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S-Noetherian Rings and Their Extensions

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Abstract. Let R be an associative ring with identity, S a multiplicative subset of R, and M a right R-module. Then M is called an S-Noetherian module if for each submodule N of M, there exist an element $s \in S$ and a finitely generated submodule F of M such that $Ns \subseteq F \subseteq N$, and R is called a right S-Noetherian ring if R_R is an S-Noetherian module. In this paper, we study some properties of right S-Noetherian rings and S-Noetherian modules. Among other things, we study Ore extensions, skew-Laurent polynomial ring extensions, and power series ring extensions of S-Noetherian rings.

1. Introduction

Due to the importance of Noetherian rings, there were several attempts to generalize the concept of Noetherian rings in order to extend the well-known results for Noetherian rings. One of them is the notion of S-Noetherian rings. A study of S-Noetherian rings was oriented in commutative algebra. In [3], Anderson, Kwak, and Zafrullah introduced the concept of "almost finitely generated" to study Querré's characterization of divisorial ideals in integrally closed polynomial rings. Later, Anderson and Dumitrescu in [1] abstracted this notion to any commutative ring and defined a general concept of Noetherian rings. Let R be an associative ring with identity, S a (not necessarily saturated) multiplicative subset of R, and M a unitary R-module. A commutative ring R is called an *S-Noetherian ring* if each ideal of R is S-finite, i.e., for each ideal I of R, there exist an element $s \in S$ and a finitely generated ideal J of R such that $Is \subseteq J \subseteq I$. They defined an R-module M to be *S-finite* if there exist an element $s \in S$ and a finitely generated R-submodule F of M such that $Ms \subseteq F$. Also, M is said to be *S-Noetherian* if each submodule of M is S-finite. It is clear that if S is a multiplicative subset consisting of units of R, then an S-Noetherian ring is Noetherian and an S-finite R-module is a finitely generated R-module. In [1], the authors gave a number of S-variants of well-known results for Noetherian rings: S-versions

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of Cohen's result, Eakin-Nagata theorem, and Hilbert basis theorem under some additional conditions. In [11], Liu studied when the ring of generalized power series is S-Noetherian. In [10], Lim and Oh completely classified when the composite ring extensions of the forms $D + E[\Gamma^*]$ and $D + [\![E^{\Gamma^*,\leq}]\!]$ are S-Noetherian. In [9], the authors studied S-Noetherian properties via the amalgamated algebra along an ideal. In [8], the local-global property of S-Noetherian rings was investigated. For more details on commutative S-Noetherian rings, the readers can refer to [1, 8–11].

In this paper, we investigate to study S-Noetherian rings and S-Noetherian modules. In Section 2, we introduce the notions of right S-Noetherian rings and S-Noetherian modules over general rings (including noncommutative rings), and study some extensions of right S-Noetherian rings. Among other things, we give an equivalent condition for the trivial extension to be S-Noetherian. We show that R is a right S-Noetherian ring and M is S-finite as a right R-module if and only if the trivial extension $T(R, M)$ is a right $T(S, M)$-Noetherian ring. In Section 3, we study Ore extensions, power series ring extensions, and composite ring extensions of right S-Noetherian rings. More precisely, we show that if R is a right S-Noetherian ring, S is a right σ-anti-Archimedean subset for an automorphism σ of R, and X is an indeterminate over R, then the Ore extension $R[X; \sigma, \delta]$ is a right S-Noetherian ring. We also prove that if R is a right S-Noetherian ring and S is a right anti-Archimedean subset of R consisting of regular elements, then the power series ring $R[\![X]\!]$ is a right S-Noetherian ring.

As mentioned before, some properties of Noetherian rings hold in S-Noethrian rings. But there are some properties of Noetherian rings which do not hold in S-Noetherian rings. In this paper, we give some examples to show that Noetherian and S-Noetherian rings have different algebraic structures.

Throughout this paper, all rings are (general) associative rings with identity, all modules are unitary right modules, and multiplicative subsets need not contain the identity element of a based ring.

2. S-Noetherian rings and modules

In this section, we define right S-Noetherian rings and S-Noetherian modules, and study some properties of them. To do this, we first give definitions of right S-Noetherian rings and S-Noetherian modules.

Definition 2.1. Let R be an associative ring with identity and S a multiplicative subset of R.

(1) A right (resp., left) ideal A of R is S-*finite* if there exist an element $s \in S$ and a finitely generated right (resp., left) ideal F of R such that $As \subseteq F \subseteq A$ (resp.,

$sA \subseteq F \subseteq A$).

(2) R is a *right* (resp., *left*) *S-Noetherian ring* if every right (resp., left) ideal of R is S-finite.

A ring R is an *S-Noetherian ring* if R is both left and right S-Noetherian.

Clearly, every right Noetherian ring is right S-Noetherian for any multiplicative subset S. We provide examples of right S-Noetherian rings in Examples 2.2 and 2.3. For a ring R and a positive integer n, $\mathrm{Mat}_n(R)$ means the ring of $n \times n$ matrices over R and $\mathcal{I}_n(R) = \{rI_n \mid r \in R\}$, where I_n is the $n \times n$ identity matrix of $\mathrm{Mat}_n(R)$.

Example 2.2. (1) Every (commutative) Noetherian ring is right S-Noetherian for any multiplicative subset S. In particular, every field and division ring are right S-Noetherian for any multiplicative subset S.

(2) Let R be a right Noetherian ring and T a subring of $\mathrm{Mat}_n(R)$ containing $\mathcal{I}_n(R) \cong R$. Then T is a right Noetherian ring [4, Proposition 1.7]; so T is a right S-Noetherian ring for any multiplicative subset S of T. In particular, $\mathrm{Mat}_n(R)$ is a right S-Noetherian ring for any multiplicative subset S of $\mathrm{Mat}_n(R)$.

The next example shows that a right S-Noetherian ring need not be right Noetherian. For a ring R and a positive integer n, $\mathrm{UTM}_n(R)$ means the ring of $n \times n$ upper triangular matrices over R.

Example 2.3. (1) Let $R = \prod_{p_n \in \mathcal{P}} \mathbb{Z}_{p_n}$ and $R^* = \prod_{p_n \in \mathcal{P}} \mathbb{Z}^*_{p_n}$, where $\mathcal{P}$ is the set of prime integers and $\mathbb{Z}^*_{p_n} = \mathbb{Z}_{p_n} \setminus \{0\}$. Consider a nonzero element a in $\bigoplus_{p_n \in \mathcal{P}} \mathbb{Z}_{p_n}$. Then $S = aR^*$ is a multiplicative subset of R. Hence R is right S-Noetherian, because As is a finitely generated right ideal of R for all right ideals A of R and for all $s \in S$. But $(\mathbb{Z}_{p_1}, 0, 0, \ldots) \subsetneq (\mathbb{Z}_{p_1}, \mathbb{Z}_{p_2}, 0, 0, \ldots) \subsetneq \cdots$ is an infinite ascending chain of right ideals in R. Thus R is not a right Noetherian ring.

(2) Let $R = \prod_{n=1}^{\infty} \mathrm{UTM}_2(\mathbb{Z})$ be the infinite direct product of $\mathrm{UTM}_2(\mathbb{Z})$ and let $T = \prod_{n=1}^{\infty} \mathrm{P}^*$, where $\mathrm{P}^* = \left(\begin{smallmatrix} \mathbb{Z}^* & 0 \\ 0 & \mathbb{Z}^* \end{smallmatrix}\right)$ and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. Let $0 \neq a \in \bigoplus_{n=1}^{\infty} \mathrm{P}$, where $\mathrm{P} = \left(\begin{smallmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$. Then $S = aT$ is a multiplicative subset of R. Hence R is right S-Noetherian, because it is easy to show that As is contained in a finitely generated right ideal of R for all right ideals A of R and for all $s \in S$. But $(\mathrm{UTM}_2(\mathbb{Z}), 0, 0, \ldots) \subsetneq (\mathrm{UTM}_2(\mathbb{Z}), \mathrm{UTM}_2(\mathbb{Z}), 0, 0, \ldots) \subsetneq \cdots$ is an infinite ascending chain of right ideals in R. Thus R is not a right Noetherian ring.

(3) Let D be a division ring, $\boldsymbol{X} = \{X_i \mid i \in \mathbb{N}\}$ a set of indeterminates over D, and $R = D[\![\boldsymbol{X}]\!] / \langle X_i X_j \mid i \neq j \rangle$ the factor ring of $D[\![\boldsymbol{X}]\!]$ by the ideal $\langle X_i X_j \mid i \neq j \rangle$. Let $\overline{X_i}$ be the image of X_i under R. Then $\langle \overline{X_1} \rangle \subsetneq \langle \overline{X_1}, \overline{X_2} \rangle \subsetneq \cdots$ is an ascending chain of ideals of R; so R is not a Noetherian ring. Fix an $i \in \mathbb{N}$ and set $S = \left\{ \overline{X_i}^{\,n} \mid n \in \mathbb{N} \right\}$. Then an easy calculation shows that R is an S-Noetherian ring.

The following result shows that for a multiplicative subset S, a right S-Noetherian ring can be right Noetherian.

Proposition 2.4. *Let S be a multiplicative set of right invertible elements of a ring R. If R is a right S-Noetherian ring, then R is right Noetherian.*

Proof. Let A be a right ideal of R. Since R is a right S-Noetherian ring, $As \subseteq F \subseteq A$ for some $s \in S$ and some finitely generated right ideal F of R. Since $A = Ass^{-1} \subseteq Fs^{-1} \subseteq F$, $A = F$; so A is finitely generated. Thus R is a right Noetherian ring. □

Recall that a right (left) ideal A of a ring is *nilpotent* if there exists a positive integer n such that $A^n = (0)$; and a right (left) ideal of a ring is *nil* if each of its elements is nilpotent. It is well known as Levitzki's theorem that every nil one-sided ideal of a right Noetherian ring is necessarily nilpotent. However, the next example shows that a nil one-sided ideal of a right S-Noetherian ring need not be nilpotent.

Example 2.5. Let p be a prime integer, $T = \prod_{n=1}^{\infty} \mathbb{Z}_{p^n}$, and $R = \langle \bigoplus_{n=1}^{\infty} \mathbb{Z}_{p^n}, 1_T \rangle$, where $1_T = (1, 1, 1, \ldots)$, a subring of T. Let $s = (1, p, p, 0, 0, 0, \ldots)$ and $S = \{s, s^2, s^3\}$. Then S is a multiplicative subset of R.

(1) Let A be an ideal of R. Note that every element of As is of the form $(a_1, a_2, a_3, 0, 0, \ldots)$; so there exists a finitely generated ideal F of R such that $As \subseteq F \subseteq A$. Thus R is an S-Noetherian ring.

(2) Let $B = \bigoplus_{n=1}^{\infty} p\mathbb{Z}_{p^n}$ be an ideal of R and $b := (pb_1, pb_2, \ldots, pb_k, 0, 0, \ldots) \in B$. Then $(b)^{p^k} = (0)$; so B is nil. But there is no positive integer m such that $B^m = (0)$. Thus B is not nilpotent.

The following example shows that every subring of a right S-Noetherian ring need not be right S-Noetherian. We give an example of a right S-Noetherian ring which is not left S-Noetherian. In addition, Example 2.6(3) illustrates that a ring R is right S_1-Noetherian but not right S_2-Noetherian for distinct multiplicative subsets S_1 and S_2 of R.

Example 2.6. (1) Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ and $S = \left\{ \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{Q} \right\}$ a multiplicative subset of R. Since R is right Noetherian [4, Exercise 1A(a)], R is right S-Noetherian. Now, consider $T = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}$ as a subring of R and $A = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}$ as a right ideal of T. Then for any $s \in S$, a right ideal B of T such that $As \subseteq B \subseteq A$ must be A. Since $\mathbb{Q}$ is not finitely generated as a right $\mathbb{Z}$-module, A is not a finitely generated right ideal. Thus T is not a right S-Noetherian ring.

(2) Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ and $S = \left\{ \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{Q} \right\}$ a multiplicative subset of R. Note that R is right S-Noetherian as in (1). Consider a left ideal $A = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$ of R. Suppose to the

contrary that A is S-finite. Then there exist an element $s \in S$ and a finitely generated left ideal F of R such that $sA \subseteq F \subseteq A$. Note that $sA = A$; so $A = F$. However it is impossible, because A is not a finitely generated left ideal of R. Hence A is not S-finite, and thus R is not left S-Noetherian.

(3) Let $R = \left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$ and $S_1 = \{\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) \mid a \in \mathbb{N}\}$ a multiplicative subset of R. Since all right ideals of R are of the forms $A_n := \left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & n\mathbb{Z} \end{smallmatrix}\right)$ for all $0 \leq n \in \mathbb{Z}$ and $A_G := \left\{\left(\begin{smallmatrix} 0 & q \\ 0 & z \end{smallmatrix}\right) \mid (q, z) \in G, \text{ which is a subgroup of } \mathbb{Q} \oplus \mathbb{Z}\right\}$ [5, Example 2.23], we can easily check that R is right S_1-Noetherian. On the other hand, take $S_2 = \left\{\left(\begin{smallmatrix} 0 & b \\ 0 & 1 \end{smallmatrix}\right) \mid b \in \mathbb{Q}\right\}$, which is a multiplicative subset of R. From (2), it is easy to see that R is not right S_2-Noetherian.

Proposition 2.7. *Let S be a multiplicative subset of a ring R and $A_1, \ldots, A_n$ be S-finite right ideals of R. Then $A_1 + \cdots + A_n$ is S-finite.*

Proof. Suppose that $A_1, \ldots, A_n$ are S-finite right ideals of R. Then for each $k \in \{1, \ldots, n\}$, there exist an element $s_k \in S$ and a finitely generated right ideal F_k of R such that $A_k s_k \subseteq F_k \subseteq A_k$. Let $s = s_1 \cdots s_n$. Then for each $k \in \{1, \ldots, n\}$, $A_k s \subseteq A_k s_k s_{k+1} \cdots s_n \subseteq F_k s_{k+1} \cdots s_n \subseteq F_k \subseteq A_k$; so we have

$$(A_1 + \cdots + A_n)s \subseteq F_1 + \cdots + F_n \subseteq A_1 + \cdots + A_n.$$

Since $F_1 + \cdots + F_n$ is a finitely generated right ideal of R, $A_1 + \cdots + A_n$ is S-finite. $\square$

The next results show that the finite direct sum preserves S-Noetherian properties but the infinite direct product does not.

Proposition 2.8. *Let $S_1, \ldots, S_n$ be multiplicative subsets of rings $R_1, \ldots, R_n$, respectively. Then the following conditions are equivalent.*

(1) *For each $i \in \{1, \ldots, n\}$, R_i is a right S_i-Noetherian ring.*

(2) *The direct sum $\prod_{i=1}^n R_i$ is a right $(\prod_{i=1}^n S_i)$-Noetherian ring.*

Example 2.9. Let $R = \prod_{n=1}^\infty \mathbb{Z}$ be the infinite direct product of $\mathbb{Z}$, $S = \prod_{n=1}^\infty \{1\}$ a multiplicative subset of R, and $A = \bigoplus_{n=1}^\infty \mathbb{Z}$ the direct sum of $\mathbb{Z}$. Then A is a nonfinitely generated right ideal of R and $A(1, 1, \ldots) = A$; so A is not S-finite. Thus R is not a right S-Noetherian ring.

Now, we define a module theoretic analogue of the S-Noetherian property for rings.

Definition 2.10. Let R be a (not necessarily commutative) ring with identity, S a multiplicative subset of R, and M a right R-module.

(1) A submodule N of M is *S-finite* if there exist an element $s \in S$ and a finitely generated submodule F of M such that $Ns \subseteq F \subseteq N$.

(2) M is *S-Noetherian* if every submodule of M is S-finite.

Remark 2.11. (1) Every submodule of an S-Noetherian module is S-Noetherian.

(2) If $S_1 \subseteq S_2$ are multiplicative subsets of a ring, then any S_1-Noetherian module is S_2-Noetherian.

(3) A homomorphic image of an S-Noetherian module is S-Noetherian.

The next example shows that the converse of Remark 2.11(2) does not hold true, in general.

Example 2.12. Let R and S be as in Example 2.3(1). Consider $S_1 = \{1_R\}$ and $S_2 = S_1 \cup S$, where $1_R = (1, 1, 1, \ldots)$. Then R is right S_2-Noetherian but not right S_1-Noetherian.

Proposition 2.13. *Let $S_1 \subseteq S_2$ be multiplicative subsets of a ring R such that for any $s \in S_2$, there exists an element $r \in S_2$ satisfying $sr \in S_1$. If R is a right S_2-Noetherian ring, then R is a right S_1-Noetherian ring.*

Proof. Let A be a right ideal of R. Since R is a right S_2-Noetherian ring, we can find $s \in S_2$ and a finitely generated right ideal F of R such that $As \subseteq F \subseteq A$. By the assumption, $sr \in S_1$ for some $r \in S_2$; so $Asr \subseteq Fr \subseteq F \subseteq A$. Hence A is S_1-finite. Thus R is right S_1-Noetherian. $\square$

Let R be a ring. It is well known that if $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules, then M is a Noetherian module if and only if so are M' and M'' [2, Proposition 10.12]. It was also shown that if L and M/L are Noetherian as right R-modules, then M is Noetherian as a right R-module [6, 1.20]; and a finitely generated module over a right Noetherian ring is a Noetherian module [2, Proposition 10.19]. (Recall that a *Noetherian module* is a module which satisfies the ascending chain condition on submodules, or equivalently, every submodule is finitely generated.) We extend these above results to S-Noetherian R-modules, where S is a multiplicative subset of R.

Lemma 2.14. *Let R be a ring, S a multiplicative subset of R, and M a right R-module. Then the following assertions hold.*

(1) *If M is S-Noetherian and N is a submodule of M, then M/N is S-Noetherian.*

(2) *If N is an S-Noetherian submodule of M such that M/N is S-Noetherian, then M is S-Noetherian.*

(3) *For a short exact sequence of right R-modules $0 \to M' \to M \to M'' \to 0$, M is S-Noetherian if and only if M' and M'' are S-Noetherian.*

(4) *If R is a right S-Noetherian ring and M is finitely generated, then M is S-Noetherian.*

(5) *If R is a right S-Noetherian ring and M is S-finite, then M is S-Noetherian.*

(6) *Let $R \subseteq E$ be an extension of rings. If R is a right S-Noetherian ring and E is S-finite as a right R-module, then E is a right S-Noetherian ring.*

Proof. (1) Let L be a submodule of M containing N. Since M is S-Noetherian, there exist an element $s \in S$ and a finitely generated submodule F of L such that $Ls \subseteq F$; so $(L/N)s \subseteq (F+N)/N \subseteq L/N$. Since $(F+N)/N$ is finitely generated, L/N is S-finite. Thus M/N is S-Noetherian.

(2) Let L be a submodule of M. If $L \subseteq N$, then there is nothing to prove, because N is S-Noetherian. Suppose that $L \nsubseteq N$. Since $(L+N)/N \cong L/(L \cap N)$ and M/N is S-Noetherian, $L/(L \cap N)$ is S-finite; so there exist $s \in S$ and $\ell_1, \ldots, \ell_n \in L$ such that $(L/(L \cap N))s \subseteq (\ell_1 + (L \cap N))R + \cdots + (\ell_n + (L \cap N))R \subseteq L/(L \cap N)$. Let $F_1 = \ell_1 R + \cdots + \ell_n R$. Then $(L/(L \cap N))s \subseteq (F_1 + (L \cap N))/(L \cap N) \subseteq L/(L \cap N)$. Also, since N is S-Noetherian, there exist an element $t \in S$ and a finitely generated submodule F_2 of $L \cap N$ such that $(L \cap N)t \subseteq F_2$. Therefore we obtain

$$Lst \subseteq (F_1 + (L \cap N))t = F_1 t + (L \cap N)t \subseteq F_1 + F_2 \subseteq L.$$

Note that $F_1 + F_2$ is finitely generated. Thus L is S-finite, which shows that M is S-Noetherian.

(3) Let $\phi \colon M' \to M$ and $\psi \colon M \to M''$ be R-module homomorphisms in the short exact sequence $0 \to M' \to M \to M'' \to 0$.

($\Rightarrow$) Suppose that M is S-Noetherian. Since M' is isomorphic to a submodule $\phi(M')$ of M, M' is S-Noetherian by Remark 2.11(1). Also, since M'' is isomorphic to $M/\operatorname{Ker}(\psi)$, M'' is S-Noetherian by (1).

($\Leftarrow$) Suppose that M' and M'' are S-Noetherian. Since $M' \cong \operatorname{Im}(\phi) = \operatorname{Ker}(\psi)$ and $M'' \cong M/\operatorname{Ker}(\psi)$, M is S-Noetherian by (2).

(4) Let M be a finitely generated module. We use the induction on the number of generators of M. Assume that $M = mR$ for some $m \in M$ and let N be a submodule of M. Set $A = \{r \in R \mid mr \in N\}$. Then A is a right ideal of R. Since R is right S-Noetherian, $As \subseteq a_1 R + \cdots + a_k R$ for some $s \in S$ and $a_1, \ldots, a_k \in A$. Let $n \in N$ be arbitrary. Then $n = mr$ for some $r \in A$; so $ns \in ma_1 R + \cdots + ma_k R$. Hence $Ns \subseteq ma_1 R + \cdots + ma_k R \subseteq N$, which says that N is S-finite.

Next, assume that every right R-module generated by fewer than n of its elements is S-Noetherian. Let $m_1, \ldots, m_n$ be minimal generators of M. Then by the induction hypothesis, $m_n R$ and $m_1 R + \cdots + m_{n-1} R$ are S-Noetherian. Consider a short exact sequence

$$0 \to m_n R \xrightarrow{\iota} M = m_1 R + \cdots + m_n R \xrightarrow{\rho} \frac{m_1 R + \cdots + m_{n-1} R}{(m_1 R + \cdots + m_{n-1} R) \cap m_n R} \to 0$$

of right modules, where ι is the natural inclusion and ρ is a surjective map given by $\rho(m_1r_1 + \cdots + m_nr_n) = (m_1r_1 + \cdots + m_{n-1}r_{n-1}) + ((m_1R + \cdots + m_{n-1}R) \cap m_nR)$ for all $r_1, \ldots, r_n \in R$. By the induction hypothesis and (1), $\frac{m_1R+\cdots+m_{n-1}R}{(m_1R+\cdots+m_{n-1}R)\cap m_nR}$ is S-Noetherian. Thus by (3), M is S-Noetherian.

(5) Suppose that M is S-finite. Then there exist an element $s \in S$ and a finitely generated submodule F of M such that $Ms \subseteq F$. Note that by (4), F is S-Noetherian. Let N be a submodule of M. Since NsR is a submodule of F, NsR is S-finite; so there exist an element $t \in S$ and a finitely generated submodule P of NsR such that $NsRt \subseteq P$. Therefore $Nst \subseteq P \subseteq N$. Hence N is S-finite, and thus M is S-Noetherian.

(6) Let A be a right ideal of E. Since E contains R, A is a submodule of E as a right R-module. Note that by (5), E is S-Noetherian as a right R-module; so we can find $s \in S$ and $a_1, \ldots, a_n \in A$ such that $As \subseteq a_1R + \cdots + a_nR \subseteq A$. Therefore $As \subseteq a_1E + \cdots + a_nE \subseteq A$, because A is a right ideal of E. Hence A is S-finite. Thus E is a right S-Noetherian ring. □

For a ring R and a positive integer n, we denote that the *standard* $n \times n$ *matrix units* in $\mathrm{Mat}_n(R)$ are the matrices e_{ij} for all $i, j = 1, \ldots, n$ such that e_{ij} has 1 for the (i, j)-entry and 0 elsewhere.

Proposition 2.15. *If R is a right S-Noetherian ring and n is a positive integer, then $\mathrm{Mat}_n(R)$ and $\mathrm{UTM}_n(R)$ are S-Noetherian as right R-modules.*

Proof. Since $\mathrm{Mat}_n(R)$ (resp., $\mathrm{UTM}_n(R)$) is generated by $\{e_{ij} \mid 1 \leq i, j \leq n\}$ (resp., $\{e_{ij} \mid 1 \leq i \leq j \leq n\}$) as a right R-module, $\mathrm{Mat}_n(R)$ (resp., $\mathrm{UTM}_n(R)$) is finitely generated. Thus the result is an immediate consequence of Lemma 2.14(4). □

For a multiplicative subset S of a ring R and a positive integer n, $\mathcal{D}_n(S)$ means the multiplicative set of $n \times n$ diagonal matrices with entries in S.

Proposition 2.16. *Let R be a right S-Noetherian ring and n a positive integer. Then the following assertions hold.*

(1) $\mathrm{UTM}_n(R)$ *is a right $\mathcal{D}_n(S)$-Noetherian ring.*

(2) $\mathrm{Mat}_n(R)$ *is a right $\mathcal{D}_n(S)$-Noetherian ring.*

Proof. (1) Assume that R is a right S-Noetherian ring. Then by Proposition 2.15, $\mathrm{UTM}_n(R)$ is S-Noetherian as a right R-module; so by Remark 2.11(2), $\mathrm{UTM}_n(R)$ is $\mathcal{D}_n(S)$-Noetherian as a right $\mathcal{I}_n(R)$-module. Let $\mathcal{A}$ be a right ideal of $\mathrm{UTM}_n(R)$. Then $\mathcal{A}$ is a right $\mathcal{I}_n(R)$-submodule of $\mathrm{UTM}_n(R)$; so there exist $\mathrm{diag}\{s_1, s_2, \ldots, s_n\} \in \mathcal{D}_n(S)$ and

$M_1, \ldots, M_k \in \mathrm{UTM}_n(R)$ such that $\mathcal{A}\,\mathrm{diag}\,\{s_1, s_2, \ldots, s_n\} \subseteq M_1\mathcal{I}_n(R) + \cdots + M_k\mathcal{I}_n(R) \subseteq \mathcal{A}$, where

$$\mathrm{diag}\,\{s_1, s_2, \ldots, s_n\} := \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_n \end{pmatrix}.$$

Therefore $\mathcal{A}\,\mathrm{diag}\,\{s_1, s_2, \ldots, s_n\} \subseteq M_1\,\mathrm{UTM}_n(R) + \cdots + M_k\,\mathrm{UTM}_n(R) \subseteq \mathcal{A}$. Hence $\mathcal{A}$ is $\mathcal{D}_n(S)$-finite, and thus $\mathrm{UTM}_n(R)$ is a right $\mathcal{D}_n(S)$-Noetherian ring.

(2) Suppose that R is a right S-Noetherian ring. Since $\mathcal{I}_n(R) \subseteq \mathrm{UTM}_n(R)$, a similar argument as in the proof of (1) shows that $\mathrm{Mat}_n(R)$ is $\mathcal{D}_n(S)$-Noetherian as a right $\mathrm{UTM}_n(R)$-module. Hence $\mathrm{Mat}_n(R)$ is $\mathcal{D}_n(S)$-finite as a right $\mathrm{UTM}_n(R)$-module. Thus by Lemma 2.14(6), $\mathrm{Mat}_n(R)$ is a right $\mathcal{D}_n(S)$-Noetherian ring. □

For a multiplicative subset S of a ring R and a positive integer n, let

$$\mathcal{V}_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & 0 & \cdots & 0 & a_1 \end{pmatrix} : a_1, \ldots, a_n \in R \right\}$$

and $\mathcal{I}_n(S) := \{sI_n \mid s \in S\}$. Then it is easy to see that $\mathcal{I}_n(S)$ is a multiplicative subset of $\mathcal{V}_n(R)$.

Proposition 2.17. *Let S be a multiplicative subset of a ring R and n a positive integer. Then the following conditions are equivalent.*

(1) *R is a right S-Noetherian ring.*

(2) *$\mathcal{V}_n(R)$ is a right $\mathcal{I}_n(S)$-Noetherian ring.*

Proof. (1) $\Rightarrow$ (2): Note that $\mathcal{V}_n(R)$ is generated by $\left\{\sum_{i=1}^{n-k} e_{i,i+k} \mid k = 0, \ldots, n-1\right\}$ as a right R-module. Since R is a right S-Noetherian ring, $\mathcal{V}_n(R)$ is S-Noetherian by Lemma 2.14(4). Since $R \cong \mathcal{I}_n(R)$, $\mathcal{V}_n(R)$ is $\mathcal{I}_n(S)$-Noetherian as a right $\mathcal{I}_n(R)$-module. Let $\mathcal{A}$ be a right ideal of $\mathcal{V}_n(R)$. Then $\mathcal{A}$ is a right $\mathcal{I}_n(R)$-submodule of $\mathcal{V}_n(R)$; so there exist $sI_n \in \mathcal{I}_n(S)$ and $M_1, \ldots, M_k \in \mathcal{A}$ such that $\mathcal{A}sI_n \subseteq M_1\mathcal{I}_n(R) + \cdots + M_k\mathcal{I}_n(R) \subseteq \mathcal{A}$. Therefore $\mathcal{A}sI_n \subseteq M_1\mathcal{V}_n(R) + \cdots + M_k\mathcal{V}_n(R) \subseteq \mathcal{A}$. Hence $\mathcal{A}$ is $\mathcal{I}_n(S)$-finite, and thus $\mathcal{V}_n(R)$ is a right $\mathcal{I}_n(S)$-Noetherian ring.

(2) ⇒ (1): Let A be a right ideal of R. Then $\mathcal{V}_n(A)$ is a right ideal of $\mathcal{V}_n(R)$. Since $\mathcal{V}_n(R)$ is a right $\mathcal{I}_n(S)$-Noetherian ring, there exist an element $sI_n \in \mathcal{I}_n(S)$ and a finitely generated right ideal $\mathcal{F}$ of $\mathcal{V}_n(R)$ such that $\mathcal{V}_n(A)sI_n \subseteq \mathcal{F} \subseteq \mathcal{V}_n(A)$. Let F be the set of $(1,1)$-entries of elements in $\mathcal{F}$. Then F is a finitely generated right ideal of R and $As \subseteq F \subseteq A$. Hence A is S-finite, and thus R is a right S-Noetherian ring. □

Let R be a ring, n a positive integer, and $A \in \mathrm{Mat}_n(R)$. From [7, Section 1], let $RA = \{rA \mid r \in R\}$ and $V = \sum_{i=1}^{n-1} e_{i,i+1}$. Note that $\mathcal{V}_n(R) = RI_n + RV + \cdots + RV^{n-1}$; so the map $\rho\colon \mathcal{V}_n(R) \to R[X]/\langle X^n \rangle$ defined by $\rho(a_0 I_n + a_1 V + \cdots + a_{n-1}V^{n-1}) = \sum_{i=0}^{n-1} a_i X^i + \langle X^n \rangle$ is a ring isomorphism. Thus by Remark 2.11(3) and Proposition 2.17, we have

Corollary 2.18. *Let S be a multiplicative subset of a ring R, n a positive integer, and $T = \{s + \langle X^n \rangle \mid s \in S\}$. Then R is a right S-Noetherian ring if and only if $R[X]/\langle X^n \rangle$ is a right T-Noetherian ring.*

For a ring R and a positive integer n, let

$$\mathcal{H}_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} : a, a_{ij} \in R \right\}.$$

Proposition 2.19. *Let S be a multiplicative subset of a ring R and n a positive integer. Then the following conditions are equivalent.*

(1) *R is a right S-Noetherian ring.*

(2) *$\mathcal{H}_n(R)$ is a right $\mathcal{I}_n(S)$-Noetherian ring.*

Proof. (1) ⇒ (2): Suppose that R is a right S-Noetherian ring. Then by Proposition 2.17, $\mathcal{V}_n(R)$ is a right $\mathcal{I}_n(S)$-Noetherian ring. Since $\mathcal{H}_n(R)$ is generated as a right $\mathcal{V}_n(R)$-module by I_n and e_{ij}, where $1 \le i < j \le n$. Thus by Lemma 2.14(6), $\mathcal{H}_n(R)$ is a right $\mathcal{I}_n(S)$-Noetherian ring.

(2) ⇒ (1): The proof is similar to that of (2) ⇒ (1) in Proposition 2.17. □

Lemma 2.20. *Let $\phi\colon R \to D$ be a ring homomorphism and S a multiplicative subset of R. If R is a right S-Noetherian ring, then $\phi(R)$ is a right $\phi(S)$-Noetherian ring.*

For a ring R and an (R,R)-bimodule M, the *trivial extension* of R by M (or *Nagata's idealization* of M in R) is the ring $T(R,M) := R \oplus M$ with the usual addition and the following multiplication:

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$$

for all $(r_1, m_1), (r_2, m_2) \in T(R, M)$. It is easy to see that $T(R, M)$ is isomorphic to the ring of matrices of the form $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in M$; and if S is a multiplicative subset of R, then $T(S, M) := \{(s, m) \mid s \in S \text{ and } m \in M\}$ is a multiplicative subset of $T(R, M)$. Note that $T(R, R) = \mathcal{H}_2(R)$; so by Proposition 2.19, R is a right S-Noetherian ring if and only if the trivial extension $T(R, R)$ of R is a right $T(S, 0)$-Noetherian ring. We generalize this result.

Proposition 2.21. *Let S be a multiplicative subset of a ring R and M an (R, R)-bimodule. Then the following assertions are equivalent.*

(1) *R is a right S-Noetherian ring and M is S-finite as a right R-module.*

(2) *$T(R, M)$ is a right $T(S, 0)$-Noetherian ring.*

(3) *$T(R, M)$ is a right $T(S, M)$-Noetherian ring.*

Proof. (1) $\Rightarrow$ (2): Note that $T(S, 0)$ is a multiplicative subset of $T(R, 0)$. Since M is S-finite, there exist $s \in S$ and $m_1, \ldots, m_k \in M$ such that $Ms \subseteq m_1R + \cdots + m_kR$; so $T(R, M)(s, 0) \subseteq (1, 0)T(R, 0) + (0, m_1)T(R, 0) + \cdots + (0, m_k)T(R, 0)$. Hence $T(R, M)$ is $T(S, 0)$-finite as a right $T(R, 0)$-module. Note that $T(R, 0)$ is a right $T(S, 0)$-Noetherian ring; so by Lemma 2.14(5), $T(R, M)$ is $T(S, 0)$-Noetherian as a right $T(R, 0)$-module. Let A be a right ideal of $T(R, M)$. Then A is $T(S, 0)$-finite as a right $T(R, 0)$-module; so there exist $t \in S$ and $a_1, \ldots, a_n \in A$ such that $A(t, 0) \subseteq a_1T(R, 0) + \cdots + a_nT(R, 0) \subseteq A$. Hence $A(t, 0) \subseteq a_1T(R, M) + \cdots + a_nT(R, M) \subseteq A$, which indicates that A is $T(S, 0)$-finite. Thus $T(R, M)$ is a right $T(S, 0)$-Noetherian ring.

(2) $\Rightarrow$ (3): This implication follows directly from Remark 2.11(2).

(3) $\Rightarrow$ (1): Suppose that $T(R, M)$ is a right $T(S, M)$-Noetherian ring. Note that a map $\varphi\colon T(R, M) \to R$ given by $\varphi(r, m) = r$ for all $(r, m) \in T(R, M)$ is a ring epimorphism and $\varphi(T(S, M)) = S$; so by Lemma 2.20, R is a right S-Noetherian ring. Since $T(0, M)$ is a right ideal of $T(R, M)$, there exist $(s, m) \in T(S, M)$ and $m_1, \ldots, m_n \in M$ such that $T(0, M)(s, m) \subseteq (0, m_1)T(R, M) + \cdots + (0, m_n)T(R, M) \subseteq T(0, M)$. Hence $Ms \subseteq m_1R + \cdots + m_nR \subseteq M$, and thus M is S-finite as a right R-module. $\square$

We end this section with some applications of Proposition 2.21.

Example 2.22. (1) Let S be any multiplicative subset of $\mathbb{Z}$. Then $\mathbb{Q}$ is not S-finite as a right $\mathbb{Z}$-module. Thus by Proposition 2.21, $T(\mathbb{Z}, \mathbb{Q})$ is not a right $T(S, \mathbb{Q})$-Noetherian ring. More precisely, consider a right ideal $T(0, \mathbb{Q})$ of $T(\mathbb{Z}, \mathbb{Q})$. Then for any $(s, q) \in T(S, \mathbb{Q})$, a right ideal J of $T(\mathbb{Z}, \mathbb{Q})$ satisfying $T(0, \mathbb{Q})(s, q) \subseteq J \subseteq T(0, \mathbb{Q})$ is only $T(0, \mathbb{Q})$, because $T(0, \mathbb{Q})(s, q) = T(0, \mathbb{Q})$. But $T(0, \mathbb{Q})$ is not finitely generated. Thus $T(\mathbb{Z}, \mathbb{Q})$ is not a right $T(S, \mathbb{Q})$-Noetherian ring.

(2) Let R be a right S-Noetherian ring, S a multiplicative subset of R, and n a positive integer. Then by Proposition 2.16(1), $\mathrm{UTM}_n(R)$ is a right $\mathcal{D}_n(S)$-Noetherian ring. Also, as in the proof of Proposition 2.16(2), $\mathrm{Mat}_n(R)$ is $\mathcal{D}_n(S)$-finite as a right $\mathrm{UTM}_n(R)$-module. Thus by Proposition 2.21, $T(\mathrm{UTM}_n(R), \mathrm{Mat}_n(R))$ is a right $T(\mathcal{D}_n(S), \mathrm{Mat}_n(R))$-Noetherian ring.

(3) Let R be a right S-Noetherian ring, S a multiplicative subset of R and n a positive integer. Then by Proposition 2.17, $\mathcal{V}_n(R)$ is a right $\mathcal{I}_n(S)$-Noetherian ring. Also, as in the proof of Proposition 2.19, $\mathcal{H}_n(R)$ is $\mathcal{I}_n(S)$-finite as a right $\mathcal{V}_n(R)$-module. Thus by Proposition 2.21, $T(\mathcal{V}_n(R), \mathcal{H}_n(R))$ is a right $T(\mathcal{I}_n(S), \mathcal{H}_n(R))$-Noetherian ring.

3. Hilbert basis theorem

In this section, we study Ore extensions, power series ring extensions, and composite ring extensions of right S-Noetherian rings. To do this, we recall some definitions. Let R be a ring. Recall that for an endomorphism σ of R, a (*left*) *σ-derivation* on R is an additive map $\delta\colon R \to R$ such that $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for all $r, s \in R$. For a ring R with a ring endomorphism $\sigma\colon R \to R$ and a σ-derivation $\delta\colon R \to R$, the *Ore extension* $R[X; \sigma, \delta]$ of R is the ring obtained by giving the polynomial ring over R with the new multiplication $Xr = \sigma(r)X + \delta(r)$ for all $r \in R$. If $\delta = 0$, then we write $R[X; \sigma]$ for $R[X; \sigma, \delta]$ and call it the *Ore extension of endomorphism type* (or the *skew polynomial ring*). If $\sigma = I_R$ (the identity function), then we write $R[X; \delta]$ for $R[X; I_R, \delta]$ and call it the *Ore extension of derivation type* (or a *differential operator ring*).

A multiplicative subset S of a ring R is said to be a *right σ-anti-Archimedean subset for an automorphism σ of R* if $\bigcap_{n\geq 1} \left(\prod_{j=0}^{n-1} \sigma^{-n+j}(s)\right) R \cap S \neq \emptyset$ for every $s \in S$. In particular, if $\sigma = I_R$, then we simply call S *right anti-Archimedean*, i.e., $\bigcap_{n\geq 1} s^n R \cap S \neq \emptyset$ for every $s \in S$. Clearly, every multiplicative set consisting of right invertible elements is right σ-anti-Archimedean for any automorphism σ of R.

Theorem 3.1. *Let S be a right σ-anti-Archimedean subset for an automorphism σ of a ring R. If R is right S-Noetherian, then so is the Ore extension $R[X; \sigma, \delta]$.*

Proof. Let $\mathcal{A}$ be a right ideal of $R[X; \sigma, \delta]$ and A the right ideal of R consisting of zero and the leading coefficients of polynomials in $\mathcal{A}$. We want to show that $\mathcal{A}$ is S-finite. Since R is right S-Noetherian, there exist $s \in S$ and $a_1, \ldots, a_n \in A$ such that $As \subseteq \sum_{i=1}^n a_i R \subseteq A$. For each $i \in \{1, \ldots, n\}$, choose an element $f_i \in \mathcal{A}$ with the leading coefficient a_i and let d_i be the degree of f_i. Set $d = \max\{d_1, \ldots, d_n\}$. Then we may assume that $\deg(f_i) = d$ for all $i = 1, \ldots, n$. Now, let $\mathcal{B} = \mathcal{A} \cap (R + RX + \cdots + RX^{d-1})$. Note that by Lemma 2.14(4), $\mathcal{B}$ is S-finite as a right R-submodule of $R[X; \sigma, \delta]$; so there exist $t \in S$ and $h_1, \ldots, h_m \in \mathcal{B}$ such that $\mathcal{B}t \subseteq \sum_{k=1}^m h_k R$.

Let $f = \sum_{j=0}^{\ell} b_j X^j \in \mathcal{A}$ be arbitrary. If $\ell < d$, then $f \in \mathcal{B}$. Otherwise, $\ell \geq d$. Since $b_\ell \in A$, $b_\ell s = \sum_{i=1}^{n} a_i r_{i0}$ for some $r_{10}, \ldots, r_{n0} \in R$. Note that $X^\ell r = \sigma^\ell(r) X^\ell + \left(\sum_{i=0}^{\ell-1} \sigma^i \delta \sigma^{\ell-1-i}(r)\right) X^{\ell-1} + \cdots + \delta^\ell(r)$ for any $r \in R$. Hence we obtain

$$\begin{aligned} X^\ell \sigma^{-\ell}(s) &= \sigma^\ell \sigma^{-\ell}(s) X^\ell + \left(\sum_{i=0}^{\ell-1} \sigma^i \delta \sigma^{\ell-1-i} \sigma^{-\ell}(s)\right) X^{\ell-1} + \cdots + \delta^\ell \sigma^{-\ell}(s) \\ &= s X^\ell + \left(\sum_{i=0}^{\ell-1} \sigma^i \delta \sigma^{-i-1}(s)\right) X^{\ell-1} + \cdots + \delta^\ell \sigma^{-\ell}(s). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} f\sigma^{-\ell}(s) &= b_\ell X^\ell \sigma^{-\ell}(s) + b_{\ell-1} X^{\ell-1} \sigma^{-\ell}(s) + \cdots + b_0 \sigma^{-\ell}(s) \\ &= b_\ell s X^\ell + \left(b_\ell \sum_{i=0}^{\ell-1} \sigma^i \delta \sigma^{-i-1}(s) + b_{\ell-1} \sigma^{-1}(s)\right) X^{\ell-1} + \cdots + \sum_{i=0}^{\ell} b_{\ell-i} \delta^{\ell-i} \sigma^{-\ell}(s). \end{aligned}$$

Let $g_0 = \sum_{i=1}^{n} f_i \sigma^{-d}(r_{i0}) X^{\ell-d}$. Then $f\sigma^{-\ell}(s) - g_0$ has the degree less than ℓ. Consider the coefficient of $(\ell-1)$-th degree term of $f\sigma^{-\ell}(s) - g_0$, say $b_{\ell 1} \in A$. Since A is S-finite, $b_{\ell 1} s = \sum_{i=1}^{n} a_i r_{i1}$ for some $r_{11}, \ldots, r_{n1} \in R$. Let $g_1 = \sum_{i=1}^{n} f_i \sigma^{-d}(r_{i1}) X^{\ell-1-d}$. Then $\left(f\sigma^{-\ell}(s) - g_0\right) \sigma^{-\ell+1}(s) - g_1$ has the degree less than $\ell - 1$. Continuing this manner, we have

$$f \prod_{j=0}^{\ell-d} \sigma^{-\ell+j}(s) = \sum_{j=0}^{\ell-d-1} \left(g_j \prod_{k=j}^{\ell-d-1} \sigma^{-\ell+k+1}(s)\right) + g_{\ell-d} + b_{\ell,\ell-d+1} X^{d-1} + \cdots + c$$

for some $b_{\ell,\ell-d+1}$ and $c \in R$. Hence $f \prod_{j=0}^{\ell-d} \sigma^{-\ell+j}(s) \in \sum_{i=1}^{n} f_i R[X;\sigma,\delta] + \mathcal{B}$.

Now, since S is a right σ-anti-Archimedean subset of R, there exists an element $w \in \bigcap_{n\geq 1} \left(\prod_{j=0}^{n-1} \sigma^{-n+j}(s)\right) R \cap S$; so $fw \in \sum_{i=1}^{n} f_i R[X;\sigma,\delta] + \mathcal{B}$ for all $f \in \mathcal{A}$. Therefore we obtain

$$\begin{aligned} fwt &\in \sum_{i=1}^{n} f_i R[X;\sigma,\delta] + \sum_{k=1}^{m} h_k R \\ &\subseteq \sum_{i=1}^{n} f_i R[X;\sigma,\delta] + \sum_{k=1}^{m} h_k R[X;\sigma,\delta]. \end{aligned}$$

Since f is arbitrary chosen in $\mathcal{A}$, we obtain

$$\mathcal{A}wt \subseteq \sum_{i=1}^{n} f_i R[X;\sigma,\delta] + \sum_{k=1}^{m} h_k R[X;\sigma,\delta] \subseteq \mathcal{A}.$$

Hence $\mathcal{A}$ is right S-finite, and thus $R[X;\sigma,\delta]$ is a right S-Noetherian ring. $\square$

By applying $\delta = 0$ or $\sigma = I_R$ to Theorem 3.1, we obtain two corollaries.

Corollary 3.2. *Let S be a multiplicative subset of a ring R, σ an automorphism of R, and δ a σ-derivation on R. If R is a right S-Noetherian ring, then the following assertions hold.*

(1) *If S is right σ-anti-Archimedean for an automorphism σ, then the skew polynomial ring $R[X;\sigma]$ is also right S-Noetherian.*

(2) *If S is right anti-Archimedean, then the differential operator ring $R[X;\delta]$ is also right S-Noetherian.*

Corollary 3.3 (Hilbert basis theorem for right S-Noetherian rings)**.** *Let R be a ring and S a right anti-Archimedean subset of R. If R is a right S-Noetherian ring, then so is the polynomial ring $R[X]$.*

For an automorphism σ of a ring R, the *skew-Laurent polynomial ring* $R[X, X^{-1};\sigma]$ consists of the polynomials in X and X^{-1} with coefficients in R written on the left, subject to the relation $X^n r = \sigma^n(r)X^n$ for all $r \in R$ and $n \in \mathbb{Z}$. If $\sigma = I_R$, then we write $R[X, X^{-1}]$ for $R[X, X^{-1};\sigma]$ and is called the *Laurent polynomial ring*.

Proposition 3.4. *Let S be a right σ-anti-Archimedean subset for an automorphism σ of a ring R. If R is a right S-Noetherian ring, then the skew-Laurent polynomial ring $R[X, X^{-1};\sigma]$ is right S-Noetherian.*

Proof. Let $\mathcal{A}$ be a right ideal of $R[X, X^{-1};\sigma]$. We want to show that $\mathcal{A}$ is S-finite.

Step 1. Let A_1 be the right ideal of R consisting of zero and the leading coefficients of Laurent polynomials in $\mathcal{A}$. Since R is right S-Noetherian, there exist $s \in S$ and $a_1, \ldots, a_{n_1} \in A_1$ such that $A_1 s \subseteq \sum_{i=1}^{n_1} a_i R \subseteq A_1$. For each $i \in \{1, \ldots, n_1\}$, choose an element $f_i = \sum_{j=p_i}^{d_i-1} \alpha_j X^j + a_i X^{d_i} \in \mathcal{A}$ with the leading coefficient a_i. Let p_i be the lowest degree of f_i and d_i be the highest degree of f_i. Set $d = \max\{d_1, \ldots, d_{n_1}\}$ and let $\mathcal{B} = \{g \in \mathcal{A} \mid \deg(g) \leq d\}$.

Let $f = \sum_{j=m}^{\ell} c_j X^j \in \mathcal{A}$ be arbitrary. If $\ell \leq d$, then $f \in \mathcal{B}$. Otherwise, $\ell > d$. Since $c_\ell \in A_1$, $c_\ell s = \sum_{i=1}^{n_1} a_i r_i$ for some $r_1, \ldots, r_{n_1} \in R$. Note that $f\sigma^{-\ell}(s) = c_\ell X^\ell \sigma^{-\ell}(s) + \cdots + c_m X^m \sigma^{-\ell}(s) = c_\ell s X^\ell + \cdots + c_m \sigma^{m-\ell}(s) X^m$. Let $h = \sum_{i=1}^{n_1} f_i \sigma^{-d_i}(r_i) X^{\ell-d_i}$. Then $f\sigma^{-\ell}(s) - h$ has the degree less than ℓ. Continuation of the process as above leads us to $fx \in \sum_{i=1}^{n_1} f_i R[X;\sigma] + \mathcal{B}$ for some $x \in \bigcap_{n\geq 1} \left(\prod_{j=0}^{n-1} \sigma^{-n+j}(s)\right) R \cap S$ because S is right σ-anti-Archimedean. Thus $\mathcal{A}x \subseteq \sum_{i=1}^{n_1} f_i R[X;\sigma] + \mathcal{B}$.

Step 2. Let A_2 be the right ideal of R consisting of zero and the coefficients of the lowest degree terms in Laurent polynomials in $\mathcal{B}$. Since R is right S-Noetherian, A_2 is S-finite. Thus, $A_2 t \subseteq \sum_{i=1}^{n_2} b_i R \subseteq A_2$ for some $t \in S$ and $b_1, \ldots, b_{n_2} \in A_2$. For each $i \in \{1, \ldots, n_2\}$, choose an element $g_i = b_i X^{k_i} + \sum_{j=k_i+1}^{q_i} \beta_j X^j \in \mathcal{B}$ with the leading coefficient b_i. Let k_i be the lowest degree of g_i and q_i be the highest degree of g_i. Set $k = \min\{k_1, \dots, k_{n_2}\}$. Now let $\mathcal{C} = \mathcal{A} \cap (RX^k + \cdots + RX^d)$.

Let $g = \sum_{j=m}^{\ell} c_j X^j \in \mathcal{B}$ be arbitrary with $\ell \le d$. If $m \ge k$, then $g \in \mathcal{C}$. Otherwise, $m < k$. Since $c_m \in A_2$, $c_m t = \sum_{i=1}^{n_2} b_i t_i$ for some $t_1, \dots, t_{n_2} \in R$. Since $g\sigma^{-m}(t) = c_m X^m \sigma^{-m}(t) + \cdots + c_\ell X^\ell \sigma^{-m}(t) = c_m t X^m + \cdots + c_\ell \sigma^{\ell-m}(t) X^\ell$, $g\sigma^{-m}(t) - \sum_{i=1}^{n_2} g_i \sigma^{-k_i}(t_i) X^{m-k_i} = g\sigma^{-m}(t) - (b_1 t_1 + \cdots + b_{n_2} t_{n_2}) X^m + [\text{upper terms}] \in \mathcal{B}$ as $\deg(g) \le d$ and $k \le \deg(g_i) \le d$. Continuation of the process as above leads us to $gy \in \sum_{i=1}^{n_2} g_i R[X^{-1}; \sigma] + \mathcal{C}$ for some $y \in \bigcap_{n \ge 1} \left(\prod_{j=0}^{n-1} \sigma^{-n+j}(s) \right) R \cap S$ as S is right σ-anti-Archimedean. Thus $\mathcal{B}y \subseteq \sum_{i=1}^{n_2} g_i R[X^{-1}; \sigma] + \mathcal{C}$.

Step 3. By Lemma 2.14(4), $\mathcal{C}$ is S-finite as a right R-submodule of $R[X, X^{-1}; \sigma]$; so there exist $z \in S$ and $h_1, \dots, h_{n_3} \in \mathcal{C}$ such that $\mathcal{C}z \subseteq \sum_{i=1}^{n_3} h_i R[X, X^{-1}; \sigma]$.

In conclusion, from Step 1, $\mathcal{A}x \subseteq \sum_{i=1}^{n_1} f_i R[X; \sigma] + \mathcal{B}$ for some $x \in S$. Also, by Step 2, $\mathcal{A}xy \subseteq \sum_{i=1}^{n_1} f_i R[X; \sigma] y + \mathcal{B}y \subseteq \sum_{i=1}^{n_1} f_i R[X; \sigma] + \sum_{i=1}^{n_2} g_i R[X^{-1}; \sigma] + \mathcal{C}$ for some $y \in S$. In addition, from Step 3, for some $z \in S$,

$$\mathcal{A}xyz \subseteq \sum_{i=1}^{n_1} f_i R[X, X^{-1}; \sigma] + \sum_{i=1}^{n_2} g_i R[X, X^{-1}; \sigma] + \sum_{i=1}^{n_3} h_i R[X, X^{-1}; \sigma] \subseteq \mathcal{A},$$

which shows that $\mathcal{A}$ is S-finite. Thus $R[X, X^{-1}; \sigma]$ is a right S-Noetherian ring. $\square$

By applying $\sigma = I_R$ to Proposition 3.4, we obtain

Corollary 3.5. *Let S be a right anti-Archimedean subset of a ring R. If R is a right S-Noetherian ring, then so is the Laurent polynomial ring $R[X, X^{-1}]$.*

Recall that a ring R is *right Ore* if given $a, b \in R$ with b regular, there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. Note that R is a right Ore ring if and only if the classical right quotient ring of R exists.

We next study the power series ring extension of right S-Noetherian rings.

Theorem 3.6. *Let R be a right Ore ring and S a right anti-Archimedean subset of R consisting of regular elements. If R is a right S-Noetherian ring, then so is the power series ring $R[\![X]\!]$.*

Proof. Let $\mathcal{A}$ be any right ideal of $R[\![X]\!]$ and A the right ideal of R consisting of zero and the lowest degree coefficients of power series of $\mathcal{A}$. Since R is right S-Noetherian, $As \subseteq \sum_{i=1}^{n} a_i R$ for some $s \in S$ and $a_1, \dots, a_n \in A$. For each $i \in \{1, \dots, n\}$, choose $f_i \in \mathcal{A}$ with the lowest degree term $a_i X^{d_i}$, where d_i is the lowest degree of f_i. Set $d = \max\{d_1, \dots, d_n\}$ and let $g = \sum_{k=\ell}^{\infty} b_k X^k \in \mathcal{A}$ with $b_\ell \neq 0$.

Case 1: $\ell > d$. Since $b_\ell \in A$, $b_\ell s = \sum_{i=1}^{n} a_i u_{i0}$ for some $u_{10}, \dots, u_{n0} \in R$. Let $g_0 = \sum_{i=1}^{n} f_i u_{i0} X^{\ell - d_i}$. Then $gs - g_0$ has the lowest degree greater than ℓ. Suppose that we have $g_0, \ldots, g_r \in f_1 R\llbracket X\rrbracket + \cdots + f_n R\llbracket X\rrbracket$ such that $gs^{r+1} - \sum_{i=0}^{r} g_i s^{r-i}$ has the lowest degree greater than $\ell + r$, say

$$gs^{r+1} - \sum_{i=0}^{r} g_i s^{r-i} = cX^{\ell+r+1} + \cdots .$$

Since $gs^{r+1} - \sum_{i=0}^{r} g_i s^{r-i} \in \mathcal{A}$, $cs \in a_1 R + \cdots + a_n R$; so $cs = \sum_{i=1}^{n} a_i u_{i,r+1}$ for some $u_{1,r+1}, \ldots, u_{n,r+1} \in R$. Let $g_{r+1} = \sum_{i=1}^{n} f_i u_{i,r+1} X^{\ell+r+1-d_i}$. Then $gs^{r+2} - \sum_{i=0}^{r+1} g_i s^{r+1-i}$ has the lowest degree greater than $\ell+r+1$. Since s is a regular element of R, we denote by s^{-1} the inverse element of s in the right quotient ring $S^{-1}R\llbracket X\rrbracket$. By using the sequence $\{g_i\}_{i\geq 0}$, we can deduce that

$$\begin{aligned} g &= \sum_{j=0}^{\infty} g_j s^{-(j+1)} \\ &= \sum_{j=0}^{\infty} \left(\sum_{i=1}^{n} f_i u_{ij} X^{\ell+j-d_i} \right) s^{-(j+1)} \\ &= \sum_{j=0}^{\infty} \sum_{i=1}^{n} f_i u_{ij} s^{-(j+1)} X^{\ell+j-d_i}. \end{aligned}$$

Since S is right anti-Archimedean, there exists an element $w \in \bigcap_{p=1}^{\infty} s^p R \cap S$; so $s^{-(j+1)} w \in R$ for all nonnegative integers j. Hence we obtain

$$\begin{aligned} gw &= \sum_{i=1}^{n} f_i \left(\sum_{j=0}^{\infty} u_{ij} s^{-(j+1)} w X^{\ell+j-d_i} \right) \\ &\in f_1 R\llbracket X\rrbracket + \cdots + f_n R\llbracket X\rrbracket . \end{aligned}$$

Case 2: $\ell \leq d$. For each $i \in \{0, \ldots, d\}$, let $B_i = \left\{ f(i) \mid f \in \mathcal{A} \cap R\llbracket X\rrbracket X^i \right\}$, where $f(i)$ denotes the coefficient of X^i in f. Then $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_d$ is a chain of right ideals of R. Since R is S-Noetherian, there exist an element $t_i \in S$ and a finitely generated right ideal $F_i := b_{i1} R + \cdots + b_{in_i} R$ such that $B_i t_i \subseteq F_i \subseteq B_i$. Therefore there exist $h_{i1}, \ldots, h_{in_i}$, where $h_{ij} = b_{ij} X^i + [\text{upper terms}]$ with $1 \leq j \leq n_i$. Since $b_\ell \in B_\ell$, $b_\ell t_\ell = \sum_{j=1}^{n_\ell} b_{\ell j} v_{\ell j}$ for some $v_{\ell 1}, \ldots, v_{\ell n_\ell} \in R$. Let $h_\ell = \sum_{j=1}^{n_\ell} h_{\ell j} v_{\ell j}$. Then $gt_\ell - h_\ell$ has the lowest degree greater than ℓ. By continuing the same process, we can get $h_{\ell+1}, \ldots, h_d$. Let

$$h = g \prod_{j=\ell}^{d} t_j - \sum_{i=\ell}^{d} \left(h_i \prod_{j=1}^{d-i} t_{i+j} \right).$$

Since h has the lowest degree greater than d, by Case 1, there exists an element $w \in \bigcap_{p=1}^{\infty} s^p R \cap S$ such that

$$g \left(\prod_{j=\ell}^{d} t_j \right) w = \sum_{i=\ell}^{d} \left(h_i \prod_{j=1}^{d-i} t_{i+j} \right) w + hw$$

$$\in \sum_{i=\ell}^{d}\sum_{j=1}^{n_i} h_{ij} R\llbracket X\rrbracket + \sum_{i=1}^{n} f_i R\llbracket X\rrbracket .$$

Let $t = \prod_{i=0}^{d} t_i$. Since g is arbitrary chosen in $\mathcal{A}$, $\mathcal{A}t \subseteq \sum_{i=0}^{d}\sum_{j=1}^{n_i} h_{ij} R\llbracket X\rrbracket + \mathcal{C}$, where $\mathcal{C} = \left\{ f \mid f \in \mathcal{A} \cap R\llbracket X\rrbracket X^{d+1} \right\}$. Therefore we obtain

$$\begin{aligned}
\mathcal{A}tw &\subseteq \sum_{i=0}^{d}\sum_{j=1}^{n_i} h_{ij} R\llbracket X\rrbracket + \mathcal{C}w \\
&\subseteq \sum_{i=0}^{d}\sum_{j=1}^{n_i} h_{ij} R\llbracket X\rrbracket + \sum_{i=1}^{n} f_i R\llbracket X\rrbracket \\
&\subseteq \mathcal{A}.
\end{aligned}$$

Hence $\mathcal{A}$ is S-finite, and thus $R\llbracket X\rrbracket$ is a right S-Noetherian ring. □

Let $R \subseteq E$ be an extension of rings, $R + XE[X] = \{f \in E[X] \mid f(0) \in R\}$ a composite polynomial ring, and $R + XE\llbracket X\rrbracket = \{f \in E\llbracket X\rrbracket \mid f(0) \in R\}$ a composite power series ring. Then $R[X] \subseteq R + XE[X] \subseteq E[X]$ and $R\llbracket X\rrbracket \subseteq R + XE\llbracket X\rrbracket \subseteq E\llbracket X\rrbracket$; so if $R \subsetneq E$, then $R + XE[X]$ (resp., $R + XE\llbracket X\rrbracket$) provides algebraic properties of polynomial (resp., power series) type rings strictly between two polynomial rings (resp., power series rings).

We first give an equivalent condition for the ring $R + XE[X]$ to be right S-Noetherian when S is a right anti-Archimedean subset of R.

Theorem 3.7. *Let $R \subseteq E$ be an extension of rings and S a right anti-Archimedean subset of R. Then the following statements are equivalent.*

(1) *$R + XE[X]$ is a right S-Noetherian ring.*

(2) *R is a right S-Noetherian ring and E is S-finite as a right R-module.*

Proof. (1) $\Rightarrow$ (2): Let A be a right ideal of R. Then $A + XE[X]$ is a right ideal of $R + XE[X]$. Since $R + XE[X]$ is a right S-Noetherian ring, we can find $s \in S$ and $f_1, \ldots, f_n \in A + XE[X]$ such that $(A + XE[X])s \subseteq f_1(R + XE[X]) + \cdots + f_n(R + XE[X])$. Therefore $As \subseteq f_1(0)R + \cdots + f_n(0)R \subseteq A$. Note that $f_1(0)R + \cdots + f_n(0)R$ is a finitely generated right ideal of R; so A is S-finite. Thus R is a right S-Noetherian ring.

We next show that E is S-finite as a right R-module. Since $XE[X]$ is a right ideal of $R + XE[X]$, there exist $t \in S$ and $g_1, \ldots, g_m \in E[X]$ such that $(XE[X])t \subseteq Xg_1(R + XE[X]) + \cdots + Xg_m(R + XE[X])$; so for any $e \in E$, we have

$$(eX)t = Xg_1h_1 + \cdots + Xg_mh_m$$

for some $h_1, \ldots, h_m \in R + XE[X]$. Hence we have

$$\begin{aligned} et &= g_1(0)h_1(0) + \cdots + g_m(0)h_m(0) \\ &\in g_1(0)R + \cdots + g_m(0)R, \end{aligned}$$

which indicates that $Et \subseteq g_1(0)R + \cdots + g_m(0)R \subseteq E$. Note that $g_1(0)R + \cdots + g_m(0)R$ is a finitely generated right R-submodule of E. Thus E is S-finite as a right R-module.

(2) $\Rightarrow$ (1): Suppose that R is a right S-Noetherian ring and E is S-finite as a right R-module. Then there exist $s \in S$ and $e_1, \ldots, e_m \in E$ such that $Es \subseteq e_1R + \cdots + e_mR \subseteq E$; so we have

$$(E[X])s \subseteq e_1R[X] + \cdots + e_mR[X] \subseteq E[X].$$

Hence $E[X]$ is S-finite as a right $R[X]$-module. Note that by Corollary 3.3, $R[X]$ is a right S-Noetherian ring; so by Lemma 2.14(5), $E[X]$ is S-Noetherian as a right $R[X]$-module. Let N be a right $(R + XE[X])$-submodule of $E[X]$. Then N is a right $R[X]$-submodule of $E[X]$; so there exist $t \in S$ and $f_1, \ldots, f_n \in N$ such that $Nt \subseteq f_1R[X] + \cdots + f_nR[X] \subseteq N$. Therefore we have

$$Nt \subseteq f_1(R + XE[X]) + \cdots + f_n(R + XE[X]) \subseteq N.$$

Hence N is S-finite as a right $(R + XE[X])$-module, which indicates that $E[X]$ is S-Noetherian as a right $(R + XE[X])$-module. Let A be a right ideal of $R + XE[X]$. Then A is a right $(R + XE[X])$-submodule of $E[X]$. Since $E[X]$ is S-Noetherian as a right $(R + XE[X])$-module, we can find $w \in S$ and $g_1, \ldots, g_k \in A$ such that $Aw \subseteq g_1(R + XE[X]) + \cdots + g_k(R + XE[X]) \subseteq A$. Hence A is S-finite. Thus $R + XE[X]$ is a right S-Noetherian ring. $\square$

We next give a necessary and sufficient condition for the ring $R + XE[\![X]\!]$ to be right S-Noetherian when S is a right anti-Archimedean subset of R consisting of regular elements.

Theorem 3.8. *Let R be a right Ore ring, E a ring extension of R, and S a right anti-Archimedean subset of R consisting of regular elements. Then $R + XE[\![X]\!]$ is a right S-Noetherian ring if and only if R is a right S-Noetherian ring and E is S-finite as a right R-module.*

Proof. If we use Lemma 2.14(5) and Theorem 3.6, then the proof is similar to that of Theorem 3.7. $\square$

We are closing this article with some applications of composite ring extensions.

Example 3.9. (1) Note that for any multiplicative subset S of $\mathbb{Z}$, $\mathbb{Q}$ is not S-finite as a (right) $\mathbb{Z}$-module. Thus by Theorems 3.7 and 3.8, neither $\mathbb{Z} + X\mathbb{Q}[X]$ nor $\mathbb{Z} + X\mathbb{Q}\llbracket X \rrbracket$ is a (right) S-Noetherian ring.

(2) Let R be a right S-Noetherian ring, S a right anti-Archimedean subset of R, and n a positive integer. Then $\mathcal{D}_n(S)$ is a right anti-Archimedean subset of $\mathrm{UTM}_n(R)$, and by Proposition 2.16(1), $\mathrm{UTM}_n(R)$ is a right $\mathcal{D}_n(S)$-Noetherian ring. Also, as in the proof of Proposition 2.16(2), $\mathrm{Mat}_n(R)$ is $\mathcal{D}_n(S)$-finite as a right $\mathrm{UTM}_n(R)$-module. Thus by Theorem 3.7, $\mathrm{UTM}_n(R) + X\,\mathrm{Mat}_n(R)[X]$ is a right $\mathcal{D}_n(S)$-Noetherian ring. Furthermore, if R is right Ore and S consists of regular elements of R, then $\mathcal{D}_n(S)$ also consists of regular elements of $\mathrm{UTM}_n(R)$; so by Theorem 3.8, $\mathrm{UTM}_n(R) + X\,\mathrm{Mat}_n(R)\llbracket X \rrbracket$ is a right $\mathcal{D}_n(S)$-Noetherian ring.

(3) Let R be a right S-Noetherian ring, S a right anti-Archimedean subset of R, and n a positive integer. Then $\mathcal{I}_n(S)$ is a right anti-Archimedean subset of $\mathcal{V}_n(R)$, and by Proposition 2.17, $\mathcal{V}_n(R)$ is a right $\mathcal{I}_n(S)$-Noetherian ring. Also, as in the proof of Proposition 2.19, $\mathcal{H}_n(R)$ is $\mathcal{I}_n(S)$-finite as a right $\mathcal{V}_n(R)$-module. Thus by Theorem 3.7, $\mathcal{V}_n(R) + X\mathcal{H}_n(R)[X]$ is a right $\mathcal{I}_n(S)$-Noetherian ring. Moreover, if R is right Ore and S consists of regular elements of R, then $\mathcal{I}_n(S)$ also consists of regular elements of $\mathcal{V}_n(R)$; so by Theorem 3.8, $\mathcal{V}_n(R) + X\mathcal{H}_n(R)\llbracket X \rrbracket$ is a right $\mathcal{I}_n(S)$-Noetherian ring.

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