacy,*
(vi) *for every $j \in J_{2\eta-4}$, the sequence $\{\mathsf{h}_{\phi^n}(x_{\eta,j})\}_{n=0}^\infty$ is H-determinate and its unique H-representing measure $P(x_{\eta,j}, \cdot)$ satisfies the following condition*

$$\eta - 2 - \lceil j/2 \rceil \leqslant \operatorname{ind}_0(P(x_{\eta,j}, \cdot)) \leqslant \eta - 2 - \lfloor j/2 \rfloor. \tag{5.5.1}$$

Proof. We split the proof into two steps.

Step 1. Let $\eta \in \mathbb{N}_2$. Then the conclusion of Theorem 5.5.2 holds if there exist measures α and β satisfying (5.4.1) and an injective sequence $\{\theta_i\}_{i=1}^{\eta-1} \subseteq \operatorname{supp}(\beta)$ such that

$$\beta(\{\theta_1, \ldots, \theta_{\eta-1}\}) > 1. \tag{5.5.2}$$

Indeed, by Lemma 2.1.1, we can extend the sequence $\{\theta_i\}_{i=1}^{\eta-1}$ to an injective sequence $\{\theta_i\}_{i=1}^{\infty}$ such that $\operatorname{supp}(\beta) = \{\theta_1, \theta_2, \ldots\}$. It follows from (5.5.2) and Lemma 5.3.2(iii) that there exists $a \in (0, \infty)$ such that the inequality (5.3.7) holds for $m = \eta - 1$. This together with (5.4.1) and Lemma 2.2.2 imply that the measures $\nu := \alpha^{(a)}$ and $\tau := \beta^{(a)}$ satisfy the conditions (5.2.1)–(5.2.4) of Procedure 5.2.1 for $\kappa = 0$, and $\operatorname{supp}(\tau) = \{\theta_1^{(a)}, \theta_2^{(a)}, \ldots\}$. Now we define a partition $\{\varDelta_i\}_{i=1}^{\eta}$ of $\operatorname{supp}(\tau)$ by

$$\varDelta_i = \begin{cases} \{\theta_i^{(a)}\} & \text{if } i \in J_{\eta-1}, \\ \{\theta_\eta^{(a)}, \theta_{\eta+1}^{(a)}, \ldots\} & \text{if } i = \eta. \end{cases} \tag{5.5.3}$$

Let X, μ, $\{P(x, \cdot)\}_{x \in X}$ and C_ϕ be as in Procedure 5.2.1 for $\kappa = 0$. Since $\eta \in \mathbb{N}_2$, it follows from (5.2.2) that

$$\begin{aligned} \sum_{i=1}^{\eta} \frac{(\int_{\varDelta_i} (t-1) \mathrm{d}\tau(t))^2}{\int_{\varDelta_i} t(t-1) \mathrm{d}\tau(t)} &> \sum_{i=1}^{\eta-1} \frac{(\int_{\varDelta_i} (t-1) \mathrm{d}\tau(t))^2}{\int_{\varDelta_i} t(t-1) \mathrm{d}\tau(t)} \\ &= \sum_{i=1}^{\eta-1} \frac{\theta_i^{(a)} - 1}{\theta_i^{(a)}}\, \tau(\{\theta_i^{(a)}\}) \\ &\overset{(5.3.7)}{>} \frac{\int_0^\infty (t-1) \mathrm{d}\tau(t)}{1 + \int_0^\infty (t-1) \mathrm{d}\tau(t)}. \end{aligned}$$

Now applying Lemma 5.3.1 we see that for every $x \in X$, $\{\mathsf{h}_{\phi^n}(x)\}_{n=0}^{\infty}$ is a Stieltjes moment sequence with the S-representing measure $P(x, \cdot)$ (in particular, (iii) holds), and C_ϕ satisfies (i), (ii) and (iv) (note that if $\eta = 1$, then, by Lemma 5.3.1(vii), C_ϕ is hyponormal). Arguing as in the proof of Theorem 5.5.1, we deduce that (v) holds as well.

Our next aim is to prove (vi). Assume that $\eta \geqslant 3$. Set $\xi_\eta = \sum_{\lambda \in \varDelta_\eta} \tau(\{\lambda\}) \delta_\lambda$. Clearly $\xi_\eta \in \mathscr{M}^+$. By [5, Theorem 3.6], ξ_η is H-determinate and

$$\operatorname{ind}_0(\xi_\eta) = \eta - 2. \tag{5.5.4}$$

Put $m_j = c_\eta \frac{\mu(x_{\eta,1})}{\mu(x_{\eta,j})}$ for $j \in \mathbb{N}$. It follows from (5.2.7) and (5.2.10) that

$$\int_\sigma t^{2k} P(x_{\eta,j}, \mathrm{d}t) = m_j \int_\sigma t^{2k+j-1} (t-1) \mathrm{d}\xi_\eta(t), \qquad \sigma \in \mathfrak{B}(\mathbb{R}), k \in \mathbb{Z}_+, j \in \mathbb{N}. \tag{5.5.5}$$

Since, by (5.2.2), $\operatorname{supp}(\xi_\eta) \subseteq (1, \infty)$, we infer from (5.5.5) that

$$\int_\sigma t^{2k} P(x_{\eta,j}, \mathrm{d}t) \leqslant m_j \int_\sigma t^{2(k+\lceil j/2\rceil)} \mathrm{d}\xi_\eta(t) \leqslant m_j \int_\sigma t^{2(\eta-2)} \mathrm{d}\xi_\eta(t),$$
$$\sigma \in \mathfrak{B}(\mathbb{R}),\ k \in \mathbb{Z}_+,\ k + \lceil j/2\rceil \leqslant \eta - 2,\ j \in \mathbb{N}. \tag{5.5.6}$$

Fix $j \in J_{2\eta-4}$. Substituting $k = 0$ into (5.5.6) and using (5.5.4) and Proposition 2.1.3, we deduce that $P(x_{\eta,j}, \cdot)$ is H-determinate. Hence, applying (5.5.4), (5.5.6) with $k = \eta - 2 - \lceil j/2\rceil$, and Proposition 2.1.3, we see that

$$\operatorname{ind}_0(P(x_{\eta,j}, \cdot)) \geqslant \eta - 2 - \lceil j/2\rceil. \tag{5.5.7}$$

Take now $k \in \mathbb{Z}_+$ such that the measure $t^{2k} P(x_{\eta,j}, \mathrm{d}t)$ is H-determinate. Since $\operatorname{supp}(\xi_\eta) \subseteq (1, \infty)$, there exists $M \in (0, 1)$ such that for every $\sigma \in \mathfrak{B}(\mathbb{R})$,

$$\int_\sigma t^{2k} P(x_{\eta,j}, \mathrm{d}t) \overset{(5.5.5)}{\geqslant} M m_j \int_\sigma t^{2k+j} \mathrm{d}\xi_\eta(t) \geqslant M m_j \int_\sigma t^{2(k+\lfloor j/2\rfloor)} \mathrm{d}\xi_\eta(t).$$

Applying (5.5.4) and Proposition 2.1.3 again, we deduce that $k + \lfloor j/2\rfloor \leqslant \eta - 2$. This implies that

$$\operatorname{ind}_0(P(x_{\eta,j}, \cdot)) \leqslant \eta - 2 - \lfloor j/2\rfloor.$$

Combining the above inequality with (5.5.7), we obtain (5.5.1).

Step 2. There exist measures α and β satisfying (5.4.1) such that

$$\beta\big(\{\inf \operatorname{supp}(\beta)\}\big) > 1.$$

Indeed, let $a \in (1, \infty)$. By Lemma 5.3.3, there exists $q \in \big(0, \frac{1}{a}\big)$ such that

$$(q/a; q)_\infty + (aq; q)_\infty > 1. \tag{5.5.8}$$

As in [8, Proposition 4.5.1] (see also [8, eq. (4.4)]), we define the Borel measures ζ and ρ on $\mathbb{R}$ (with different notation) by

$$\begin{aligned} \zeta &:= \beta^{(a;q)} \circ \psi_{1,-1}^{-1} \overset{(2.3.6)}{=} (aq; q)_\infty \sum_{n=0}^\infty \frac{a^n q^{n^2}}{(aq; q)_n (q; q)_n} \delta_{q^{-n}-1}, \\ \rho &:= \gamma^{(a;q)} \circ \psi_{1,-1}^{-1} \overset{(2.3.7)}{=} (q/a; q)_\infty \sum_{n=0}^\infty \frac{a^{-n} q^{n^2}}{(q/a; q)_n (q; q)_n} \delta_{aq^{-n}-1}. \end{aligned} \tag{5.5.9}$$

(Another possible choice of measures ζ and ρ is discussed in Remark 5.5.3.) In view of the proof of [37, eq. (5.10)], ζ and ρ are probability measures. It follows from

[8, Proposition 4.5.1] that ζ and ρ are N-extremal measures of the same Stieltjes moment sequence (see also Section 2.3 for a detailed discussion of this matter). Note that $\zeta(\{0\}) = (aq;q)_\infty \in (0,1)$. Following the proof of Theorem 5.4.1, we set $\alpha = r\zeta$ and $\beta = r\rho$ with $r = (1-\zeta(\{0\}))^{-1}$. It is easily seen that the measures α and β satisfy (5.4.1). Now, combining (5.5.8) with (5.5.9), we get

$$\beta(\{\inf \operatorname{supp}(\beta)\}) = \frac{(q/a;q)_\infty}{1-(aq;q)_\infty} > 1.$$

To finish the proof of the theorem, take measures α and β as in Step 2. Then $\operatorname{supp}(\beta) = \{\theta_1, \theta_2, \ldots\}$, where $\{\theta_i\}_{i=1}^\infty$ is a strictly increasing sequence (see Lemma 2.1.1). This implies that $\beta(\{\theta_1\}) > 1$, and thus the inequality (5.5.2) holds for every $\eta \in \mathbb{N}_2$. Applying Step 1 completes the proof. $\square$

To the best of our knowledge, there are few explicit examples (or rather classes of examples) of N-extremal measures. The ones used in the proof of Theorem 5.5.2 are related to the Al-Salam–Carlitz q-polynomials (see Section 2.3). In turn, the measures discussed below come from an S-indeterminate Stieltjes moment problem associated with a quartic birth and death process (see [8]).

Remark 5.5.3. Define the Borel measures ζ and ρ on $\mathbb{R}$ by

$$\begin{aligned} \zeta &= \frac{\pi}{K_0^2}\delta_{x_0} + \frac{4\pi}{K_0^2}\sum_{n=1}^\infty \frac{2n\pi}{\sinh(2n\pi)}\delta_{x_{2n}}, \\ \rho &= \frac{4\pi}{K_0^2}\sum_{n=0}^\infty \frac{(2n+1)\pi}{\sinh\big((2n+1)\pi\big)}\delta_{x_{2n+1}}, \end{aligned} \tag{5.5.10}$$

where

$$K_0 = \frac{\Gamma(\frac14)^2}{4\sqrt{\pi}} \quad \text{and} \quad x_k = \left(\frac{k\pi}{K_0}\right)^4 \text{ for } k \in \mathbb{Z}_+.$$

($\Gamma(\cdot)$ stands for the Euler gamma function.) It was proved in [8, Proposition 3.4.1] that ζ and ρ are N-extremal measures of the same Stieltjes moment sequence. A careful inspection of [8] reveals that these measures are probabilistic. As a consequence, $\zeta(\{0\}) = \pi K_0^{-2} \in (0,1)$. Note that $\beta(\{\theta_1\}) > 1$, where, as in the proof of Step 2 of Theorem 5.5.2, $\beta = r\rho$ with $r = (1-\zeta(\{0\}))^{-1}$ and $\theta_1 = \inf \operatorname{supp}(\beta)$. Indeed, since $\pi > 3$, $\Gamma(\frac14) < 4$ and $\sinh(\pi) < 12$, we get

$$\beta(\{\theta_1\}) = \frac{64\pi^3}{(\Gamma(\frac14)^4 - 16\pi^2)\sinh(\pi)} > \frac{9}{7} > 1.$$

(In fact, $\beta(\{\theta_1\}) > \frac{23}{2}$.) Putting all this together, we see that the proof of Theorem 5.5.2 goes through without change if the measures ζ and ρ are defined by (5.5.10) instead of (5.5.9).

Below we make an important remark on the measures ζ and ρ and their relatives ν and τ appearing in the proof of Theorem 5.5.2 and in Remark 5.5.3.

Remark 5.5.4. Let ζ and ρ be measures as in (5.5.9). Recall that they are representing measures of the same Stieltjes moment sequence, say $\boldsymbol{\gamma}$. Since $\inf \operatorname{supp}(\zeta) = 0$, we infer from (2.1.1) that ζ is the Krein measure of $\boldsymbol{\gamma}$. In turn, the measure ρ is the Friedrichs measure of $\boldsymbol{\gamma}$. Indeed, this can be deduced by combining (2.1.1) and [52, Proposition 3.1] with [52, Proposition 3.2] and [8, Lemma 4.4.2] (that this particular ρ is the Friedrichs measure of $\boldsymbol{\gamma}$ follows also from Theorem 2.2.3(i) and the discussion in the paragraph containing (2.3.7)). Hence, by Corollary 2.2.4 (see also Theorem 2.2.3), the measures β and τ are the Friedrichs measures of appropriate S-indeterminate Stieltjes moment sequences (see Step 1 of the proof of Theorem 5.5.2); however, ν is not the Krein measure (of the S-indeterminate Stieltjes moment sequence $\{\int_0^\infty t^n \mathrm{d}\nu(t)\}_{n=0}^\infty$). It is easily seen that the above discussion applies to the measures ζ and ρ defined by (5.5.10) and the corresponding measures ν and τ (but now we have to use [8, Corollary 3.3.3] in place of [8, Lemma 4.4.2]).

It is worth pointing out that [52, Proposition 3.1] is essentially due to Chihara (see [21, Theorem 5]) and that there is another way of parameterizing N-extremal measures in which the traditional interval $[\alpha, 0]$ is replaced by $[-1/\alpha, \infty) \cup \{\infty\}$, where α is as in [52, Proposition 3.2] (see [57, Theorem 4.18, Proposition 5.20, and Remark on p. 127]).

We end this section with yet another remark.

Remark 5.5.5. Since Lemma 5.3.1 was used as one of the main tools to prove Theorems 5.5.1 and 5.5.2, the following statement can be added to their conclusions:

there exists a family $\{P(x_{i,1}, \cdot)\}_{i \in J_\eta}$ *of Borel probability measures on* $\mathbb{R}_+$ *that satisfies the conditions* (i-b), (i-c) *and* (i-d′) *of Theorems* 4.4.1 and 4.4.2, *and does not satisfy the condition* (i-d) *of Theorem* 4.4.1.

5.6. Addendum

The construction of the composition operator C_ϕ that appears in the proofs of theorems of Section 5 depends on the choice of a partition $\{\varDelta_i\}_{i=1}^\eta$ of $\operatorname{supp}(\tau)$ (see Procedure 5.2.1). In particular, the hyponormality of C_ϕ requires finding a partition $\{\varDelta_i\}_{i=1}^\eta$ satisfying the inequality (5.3.1). Hence, it seems to be of some independent interest to calculate the infimum and the supremum of the quantity appearing on the left-hand side of (5.3.1) over all partitions $\{\varDelta_i\}_{i=1}^\eta$ of $\operatorname{supp}(\tau)$. The following general proposition sheds more light on this issue.

Proposition 5.6.1. *Let* τ *be an N-extremal measure such that* $1 < \inf \operatorname{supp}(\tau)$. *Then for every* $\eta \in \mathbb{N} \cup \{\infty\}$, *the following two conditions hold:*

$$\inf\Big\{\varLambda(\{\varDelta_i\}_{i=1}^{\eta})\colon \{\varDelta_i\}_{i=1}^{\eta} \text{ satisfies (5.2.5)}\Big\} = \frac{\big(\int_0^\infty (t-1)\mathrm{d}\tau(t)\big)^2}{\int_0^\infty t(t-1)\mathrm{d}\tau(t)}, \tag{5.6.1}$$

$$\sup\Big\{\varLambda(\{\varDelta_i\}_{i=1}^{\eta})\colon \{\varDelta_i\}_{i=1}^{\eta} \text{ satisfies (5.2.5)}\Big\} \leqslant \int_0^\infty \frac{t-1}{t}\mathrm{d}\tau(t), \tag{5.6.2}$$

where

$$\varLambda(\{\varDelta_i\}_{i=1}^{\eta}) = \sum_{i=1}^{\eta} \frac{(\int_{\varDelta_i}(t-1)\mathrm{d}\tau(t))^2}{\int_{\varDelta_i} t(t-1)\mathrm{d}\tau(t)}, \quad \{\varDelta_i\}_{i=1}^{\eta} \text{ satisfies (5.2.5)}. \tag{5.6.3}$$

If $\eta = \infty$, then the inequality in (5.6.2) becomes an equality. Moreover, we have

$$\sup\Big\{\varLambda(\{\varDelta_i\}_{i=1}^{\eta})\colon \{\varDelta_i\}_{i=1}^{\eta} \text{ satisfies (5.2.5)}, \eta\in\mathbb{N}\Big\} = \int_0^\infty \frac{t-1}{t}\mathrm{d}\tau(t). \tag{5.6.4}$$

Proof. We begin with two observations. First, if $\{\varDelta_i\}_{i=1}^{\eta}$ is a partition of $\operatorname{supp}(\tau)$, then by Lemma 2.1.1, $\int_{\varDelta_i} t(t-1)\mathrm{d}\tau(t) \in (0,\infty)$ for every $i \in J_\eta$. Second, due to the Cauchy–Schwarz inequality, the following two inequalities hold

$$\Big(\sum_{i=1}^{\eta} x_i\Big)^2 \leqslant \Big(\sum_{i=1}^{\eta} y_i\Big)\Big(\sum_{i=1}^{\eta} \frac{x_i^2}{y_i}\Big), \quad \{x_i\}_{i=1}^{\eta} \subseteq \mathbb{R}_+, \ \{y_i\}_{i=1}^{\eta} \subseteq (0,\infty), \tag{5.6.5}$$

$$\Big(\int_{\varDelta} (t-1)\mathrm{d}\tau(t)\Big)^2 \leqslant \int_{\varDelta} t(t-1)\mathrm{d}\tau(t) \int_{\varDelta} \frac{t-1}{t}\mathrm{d}\tau(t), \quad \varDelta \in \mathfrak{B}(\mathbb{R}). \tag{5.6.6}$$

Therefore, if $\{\varDelta_i\}_{i=1}^{\eta}$ is a partition of $\operatorname{supp}(\tau)$, then

$$\begin{aligned}
\frac{\big(\int_0^\infty (t-1)\mathrm{d}\tau(t)\big)^2}{\int_0^\infty t(t-1)\mathrm{d}\tau(t)} &= \frac{\big(\sum_{i=1}^{\eta}\int_{\varDelta_i} (t-1)\mathrm{d}\tau(t)\big)^2}{\sum_{i=1}^{\eta}\int_{\varDelta_i} t(t-1)\mathrm{d}\tau(t)} \\
&\overset{(5.6.5)}{\leqslant} \sum_{i=1}^{\eta} \frac{(\int_{\varDelta_i}(t-1)\mathrm{d}\tau(t))^2}{\int_{\varDelta_i} t(t-1)\mathrm{d}\tau(t)} \overset{(5.6.3)}{=} \varLambda(\{\varDelta_i\}_{i=1}^{\eta}) \\
&\overset{(5.6.6)}{\leqslant} \sum_{i=1}^{\eta} \int_{\varDelta_i} \frac{t-1}{t}\mathrm{d}\tau(t) \\
&= \int_0^\infty \frac{t-1}{t}\mathrm{d}\tau(t).
\end{aligned} \tag{5.6.7}$$

This proves (5.6.2) and the inequality "$\geqslant$" in (5.6.1). To show the reverse inequality, we may assume that $\eta \geqslant 2$. Let us define for every $k \in \mathbb{N}$ a partition $\{\varDelta_{k,i}\}_{i=1}^{\eta}$ of $\operatorname{supp}(\tau)$ by

$$\varDelta_{k,i} = \begin{cases} \{\theta_1, \ldots, \theta_k\} & \text{if } i = 1, \\ \{\theta_{i+k-1}\} & \text{if } 2 \leqslant i < \eta, \\ \{\theta_{\eta+k-1}, \theta_{\eta+k}, \ldots\} & \text{if } \eta < \infty \text{ and } i = \eta, \end{cases}$$

where $\{\theta_j\}_{j=1}^\infty$ is a strictly increasing sequence such that $\operatorname{supp}(\beta) = \{\theta_1, \theta_2, \ldots\}$. By Lemma 2.1.1 such a sequence exists and $\lim_{j\to\infty} \theta_j = \infty$. Then

$$\begin{aligned} \varLambda(\{\varDelta_{k,i}\}_{i=1}^\eta) &= \frac{(\int_0^{\theta_k} (t-1)\mathrm{d}\tau(t))^2}{\int_0^{\theta_k} t(t-1)\mathrm{d}\tau(t)} + \sum_{i=2}^{\eta} \frac{(\int_{\varDelta_{k,i}} (t-1)\mathrm{d}\tau(t))^2}{\int_{\varDelta_{k,i}} t(t-1)\mathrm{d}\tau(t)} \\ &\overset{(5.6.6)}{\leqslant} \frac{(\int_0^{\theta_k} (t-1)\mathrm{d}\tau(t))^2}{\int_0^{\theta_k} t(t-1)\mathrm{d}\tau(t)} + \sum_{i=2}^{\eta} \int_{\varDelta_{k,i}} \frac{t-1}{t}\mathrm{d}\tau(t) \\ &= \frac{(\int_0^{\theta_k} (t-1)\mathrm{d}\tau(t))^2}{\int_0^{\theta_k} t(t-1)\mathrm{d}\tau(t)} + \int_{\theta_{k+1}}^{\infty} \frac{t-1}{t}\mathrm{d}\tau(t), \quad k \in \mathbb{N}. \end{aligned}$$

Hence, by applying the Lebesgue monotone and dominated convergence theorems, we get the inequality "$\leqslant$" in (5.6.1). This completes the proof of (5.6.1).

To prove the remaining part of the conclusion, we define for every $\eta \in \mathbb{N}_2 \cup \{\infty\}$ a partition $\{\tilde{\varDelta}_{\eta,i}\}_{i=1}^\eta$ of $\operatorname{supp}(\tau)$ by (compare with (5.5.3))

$$\tilde{\varDelta}_{\eta,i} = \begin{cases} \{\theta_i\} & \text{if } i \in J_{\eta-1}, \\ \{\theta_\eta, \theta_{\eta+1}, \ldots\} & \text{if } \eta < \infty \text{ and } i = \eta. \end{cases}$$

Then

$$\begin{aligned} \varLambda(\{\tilde{\varDelta}_{\eta,i}\}_{i=1}^\eta) = \sum_{i=1}^{\eta-1} \frac{\theta_i - 1}{\theta_i}\tau(\{\theta_i\}) + \frac{(\int_{\theta_\eta}^\infty (t-1)\mathrm{d}\tau(t))^2}{\int_{\theta_\eta}^\infty t(t-1)\mathrm{d}\tau(t)} \\ \geqslant \int_0^{\theta_{\eta-1}} \frac{t-1}{t}\mathrm{d}\tau(t), \quad \eta \in \mathbb{N}_2. \end{aligned}$$

Applying the Lebesgue monotone convergence theorem and using (5.6.2), we get (5.6.4). In the case of $\eta = \infty$, we have

$$\varLambda(\{\tilde{\varDelta}_{\infty,i}\}_{i=1}^\infty) = \int_0^\infty \frac{t-1}{t}\mathrm{d}\tau(t). \tag{5.6.8}$$

This completes the proof of the proposition. □

Remark 5.6.2. It is worth pointing out that under the assumptions of Proposition 5.6.1, the following two inequalities hold

$$\frac{\left(\int_{\Delta}(t-1)\mathrm{d}\tau(t)\right)^2}{\int_{\Delta} t(t-1)\mathrm{d}\tau(t)} \leqslant \sum_{i=1}^{n} \frac{(\int_{\Delta_i}(t-1)\mathrm{d}\tau(t))^2}{\int_{\Delta_i} t(t-1)\mathrm{d}\tau(t)} \leqslant \int_{\Delta} \frac{t-1}{t}\mathrm{d}\tau(t) \tag{5.6.9}$$

whenever $\{\Delta_i\}_{i=1}^n$ is a finite or infinite partition of a nonempty subset Δ of $\operatorname{supp}(\tau)$ (argue as in the proof of (5.6.7)). We will show that if, in addition, $\operatorname{card}(\Delta_i) \geqslant 2$ for at least one $i \in J_n$, then the second inequality (counting from the left) in (5.6.9) is strict. Indeed, otherwise taking a close look at (5.6.7) shows that the Cauchy–Schwarz inequality (5.6.6) becomes an equality for every $\Delta \in \{\Delta_i \colon i \in J_n\}$. As a consequence, for every $i \in J_n$, there exists $\alpha_i \in (0, \infty)$ such that $\sqrt{(t-1)t} = \alpha_i \sqrt{\frac{t-1}{t}}$ for every $t \in \Delta_i$ (recall that $\Delta_i \subseteq \operatorname{supp}(\tau)$). This implies that $\operatorname{card}(\Delta_i) = 1$ for every $i \in J_n$, which contradicts our assumption.

It follows from the previous paragraph that for every $\eta \in \mathbb{N} \cup \{\infty\}$ and for every partition $\{\Delta_i\}_{i=1}^{\eta}$ of $\operatorname{supp}(\tau)$ such that $\sup\{\operatorname{card}(\Delta_i) \colon i \in J_\eta\} \geqslant 2$ we have

$$\Lambda(\{\Delta_i\}_{i=1}^{\eta}) < \int_0^{\infty} \frac{t-1}{t}\mathrm{d}\tau(t). \tag{5.6.10}$$

However, if $\sup\{\operatorname{card}(\Delta_i) \colon i \in J_\eta\} = 1$ (and consequently $\eta = \infty$), then the strict inequality in (5.6.10) becomes an equality (see (5.6.8)).

Applying (5.6.4) and (5.6.9), we deduce that the sequence

$$\left\{\sup\left\{\Lambda(\{\Delta_i\}_{i=1}^{\eta}) \colon \{\Delta_i\}_{i=1}^{\eta} \text{ satisfies (5.2.5)}\right\}\right\}_{\eta=1}^{\infty}$$

is monotonically increasing to $\int_0^\infty \frac{t-1}{t}\mathrm{d}\tau(t)$. Hence, there arises the question whether or not there exists $\eta_0 \in \mathbb{N}$ at which the above sequence stabilizes.

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