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Real hypersurfaces in the complex quadric with harmonic curvature ☆

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We introduce the notion of harmonic curvature for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. We give a complete classification, in terms of their $\mathfrak{A}$-principal or their $\mathfrak{A}$-isotropic unit normal vector fields, of real hypersurfaces in $Q^m = SO_{m+2}/SO_mSO_2$ having harmonic curvature tensor.

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R É S U M É

On introduit la notion de courbure harmonique pour les hypersurfaces réelles de la quadrique complexe $Q^m = SO_{m+2}/SO_mSO_2$. On en déduit une classification complète des hypersurfaces réelles de $Q^m = SO_{m+2}/SO_mSO_2$ à tenseur de courbure harmonique. Cette classification utilise leur $\mathfrak{A}$-principal ou leur champ $\mathfrak{A}$-isotrope de vecteurs unités normaux.

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1. Introduction

Usually, for a Riemannian manifold (N, g) the Ricci tensor Ric can be regarded as a 1-form with values in the cotangent bundle T^*N. Then a Riemannian manifold N is said to have *harmonic curvature* or *harmonic Weyl tensor*, if Ric_N or $Ric_N - r_N g_N/2(n-1)$ for the scalar curvature r is a Codazzi tensor, that is, it satisfies

$$dRic = 0 \quad \text{or} \quad d\{Ric - rg/2(n-1)\} = 0,$$

where d denotes the exterior differential. For the harmonic Weyl tensor, it is seen that in the case of $n \geq 4$ the Weyl curvature tensor W which is regarded as a 2-form with values in the bundle $\Lambda^2 T^*N$ is closed and

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coclosed, namely it is harmonic. In the case of $n = 3$ the Riemannian manifold N is conformally flat (see Besse [2]).

In the geometry of real hypersurfaces in complex space forms or in quaternionic space forms it can be easily checked that there does not exist any real hypersurface with parallel shape operator A by virtue of the equation of Codazzi.

From this point of view many differential geometers have considered a notion weaker than the parallel Ricci tensor, that is, $\nabla Ric = 0$. In particular, Kwon and Nakagawa [9] have proved that there are no Hopf real hypersurfaces M in a complex projective space $\mathbb{C}P^m$ with *harmonic curvature*, that is, $(\nabla_X Ric)Y = (\nabla_Y Ric)X$ for any X, Y in M. Moreover, Ki, Nakagawa and Suh [7] have also proved that there are no real hypersurface with *harmonic Weyl tensor* in non-flat complex space forms $M_n(c)$, $c \neq 0$, $n \geq 3$.

Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Then the above situation is not so simple if we consider a real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. Suh [16] has shown that there does not exist any hypersurface in $G_2(\mathbb{C}^{m+2})$ with parallel Ricci tensor, that is, $\nabla Ric = 0$, and have investigated the problem related to the Reeb parallel Ricci tensor Ric for real hypersurfaces M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, that is, $\nabla_\xi Ric = 0$ for the Reeb vector field ξ tangent to M (see [18]).

The ambient space $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J} = \mathrm{Span}\{J_1, J_2, J_3\}$ not including J (see Berndt and Suh [3]).

In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperkähler manifold. So, we have considered the two natural geometric conditions for real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with unit normal vector field N, that the 1-dimensional distribution $[\xi] = \mathrm{Span}\{\xi\}$, $\xi = -JN$ and the 3-dimensional distribution $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$, $\xi_i = -J_iN$, $i = 1, 2, 3$ both are invariant under the shape operator. By using such two geometric conditions and the results in Alekseevskii [1], Berndt and Suh [3] proved the following:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*
(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

In the proof of Theorem A we proved that the 1-dimensional distribution $[\xi]$ is contained in either the 3-dimensional distribution $\mathfrak{D}^\perp$ or in the orthogonal complement $\mathfrak{D}$ such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$. The case (A) in Theorem A is just the case that the 1-dimensional distribution $[\xi]$ is contained in the distribution $\mathfrak{D}^\perp$. Of course, it is not difficult to check that the Ricci tensor of any real hypersurface mentioned in Theorem A is not parallel. Then it must be a natural question to ask whether real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with conditions more weaker than parallel Ricci tensor can exist or not.

From such a view point, Besse [2] has introduced a notion of harmonic curvature which is given by $\triangle Ric = (d\delta + \delta d)Ric = 0$ for the Ricci tensor Ric. Then the notion of harmonic curvature is equivalent to $\delta Ric = 0$, because $dRic = 0$ always holds from the contraction of the 2nd Bianchi identity.

Then a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with harmonic curvature satisfies

$$(\nabla_X Ric)Y = (\nabla_Y Ric)X$$

for any tangent vector fields X and Y on M in $G_2(\mathbb{C}^{m+2})$.

But considering real hypersurfaces of *harmonic curvature* in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, the situation is quite different from the complex projective space $\mathbb{C}P^m$. Instead of the non-existence results in $\mathbb{C}P^m$, we [17] gave a classification of all Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with harmonic or Weyl harmonic tensor. First for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with harmonic curvature tensor, we asserted the following:

Theorem B. *Let M be a Hopf real hypersurface of harmonic curvature with constant scalar and mean curvatures. If the shape operator commutes with the structure tensor on the distribution $\mathfrak{D}^{\perp}$, then M is locally congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius r, $\cot^2 \sqrt{2}r = \frac{4}{3}(m-1)$.*

On the other hand, a $(4m-1)$-dimensional real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to have *harmonic Weyl tensor* if $\Delta W = 0$ for Weyl curvature tensor W defined by $W = Ric - rg/4(2m-1)$, where Ric and r denotes respectively the Ricci tensor and the scalar curvature of M in $G_2(\mathbb{C}^{m+2})$. Then from the 2nd Bianchi identity this is equivalent to $\delta W = 0$, that is, $(\nabla_X W)Y = (\nabla_Y W)X$. Naturally it means that

$$(\nabla_X Ric)Y - (\nabla_Y Ric)X = \{dr(X)Y - dr(Y)X\}/4(2m-1).$$

When we consider some Hermitian symmetric spaces of rank 2, we can usually give examples of Riemannian symmetric spaces $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians respectively (see [3] and [19]). Those are said to be Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$ on $SU_{2,m}/S(U_2U_m)$. The rank of $SU_{2,m}/S(U_2U_m)$ is 2 and there are exactly two types of singular tangent vectors X of $G_2(\mathbb{C}^{m+2})$ which are characterized by the geometric properties $JX \in \mathfrak{J}X$ and $JX \perp \mathfrak{J}X$ respectively.

As another kind of Hermitian symmetric space with rank 2 of compact type different from the above ones, we can give the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$, which is a complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$ (see Berndt and Suh [4], [5] and Smyth [15]). The complex quadric can also be regarded as a kind of real Grassmann manifolds of compact type with rank 2 (see Kobayashi and Nomizu [10]). Accordingly, the complex quadric admits two important geometric structures, that is, a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple (Q^m, J, g) is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [6] and Reckziegel [14]).

For the complex projective space $\mathbb{C}P^{m+1}$ and the quaternionic projective space $\mathbb{Q}P^{m+1}$ some classifications related to parallel Ricci tensor were investigated in Kimura [8], and Pérez and Suh [11], respectively. For the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_mU_2)$ a new classification was obtained by Berndt and Suh [3]. By using this classification Pérez and Suh [12] proved a non-existence property for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel and commuting Ricci tensor and Pérez, Suh and Watanabe [13] considered the notion of generalized Einstein hypersurfaces in $G_2(\mathbb{C}^{m+2})$. Suh [19] strengthened this result to hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel Ricci tensor. Moreover, Suh and Woo [22] studied another non-existence property for Hopf hypersurfaces in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2U_m)$ with parallel Ricci tensor.

When we consider a hypersurface M in the complex quadric Q^m, the unit normal vector field N of M in Q^m can be divided into two classes according to N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [4], [5], and [20]). In the first case where N is $\mathfrak{A}$-isotropic, we have shown in [4] that M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k}. In the second case, when the unit normal N is $\mathfrak{A}$-principal, we proved that a contact hypersurface M in Q^m is locally congruent to a tube over a totally geodesic and totally real submanifold S^m in Q^m (see [5]). In this paper we consider the notion of *harmonic curvature* for hypersurfaces in Q^m, that is, $(\nabla_X Ric)Y = (\nabla_Y Ric)X$ for any vector fields X and Y on M in Q^m. Then motivated by the result in the case of $\mathfrak{A}$-principal normal for contact hypersurfaces in Q^m, we assert the following:

Theorem 1. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 4$, with harmonic curvature. If the unit normal N is $\mathfrak{A}$-principal, then M has at most 5 distinct constant principal curvatures, five of which are given by*

$$\alpha, \quad \lambda_1, \quad \mu_1, \quad \lambda_2, \quad \textit{and} \quad \mu_2$$

with corresponding principal curvature spaces

$$T_\alpha = [\xi], \quad \phi T_{\lambda_1} = T_{\mu_1}, \quad \phi T_{\lambda_2} = T_{\mu_2},$$
$$\dim T_{\lambda_1} + \dim T_{\lambda_2} = m-1, \quad \dim T_{\mu_1} + \dim T_{\mu_2} = m-1.$$

Here four roots λ_i and μ_i, $i = 1, 2$ satisfy the quadratic equation

$$2x^2 - 2\beta x + 2 + \alpha\beta = 0,$$

where the function β is denoted by $\beta = \frac{\alpha^2+1\pm\sqrt{(\alpha^2+1)^2+4\alpha h}}{\alpha}$ and the function h denotes the mean curvature of M in Q^m.

In section 4 we will give the proof of this theorem in detail. Now at each point $z \in M$ let us consider a maximal $\mathfrak{A}$-invariant subspace $\mathcal{Q}_z$ of T_zM, $z \in M$, defined by

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}$$

of T_zM, $z \in M$. Thus *for a case where* the unit normal vector field N is $\mathfrak{A}$-isotropic it can be easily checked that the orthogonal complement $\mathcal{Q}_z^\perp = \mathcal{C}_z \ominus \mathcal{Q}_z$, $z \in M$, of the distribution $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$, becomes $\mathcal{Q}_z^\perp = \text{Span } [A\xi, AN]$. Here it can be easily checked that the vector fields $A\xi$ and AN belong to the tangent space T_zM, $z \in M$ if the unit normal vector field N becomes $\mathfrak{A}$-isotropic. Then motivated by the above result, in this paper we give another theorem for real hypersurfaces in the complex quadric Q^m with harmonic curvature and $\mathfrak{A}$-isotropic unit normal as follows:

Theorem 2. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geq 4$, with harmonic curvature and $\mathfrak{A}$-isotropic unit normal N. If the shape operator commutes with the structure tensor on the distribution $\mathcal{Q}^\perp$, then M is locally congruent to an open part of a tube around k-dimensional complex projective space $\mathbb{C}P^k$ in Q^m, $m = 2k$, or M has at most 6 distinct constant principal curvatures given by*

$$\alpha, \quad \gamma = 0(\alpha), \quad \lambda_1, \quad \mu_1, \quad \lambda_2 \quad \textit{and} \quad \mu_2$$

with corresponding principal curvature spaces

$$T_\alpha = [\xi], \quad T_\gamma = [A\xi, AN], \quad \phi(T_{\lambda_1}) = T_{\mu_1}, \quad \phi T_{\lambda_2} = T_{\mu_2}.$$
$$\dim T_{\lambda_1} + \dim T_{\lambda_2} = m-2, \quad \dim T_{\mu_1} + \dim T_{\mu_2} = m-2.$$

Here four roots λ_i and μ_i, $i = 1, 2$ satisfy the equation

$$2x^2 - 2\beta x + 2 + \alpha\beta = 0,$$

where the function β denotes $\beta = \frac{\alpha^2+2\pm\sqrt{(\alpha^2+2)^2+4\alpha h}}{\alpha}$. In particular, $\alpha = \sqrt{\frac{2m-1}{2}}$, $\gamma(=\alpha) = \sqrt{\frac{2m-1}{2}}$, $\lambda = 0$, $\mu = -\frac{2\sqrt{2}}{\sqrt{2m-1}}$, with multiplicities 1, 2, $m-2$ and $m-2$ respectively.

The particular case mentioned in Theorem 2 can occur for real hypersurfaces in Q^m with parallel Ricci tensor, that is, $\nabla Ric = 0$. Naturally harmonic curvature $\delta Ric = 0$ includes the notion of Ricci parallel.

Our paper is organized as follows. In section 2 we present basic material about the complex quadric Q^m, including its Riemannian curvature tensor and a description of its singular vectors for $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic unit normal vector field. Apart from the complex structure J there is another distinguished geometric structure on Q^m, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an S^1-bundle of real structures, that is, complex conjugations A on the tangent spaces of Q^m. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^m. Accordingly, in section 3, we investigate the geometry of this subbundle $\mathcal{Q}$ for hypersurfaces in Q^m and some equations including Codazzi and fundamental formulas related to the vector fields ξ, N, $A\xi$, and AN for the complex conjugation A of M in Q^m.

Finally, in sections 4 and 5, we present the proof of Theorems 1 and 2, respectively. In section 4, the first step is to derive the Ricci tensor from the equation of Gauss for real hypersurfaces M in Q^m and to get some formulas by the assumption of harmonic curvature and $\mathfrak{A}$-principal normal vector field, and show that a real hypersurface in Q^m which is a tube over an m-dimensional unit sphere S^m does not admit a harmonic curvature tensor. The next step is to show that there do not exist any real hypersurfaces in the complex quadric Q^m with harmonic curvature and $\mathfrak{A}$-principal normal vector field.

In section 5, we give a complete proof of Theorem 2. The first part of this proof is devoted to give some fundamental formulas from harmonic curvature and $\mathfrak{A}$-isotropic unit normal vector field. Then in the latter part of the proof we will use the expression of the shape operator for real hypersurfaces in Q^m when the shape operator and the structure tensor commutes on the distribution $\mathcal{Q}^\perp$, where $\mathcal{Q}^\perp$ denotes an orthogonal complement of the maximal $\mathfrak{A}$-invariant subspace $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$ of T_zM, $z \in M$ in Q^m.

2. The complex quadric

For more details in this section we refer to [4–6,10,14,20,21]. The complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ which is defined by the equation $z_0^2 + \cdots + z_{m+1}^2 = 0$, where $z_0, \ldots, z_{m+1}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric g which is induced from the Fubini–Study metric $\bar{g}$ on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. The Fubini–Study metric $\bar{g}$ is defined by $\bar{g}(X,Y) = \Phi(JX,Y)$ for any vector fields X and Y on $\mathbb{C}P^{m+1}$ and a globally closed $(1,1)$-form Φ given by $\Phi = -4i\partial\bar{\partial}\log f_j$ on an open set $U_j = \{[z^0, z^1, \cdots, z^{m+1}] \in \mathbb{C}P^{m+1} | z^j \neq 0\}$, where the function f_j denotes $f_j = \sum_{k=0}^{m+1} t_j^k \bar{t}_j^k$, and $t_j^k = \frac{z^k}{z^j}$ for j, $k = 0, \cdots, m+1$. Then naturally the Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J,g) on the complex quadric Q^m.

The complex projective space $\mathbb{C}P^{m+1}$ is a Hermitian symmetric space of the special unitary group SU_{m+2}, namely $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. We denote by $o = [0, \ldots, 0, 1] \in \mathbb{C}P^{m+1}$ the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The special orthogonal group $SO_{m+2} \subset SU_{m+2}$ acts on $\mathbb{C}P^{m+1}$ with cohomogeneity one. The orbit containing o is a totally geodesic real projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$. The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. This homogeneous space model leads to the geometric interpretation of the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in $\mathbb{R}^{m+2}$. It also gives a model of Q^m as a Hermitian symmetric space of rank 2. The complex quadric Q^1 is isometric to a sphere S^2 with constant curvature, and Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume $m \geq 3$ from now on.

For a nonzero vector $z \in \mathbb{C}^{m+2}$ we denote by $[z]$ the complex span of z, that is, $[z] = \{\lambda z \mid \lambda \in \mathbb{C}\}$. Note that by definition $[z]$ is a point in $\mathbb{C}P^{m+1}$. As usual, for each $[z] \in \mathbb{C}P^{m+1}$ we identify $T_z\mathbb{C}P^{m+1}$ with the orthogonal complement $\mathbb{C}^{m+2} \ominus [z]$ of $[z]$ in $\mathbb{C}^{m+2}$. For $[z] \in Q^m$ the tangent space T_zQ^m can then be identified canonically with the orthogonal complement $\mathbb{C}^{m+2} \ominus ([z] \oplus [\bar{z}])$ of $[z] \oplus [\bar{z}]$ in $\mathbb{C}^{m+2}$ (see Kobayashi and Nomizu [10]). Note that $\bar{z} \in \nu_z Q^m$ is a unit normal vector of Q^m in $\mathbb{C}P^{m+1}$ at the point $[z]$.

We denote by $A_{\bar{z}}$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to the unit normal $\bar{z}$. It is defined by $A_{\bar{z}}w = \tilde{\nabla}_w \bar{z} = \bar{w}$ for a complex Euclidean connection $\tilde{\nabla}$ and all $w \in T_zQ^m$. That is, the shape operator $A_{\bar{z}}$ is just complex conjugation restricted to T_zQ^m. The shape operator $A_{\bar{z}}$ is an anti-commuting involution such that $A_{\bar{z}}^2 = I$ and $AJ = -JA$ on the complex vector space T_zQ^m and

$$T_zQ^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \mathbb{R}^{m+2} \cap T_zQ^m$ is the $(+1)$-eigenspace and $JV(A_{\bar{z}}) = i\mathbb{R}^{m+2} \cap T_zQ^m$ is the (-1)-eigenspace of $A_{\bar{z}}$. That is, $A_{\bar{z}}X = X$ and $A_{\bar{z}}JX = -JX$, respectively, for any $X \in V(A_{\bar{z}})$.

Geometrically this means that the shape operator $A_{\bar{z}}$ defines a real structure on the complex vector space T_zQ^m, or equivalently, is a complex conjugation on T_zQ^m. Since the real codimension of Q^m in $\mathbb{C}P^{m+1}$ is 2, this induces an S^1-subbundle $\mathfrak{A}$ of the endomorphism bundle $\mathrm{End}(TQ^m)$ consisting of complex conjugations.

There is a geometric interpretation of these conjugations. The complex quadric Q^m can be viewed as the complexification of the m-dimensional sphere S^m. Through each point $z \in Q^m$ there exists a one-parameter family of real forms of Q^m which are isometric to the sphere S^m. These real forms are congruent to each other under action of the center SO_2 of the isotropy subgroup of SO_{m+2} at z. The isometric reflection of Q^m in such a real form S^m is an isometry, and the differential at z of such a reflection is a conjugation on T_zQ^m. In this way the family $\mathfrak{A}$ of conjugations on T_zQ^m corresponds to the family of real forms S^m of Q^m containing z, and the subspaces $V(A) \subset T_zQ^m$ correspond to the tangent spaces T_zS^m of the real forms S^m of Q^m.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugations $A \in \mathfrak{A}$:

$$\begin{aligned}\bar{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY.\end{aligned}$$

Note that J and each complex conjugation A anti-commute, that is, $AJ = -JA$ for each $A \in \mathfrak{A}$.

Recall that a nonzero tangent vector $W \in T_zQ^m$ is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m :

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/\|W\| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

For every unit tangent vector $W \in T_zQ^m$ there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \pi/4]$. The singular tangent vectors correspond to the values $t = 0$ and $t = \pi/4$. If $0 < t < \pi/4$ then the unique maximal flat containing W is $\mathbb{R}X \oplus \mathbb{R}JY$. Later we will need the eigenvalues and eigenspaces of the Jacobi operator $R_W = R(\cdot, W)W$ for a singular unit tangent vector W.

1. If W is an $\mathfrak{A}$-principal singular unit tangent vector with respect to $A \in \mathfrak{A}$, then the eigenvalues of R_W are 0 and 2 and the corresponding eigenspaces are $\mathbb{R}W \oplus J(V(A) \ominus \mathbb{R}W)$ and $(V(A) \ominus \mathbb{R}W) \oplus \mathbb{R}JW$, respectively.

2. If W is an $\mathfrak{A}$-isotropic singular unit tangent vector with respect to $A \in \mathfrak{A}$ and $X, Y \in V(A)$, then the eigenvalues of R_W are 0, 1 and 4 and the corresponding eigenspaces are $\mathbb{R}W \oplus \mathbb{C}(JX + Y)$, $T_zQ^m \ominus (\mathbb{C}X \oplus \mathbb{C}Y)$ and $\mathbb{R}JW$, respectively.

3. Some general equations

Let M be a real hypersurface in Q^m and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M and η the corresponding 1-form defined by $\eta(X) = g(\xi, X)$ for any tangent vector field X on M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$ we define a maximal $\mathfrak{A}$-invariant subspace of T_zM, $z \in M$ as follows:

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Lemma 3.1. *(See [20].) For each $z \in M$ we have*

(i) *If N_z is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*
(ii) *If N_z is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

We now assume that M is a Hopf hypersurface. Then the shape operator S of M in Q^m satisfies

$$S\xi = \alpha\xi$$

for the Reeb vector field ξ and the smooth function $\alpha = g(S\xi, \xi)$ on M. When we consider a transform JX of the Kähler structure J on Q^m for any vector field X on M in Q^m, we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal N to M. Then we now consider the Codazzi equation

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) = {} & \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) + g(X, AN)g(AY, Z) \\ & - g(Y, AN)g(AX, Z) + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \end{aligned} \tag{3.1}$$

Putting $Z = \xi$ in (3.1) we get

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = {} & -2g(\phi X, Y) + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ & - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) &= g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y). \end{aligned}$$

Comparing the previous two equations and putting $X = \xi$ yields

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi). \tag{3.2}$$

Reinserting this into the previous equation yields

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = &-2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ &+ 2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X) \\ &+ \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y). \end{aligned} \tag{3.3}$$

Altogether this implies

$$\begin{aligned} 0 = &\, 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ &+ g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ &- g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ &+ 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ &- 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned} \tag{3.4}$$

At each point $z \in M$ we can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \le t \le \frac{\pi}{4}$ (see Proposition 3 in [14]). Note that t is a function on M. First of all, since $\xi = -JN$, we have

$$\begin{aligned} N &= \cos(t)Z_1 + \sin(t)JZ_2, \\ AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1. \end{aligned} \tag{3.5}$$

This implies $g(\xi, AN) = 0$ and hence

$$\begin{aligned} 0 = &\, 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ &+ g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ &- g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ &- 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned} \tag{3.6}$$

We have $JA\xi = -AJ\xi = -AN$, and inserting this into the previous equation implies

Lemma 3.2. *Let M be a Hopf hypersurface in Q^m with (local) unit normal vector field N. For each point in $z \in M$ we choose $A \in \mathfrak{A}_z$ such that $N_z = \cos(t)Z_1 + \sin(t)JZ_2$ holds for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \le t \le \frac{\pi}{4}$. Then*

$$\begin{aligned} 0 = &\, 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) + 2g(X, AN)g(Y, A\xi) \\ &- 2g(Y, AN)g(X, A\xi) + 2g(\xi, A\xi)\{g(Y, AN)\eta(X) - g(X, AN)\eta(Y)\} \end{aligned}$$

holds for all vector fields X, Y on M.

We will now apply this result to get more information on Hopf hypersurfaces.

Lemma 3.3. *(See [20].) Let M be a Hopf hypersurface in Q^m such that the normal vector field N is $\mathfrak{A}$-principal everywhere. Then α is constant. Moreover, if $X \in \mathcal{C}$ is a principal curvature vector of M with principal curvature λ, then $2\lambda \neq \alpha$ and ϕX is a principal curvature vector of M with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

Lemma 3.4. *(See [20].) Let M be a Hopf hypersurface in Q^m, $m \geq 3$, such that the normal vector field N is $\mathfrak{A}$-isotropic everywhere. Then α is constant.*

4. Proof of Theorem 1

By the equation of Gauss, the curvature tensor $R(X,Y)Z$ for a real hypersurface M in Q^m induced from the curvature tensor $\bar{R}$ of Q^m can be described in terms of the complex structure J and the complex conjugation $A \in \mathfrak{A}$ as follows:

$$\begin{aligned}R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ & + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY \\ & + g(SY,Z)SX - g(SX,Z)SY\end{aligned}$$

for any X, Y, $Z \in T_zM$, $z \in M$. From this, contracting Y and Z on M in Q^m, we have

$$\begin{aligned}Ric(X) = {} & (2m-1)X - X - \phi^2X - 2\phi^2X \\ & - g(AN,N)AX - X + g(AX,N)AN - g(JAN,N)JAX \\ & - X + g(JAX,N)JAN + (trS)SX - S^2X \\ = {} & (2m-1)X - 3\eta(X)\xi - g(AN,N)AX + g(AX,N)AN \\ & - g(JAN,N)JAX + g(JAX,N)JAN + hSX - S^2X, \end{aligned} \tag{4.1}$$

where $h = trS$ denotes the mean curvature and is defined by the trace of the shape operator S of M in Q^m. Here we have used the following

$$\begin{aligned}\sum_{i=1}^{2m-1} g(Ae_i,e_i) &= \mathrm{Tr}A - g(AN,N) = -g(AN,N), \\ \sum_{i=1}^{2m-1} g(AX,e_i)Ae_i &= \sum_{i=1}^{2m} g(AX,e_i)Ae_i - g(AX,N)AN = X - g(AX,N)AN, \\ \sum_{i=1}^{2m-1} g(JAe_i,e_i)JAX &= \sum_{i=1}^{2m} g(JAe_i,e_i)JAX - g(JAN,N)JAX \\ &= -g(JAN,N)JAX,\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{2m-1} g(JAX,e_i)JAe_i &= \sum_{i=1}^{2m} g(JAX,e_i)JAe_i - g(JAX,N)JAN \\ &= JAJAX - g(JAX,N)JAN \\ &= X - g(JAX,N)JAN.\end{aligned}$$

Now in this section we consider only an $\mathfrak{A}$-principal normal vector field N, that is, $AN = N$, for a real hypersurface M in Q^m with parallel Ricci tensor. Then (4.1) becomes

$$Ric(X) = (2m-1)X - 2\eta(X)\xi - AX + hSX - S^2X. \tag{4.2}$$

Then the covariant derivative of (4.2) is given by

$$(\nabla_X Ric)Y = -2g(\phi SX, Y)\xi - 2\eta(Y)\phi SX - (\nabla_X A)Y + (Xh)SY + h(\nabla_X S)Y - (\nabla_X S^2)Y, \tag{4.3}$$

where $(\nabla_X A)Y = \nabla_X(AY) - A\nabla_X Y$. Here, AY belongs to T_zM, $z \in M$, from the fact that $g(AY, N) = g(Y, AN) = g(Y, N) = 0$ for any tangent vector Y on M.

Now let us suppose that M has harmonic curvature, that is, $\delta Ric = 0$, i.e., $(\nabla_X Ric)Y = (\nabla_Y Ric)X$. Then it follows that

$$\begin{aligned} 0 &= (\nabla_X Ric)Y - (\nabla_Y Ric)X \\ &= -2g((\phi S + S\phi)X, Y)\xi - 2\eta(Y)\phi SX + 2\eta(X)\phi SY - \{(\nabla_X A)Y - (\nabla_Y A)X\} \\ &\quad + (Xh)SY + (Yh)SX + h\{(\nabla_X S)Y - (\nabla_Y S)X\} - \{(\nabla_X S^2)Y - (\nabla_Y S^2)X\}. \end{aligned} \tag{4.4}$$

Now let us consider the following equation

$$\begin{aligned} (\nabla_X A)\xi &= \nabla_X(A\xi) - A\nabla_X\xi \\ &= (\bar\nabla_X(A\xi))^T - A\nabla_X\xi \\ &= \Big\{(\bar\nabla_X A)\xi + A\bar\nabla_X\xi\Big\}^T - A\phi SX \\ &= \{-q(X)AN + \alpha\eta(X)AN\}^T = 0, \end{aligned}$$

where $(\cdots)^T$ denotes the tangential component of the vector $(\cdots)$ in Q^m, and we have used the following

$$(\bar\nabla_X A)Y = q(X)JAY$$

for a certain 1-form q and any vector fields X and Y defined on T_zQ^m, $z \in Q^m$, (see Smyth [15]) and the Gauss formula

$$\bar\nabla_X\xi = \nabla_X\xi + g(SX, \xi)N$$

for connections $\bar\nabla$ on T_zQ^m, $z \in Q^m$ and ∇ on M in Q^m respectively, and S denotes the shape operator of M in Q^m.

Then let us take the inner product of (4.4) with ξ, and using the above fact, we have:

$$\begin{aligned} &\alpha\eta(Y)(Xh) - \alpha\eta(X)(Yh) - 2g((\phi S + S\phi)X, Y) \\ &\quad + hg((\nabla_X S)Y - (\nabla_Y S)X, \xi) - g((\nabla_X S^2)Y - (\nabla_Y S^2)X, \xi) = 0. \end{aligned} \tag{4.5}$$

Moreover, we get

$$(\nabla_X S)\xi = \nabla_X(S\xi) - S\nabla_X\xi = (X\alpha)\xi + \alpha\phi SX - S\phi SX,$$

and

$$(\nabla_X S^2)\xi = \nabla_X(S^2\xi) - S^2\nabla_X\xi = (X\alpha^2)\xi + \alpha^2\phi SX - S^2\phi SX.$$

Then it follows that

$$g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y)$$

and

$$\begin{aligned} g((\nabla_X S^2)Y - (\nabla_Y S^2)X, \xi) = \eta(Y)(X\alpha^2) - \eta(X)(Y\alpha^2) \\ + \alpha^2 g((\phi S + S\phi)X, Y) - 2g((S^2\phi S + S\phi S^2)X, Y). \end{aligned}$$

Then from (4.5), together with the above formulas, we have

$$\begin{aligned} &\alpha\eta(Y)(Xh) - \alpha\eta(X)(Yh) - 2g((\phi SX + S\phi)X, Y) \\ &\quad + h\Big[(X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y)\Big] \\ &\quad - \Big[(X\alpha^2)\eta(Y) - (Y\alpha^2)\eta(X) + \alpha^2 g((\phi S + S\phi)X, Y) \\ &\quad - 2g((S^2\phi S + S\phi S^2)X, Y)\Big] = 0. \end{aligned} \tag{4.6}$$

By putting $X = \xi$ into (4.5) and using (3.2), (4.6) can be arranged as follows:

$$(h\alpha - \alpha^2 - 2)(\phi S + S\phi)X - 2hS\phi SX + (S^2\phi S + S\phi S^2)X = 0. \tag{4.7}$$

On the other hand, by (3.6) in section 3, we know that

$$S\phi SX = \frac{\alpha}{2}(S\phi + \phi S)X + \phi X.$$

From this it follows that

$$(S^2\phi S + S\phi S^2)X = \frac{\alpha}{2}(S^2\phi + \phi S^2)X + \frac{\alpha^2}{2}(S\phi + \phi S)X + \alpha\phi X + (S\phi + \phi S)X.$$

Using these formulas, (4.7) becomes

$$(-\frac{\alpha^2}{2} - 1)(S\phi + \phi S)X + (\alpha - 2h)\phi X + \frac{\alpha}{2}(S^2\phi + \phi S^2)X = 0. \tag{4.8}$$

Then if we put $SX = \lambda X$, then $S\phi X = \mu\phi X$, $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. So (4.8) becomes

$$\frac{\alpha}{2}(\lambda + \mu)^2 - (\alpha^2 + 1)(\lambda + \mu) - 2h = 0. \tag{4.9}$$

Then $\lambda + \mu$ becomes

$$\lambda + \mu = \frac{\alpha^2 + 1 \pm \sqrt{(\alpha^2+1)^2 + 4\alpha h}}{\alpha}.$$

Here let us denote by $\beta = \frac{\alpha^2+1\pm\sqrt{(\alpha^2+1)^2+4\alpha h}}{\alpha}$. Then the functions λ and μ satisfy the following equations

$$2\lambda^2 - 2\beta\lambda + 2 + \alpha\beta = 0 \tag{4.10}$$

and

$$2\mu^2 - 2\beta\mu + 2 + \alpha\beta = 0. \tag{4.11}$$

From (4.10) and (4.11) we have

$$(\lambda - \mu)(\lambda + \mu - \beta) = 0.$$

Then we can divide the problem into two cases, that the principal curvatures λ and μ satisfy $\lambda = \mu$ or $\lambda + \mu = \beta$.

The first case becomes $x^2 - \alpha x - 1 = 0$. By Lemma 3.3, the function α is constant. In this case the shape operator has three constant principal curvatures $\alpha = 2\cot 2r$, $\cot r$ and $-\tan r$ with multiplicities 1, $m-1$ and $m-1$ respectively.

In the latter part, by (4.10) and (4.11) two roots λ and μ satisfy the equation

$$2x^2 - 2\beta x + 2 + \alpha\beta = 0.$$

Then the discriminant D of this equation is given by $D = 4\beta^2 - 8(2 + \alpha\beta)$. So there exists three distinct constant principal curvatures α, $\frac{\beta+\sqrt{\beta^2-2(2+\alpha\beta)}}{2}$, $\frac{\beta-\sqrt{\beta^2-2(2+\alpha\beta)}}{2}$ with multiplicities 1, $m-1$ and $m-1$ respectively, provided with $\beta > \alpha + \sqrt{\alpha^2+4}$ or $\beta < \alpha - \sqrt{\alpha^2+4}$.

When the function $\alpha = 0$, by (4.9) we know that $2h = -(\lambda + \mu) = -\lambda - \frac{1}{\lambda}$. This gives that two roots λ and μ satisfy the quadratic equation $x^2 + 2hx + 1 = 0$. That is, $\lambda = -h + \sqrt{h^2-1}$ and $\mu = -h - \sqrt{h^2-1}$ with multiplicities $m-1$ and $m-1$ respectively. So the trace of the shape operator S becomes

$$h = \mathrm{Tr}\, S = (m-2)\{-h + \sqrt{h^2-1} - h - \sqrt{h^2-1}\} = -2(m-2)h.$$

This gives $h = 0$, which implies $\lambda^2 + 1 = 0$. This gives a contradiction. Summing up above discussions, we give a complete proof of Theorem 1 in the introduction.

Remark 4.1. It is known that in Berndt and Suh [5] a real hypersurface M is a tube over S^m in Q^m if and only if the shape operator S of M satisfies $S\phi + \phi S = k\phi$ for a non-zero constant k. Then let us check whether a tube of radius r, $0 < r < \frac{\pi}{2\sqrt{2}}$, over S^m could satisfy (4.8) or not. Then (4.8) gives

$$-(\frac{\alpha^2}{2} + 1)k\phi X + (\alpha - 2h)\phi X + \frac{\alpha}{2}(S\phi^2 + \phi S^2)X = 0.$$

If we consider an eigenvector such that $SX = \lambda X$, then $(S\phi + \phi S)X = k\phi X$ gives that $S\phi X = (k - \lambda)\phi X$. From this, together with (4.8) using $\alpha k = -2$, the principal curvatures satisfy a quadratic equation such that

$$\alpha x^2 + x + 2\{k - \alpha + h\} = 0.$$

On the other hand, the tube over S^m in Q^m has 3 distinct constant principal curvatures such that $\alpha = -\sqrt{2}\cot(\sqrt{2}r)$, $\lambda = 0$ and $\mu = \sqrt{2}\tan\sqrt{2}r$ with multiplicities 1, $m-1$ and $m-1$ respectively (see [5]). Then $k\alpha = -2$ gives that

$$k = -\frac{2}{\alpha} = \frac{2}{\sqrt{2}\cot(\sqrt{2}r)} = \sqrt{2}\tan(\sqrt{2}r).$$

Since the quadratic equation has a root $\lambda = 0$, we should have $k - \alpha + h = 0$. From this, together with the fact $k\alpha = -2$, the trace h of the shape operator becomes

$$
\begin{aligned}
h &= -\sqrt{2}\cot(\sqrt{2}r) - \sqrt{2}\tan(\sqrt{2}r) \\
&= -\sqrt{2}\cot(\sqrt{2}r) + \frac{1}{\sqrt{2}}(m-1)\tan(\sqrt{2}r).
\end{aligned}
$$

This implies $(\frac{m-1}{\sqrt{2}} + \sqrt{2})\tan\sqrt{2}r = 0$, which gives a contradiction. So we conclude that a real hypersurface in Q^m which is a tube over an m-dimensional sphere S^m with radius r satisfying $0 < r < \frac{\pi}{2\sqrt{2}}$ does not admit harmonic curvature. Of course, in this case the unit normal N is $\mathfrak{A}$-principal.

5. Proof of Theorem 2

In this section we want to prove Theorem 2 for real hypersurfaces with harmonic curvature and $\mathfrak{A}$-isotropic unit normal vector field. Since we assumed that the unit normal N is $\mathfrak{A}$-isotropic, by the definition in section 3 we know that $t = \frac{\pi}{4}$. Then by the expression of the $\mathfrak{A}$-isotropic unit normal vector field, (3.3) gives $N = \frac{1}{\sqrt{2}}Z_1 + \frac{1}{\sqrt{2}}JZ_2$. This implies that

$$
g(\xi, A\xi) = 0,\ g(\xi, AN) = 0,\ g(AN, N) = 0,\ g(A\xi, N) = 0,
$$

and

$$
g(JAN, \xi) = -g(AN, N) = 0.
$$

Then the vector fields AN and $A\xi$ become tangent vector fields on M in Q^m.

The Ricci tensor (4.1) for a real hypersurface M in Q^m with $\mathfrak{A}$-isotropic unit normal gives

$$
\begin{aligned}
Ric(X) &= (2m-1)X - 3\eta(X)\xi - g(AN, N)AX + g(AX, N)AN \\
&\quad - g(JAN, N)JAX + g(JAX, N)JAN + (\mathrm{Tr}S)SX - S^2X \\
&= (2m-1)X - 3\eta(X)\xi + g(X, AN)AN + g(X, A\xi)A\xi + hSX - S^2X.
\end{aligned}
$$

Then in this case we want to make the derivative of the Ricci tensor as follows:

$$
\begin{aligned}
(\nabla_Y Ric)X &= \nabla_Y(Ric(X)) - Ric(\nabla_Y X) \\
&= -3(\nabla_Y \eta)(X)\xi - 3\eta(X)\nabla_Y\xi + g(X, \nabla_Y(AN))AN - g(AX, N)\nabla_Y(AN) \\
&\quad + g((\nabla_Y(A\xi), X)A\xi + \eta(AX)\nabla_Y(A\xi) + (Yh)SX + h(\nabla_Y S)X - (\nabla_Y S^2)X.
\end{aligned}
\tag{5.1}
$$

Then it follows that

$$
\nabla_Y(AN) = \{(\bar{\nabla}_Y A)N + A\bar{\nabla}_Y N\}^T = \{q(Y)JAN - ASY\}^T,
$$

and

$$
\nabla_Y(A\xi) = \{(\bar{\nabla}_Y A)\xi + A\bar{\nabla}_Y \xi\}^T = \{q(Y)JA\xi + A\phi SY\}^T,
$$

where $(\cdots)^T$ denotes the tangential component of the vector $(\cdots)$ in Q^m.

By our assumption of harmonic curvature, the above formula becomes

$$
\begin{aligned}
0 &= -3g(\phi SY, X)\xi - 3\eta(X)\phi SY + 3g(\phi SX, Y)\xi + 3\eta(Y)\phi SX \\
&\quad + \{q(Y)g(JAN, X) - g(ASY, X) - q(X)g(JAN, Y) + g(ASX, Y)\}AN
\end{aligned}
$$

$$
\begin{aligned}
&- g(AX,N)\{q(Y)JAN - ASY\}^T + g(AY,N)\{q(X)JAN - ASX\}^T \\
&+ \{q(Y)g(JA\xi,X) + g(A\phi SY,X) - q(X)g(JA\xi,Y) - g(A\phi SX,Y)\}A\xi \\
&+ \eta(AX)\{q(Y)JA\xi + A\phi SY\}^T - \eta(AY)\{q(X)JA\xi + A\phi SX\}^T \\
&+ (Yh)SX - (Xh)SY + h\{(\nabla_Y S)X - (\nabla_X S)Y\} \\
&- \{(\nabla_Y S^2)X - (\nabla_X S^2)Y\}. \qquad (5.2)
\end{aligned}
$$

By taking inner product of (5.2) with the Reeb vector field ξ and using $g(AN,N) = 0$ and $g(A\xi,\xi) = 0$, we have

$$
\begin{aligned}
0 = &-3g(\phi SY,X) + g(AX,N)\eta(ASY) - 3g(\phi SX,Y) \\
&+ g(AY,N)\eta(ASXY) + \eta(AX)\eta(A\phi SY) - \eta(AY)\eta(A\phi SX) \\
&+ (Yh)\alpha\eta(X) - (Xh)\alpha\eta(Y) + hg((\nabla_Y S)X - (\nabla_X S)Y,\xi) \\
&- g((\nabla_Y S^2)X - (\nabla_X S^2)Y,\xi) \\
&- 3g(\phi SY,X) + g(AX,N)\eta(ASY) - 3g(\phi SX,Y) + g(AY,N)\eta(ASXY) \\
&+ \eta(AX)\eta(A\phi SY) - \eta(AY)\eta(A\phi SX) \\
&+ (Yh)\alpha\eta(X) - (Xh)\alpha\eta(Y) \\
&+ h\{(Y\alpha)\eta(X) - (X\alpha)\eta(Y) + \alpha g((\phi S + S\phi)Y,X) - 2g(S\phi SY,X)\} \\
&- \big\{(Y\alpha^2)\eta(X) - (X\alpha^2)\eta(Y) + \alpha^2 g((\phi S + S\phi)Y,X) \\
&- 2g((S^2\phi S + S\phi S^2)Y,X)\big\}. \qquad (5.3)
\end{aligned}
$$

From this, putting $Y = \xi$, we have

$$
(\xi h)\alpha\eta(X) - (Xh)\alpha + h\{(\xi\alpha)\eta(X) - X\alpha\} - \{\eta(X)\xi\alpha^2 - X\alpha^2\} = 0. \qquad (5.4)
$$

Since the unit normal N is $\mathfrak{A}$-isotropic, by (3.2) we know that $Y\alpha = (\xi\alpha)\eta(Y)$, because $g(\xi, A\xi) = 0$ and $g(\xi, AN) = 0$. Then $grad^M\alpha = d\alpha(\xi)\xi$ gives

$$
(\mathrm{Hess}^M\alpha)(X,Y) = g(\nabla_X \mathrm{grad}^M\alpha, Y) = d(d\alpha(\xi))(X)\eta(Y) + d\alpha(\xi)g(\phi SX,Y).
$$

By using the symmetry of Hessian, we have $d\alpha(\xi)g((S\phi+\phi S)X,Y) = 0$ for any vector fields X and Y on M in Q^m. Then by using the same method as in [4] due to Berndt and Suh, we could prove that the function α is constant (see Lemma 3.4). Moreover, (5.4) gives $(Xh)\alpha = \alpha(\xi h)\eta(X)$. In this case we can also prove that the mean curvature h is constant, by the help of the constant function α. Accordingly, the equation (5.3) becomes

$$
\begin{aligned}
(2 + \frac{\alpha^2}{2})g((\phi S + S\phi)Y,X) = &\ 2g(AX,N)\eta(ASY) - 2g(AY,N)\eta(ASX) \\
&+ (\alpha - 2h)g(\phi Y,X) - (\alpha - 2h)g(Y,AN)g(A\xi,X) \\
&+ (\alpha - 2h)g(Y,A\xi)g(AN,X) + \frac{\alpha}{2}g((S^2\phi + \phi S^2)Y,X). \qquad (5.5)
\end{aligned}
$$

Here, in order to get the equation (5.5) we have used two important formulas from (3.6) in section 3 for $\mathfrak{A}$-isotropic unit normal as follows:

$$
2S\phi SX = \alpha(\phi S + S\phi)X + 2\phi X - 2g(X,AN)A\xi + 2g(X,A\xi)AN \qquad (5.6)
$$

and

$$\begin{aligned}(S^2\phi S+S\phi S^2)X=&\frac{\alpha}{2}(S^2\phi+\phi S^2)X+\frac{\alpha^2}{2}(\phi S+S\phi)X+\alpha\phi X\\&-\alpha g(X,AN)A\xi+\alpha g(X,A\xi)AN\\&+(S\phi+\phi S)X-g(X,AN)SA\xi+g(X,A\xi)SAN\\&-g(SX,AN)A\xi+g(SX,A\xi)AN.\end{aligned}\tag{5.7}$$

Now let us consider the distribution $\mathcal{Q}^{\perp}$, which is an orthogonal complement of the maximal $\mathfrak{A}$-invariant subspace $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$ of T_zM, $z\in M$ in Q^m. For any vector fields Y and ϕY belonging to the distribution Q such that $SY=\lambda Y$ and $S\phi Y=\mu\phi Y$, where $\mu=\frac{\alpha\lambda+2}{2\lambda-\alpha}$, by putting $X=\phi Y$ into the equation (5.5), we have the following

$$(2+\frac{\alpha^2}{2})(\lambda+\mu)=(\alpha-2h)+\frac{\alpha}{2}(\lambda^2+\mu^2).\tag{5.8}$$

On the other hand, the formula (5.6) on the distribution Q for $SX=\lambda X$ and $S\phi X=\mu\phi X$ gives the following:

$$\lambda\mu=\frac{\alpha}{2}(\lambda+\mu)+1.\tag{5.9}$$

From (5.8) and (5.9) it follows that

$$\frac{\alpha}{2}(\lambda+\mu)^2-(\alpha^2+2)(\lambda+\mu)-2h=0.$$

So if we put $\lambda+\mu=\beta$, then the function β becomes a solution of the quadratic equation $\frac{\alpha}{2}x^2-(\alpha^2+2)x-2h=0$ and it is given by

$$\beta=\frac{\alpha^2+2\pm\sqrt{(\alpha^2+2)^2+4\alpha h}}{\alpha}.\tag{5.10}$$

Here, for the case $\alpha=0$, (5.8) gives $h=-(\lambda+\mu)=-\lambda-\frac{1}{\lambda}$. Then the principal curvature λ satisfies the equation

$$x^2+hx+1=0.$$

Then the shape operator S has the principal curvatures $\alpha=0$, $\gamma=0$, $\lambda=\frac{-h+\sqrt{h^2-4}}{2}$ and $\mu=\frac{-h-\sqrt{h^2-4}}{2}=\frac{1}{\lambda}$. Then the trace h of the shape operator S becomes

$$h=(m-2)(\lambda+\mu).$$

From this, together with $h=-(\lambda+\mu)$ of the quadratic equation, it gives $h=0$, that is, $\lambda^2+1=0$, which gives a contradiction. So such a case $\alpha=0$ cannot be considered for M in Q^m with harmonic curvature.

Now we consider the distribution $\mathcal{Q}^{\perp}$. Then by Lemma 3.1 in section 3, the orthogonal complement $\mathcal{Q}^{\perp}=\mathcal{C}\ominus\mathcal{Q}$ becomes $\mathcal{C}\ominus\mathcal{Q}=\text{Span}[AN,A\xi]$. From the assumption of $S\phi=\phi S$ on the distribution $\mathcal{Q}^{\perp}$ it can be easily checked that the distribution $\mathcal{Q}^{\perp}$ is invariant by the shape operator S, because $Q^{\perp}=span[AN,\phi AN]$. Then we may put $SAN=\lambda AN$, from this together with Lemma 3.2, we have the following:

$$\begin{aligned}(2\lambda-\alpha)S\phi AN &= (\alpha\lambda+2)\phi AN - 2A\xi \\ &= (\alpha\lambda+2)\phi AN - 2\phi AN \\ &= \alpha\lambda\phi AN.\end{aligned}$$

Then $A\xi = \phi AN$ gives the following

$$SA\xi = \frac{\alpha\lambda}{2\lambda-\alpha}A\xi. \tag{5.11}$$

Then from the assumption $S\phi = \phi S$ on $\mathcal{Q}^\perp = \mathcal{C} \ominus \mathcal{Q}$ it follows that $\lambda = \frac{\alpha\lambda}{2\lambda-\alpha}$ gives

$$\lambda = 0 \text{ or } \lambda = \alpha. \tag{5.12}$$

On the other hand, on the distribution $\mathcal{Q}$ we know that $AX \in T_zM$, $z \in M$, because $AN \in Q$. So (5.6), together with the fact that $g(X, A\xi) = 0$ and $g(X, AN) = 0$ for any $X \in \mathcal{Q}$, imply that

$$2S\phi SX = \alpha(S\phi + \phi S)X + 2\phi X. \tag{5.13}$$

Then we can take an orthonormal basis $X_1, \dots, X_{2(m-2)} \in \mathcal{Q}$ such that $AX_i = \lambda_i X_i$ for $i = 1, \dots, m-2$. Then by (5.6) we know that

$$S\phi X_i = \frac{\alpha\lambda_i + 2}{2\lambda_i - \alpha}\phi X_i.$$

Accordingly, by (5.12) the shape operator S can be expressed as

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0(\alpha) & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0(\alpha) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{m-2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mu_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \mu_{m-2} \end{bmatrix}$$

From $\lambda + \mu = \beta$, and $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$, we have

$$2\lambda^2 - 2\beta\lambda + 2 + \alpha\beta = 0,$$

and

$$2\mu^2 - 2\beta\mu + 2 + \alpha\beta = 0.$$

Subtracting these two equations gives

$$(\lambda - \mu)(\lambda + \mu - \beta) = 0.$$

When $\lambda = \mu$, the shape operator S has at most 4 distinct constant principal curvatures $2\cot 2r$, $0(\alpha)$, $\cot r$ and $-\tan r$ with multiplicities 1, 2, $m-2$ and $m-2$ respectively. Then M is locally congruent to a tube around $\mathbb{C}P^k$ in Q^{2k}, $m = 2k$. Now let us check whether this kind of tube could admit harmonic curvature or not. Here we denote by $\lambda = \cot r$. Then by putting $\lambda = \mu \cot r$ in (5.8), it becomes

$$\alpha\lambda^2 - (4+\alpha^2)\lambda + (\alpha - 2h) = 0.$$

On the other hand, the trace of the shape operator this tube is given by $h = \alpha + (2k-2)(\cot r - \tan r) = (2k-1)\alpha$. Then $\alpha - 2h = (-4k+3)\alpha$. From this, together with $\lambda^2 - \alpha\lambda - 1 = 0$, the equation becomes

$$\lambda = -(k-1)\alpha = -(k-1)(\lambda - \frac{1}{\lambda}).$$

This gives $k\lambda^2 = k-1$, which means that the radius of this tube is given by $r = \cot^{-1}\sqrt{\frac{k-1}{k}}$. So M admits harmonic curvature.

For the case where $\lambda \neq \mu$, then the function $\beta = \lambda + \mu$, where the function β becomes

$$\beta = \frac{\alpha^2 + 2 \pm \sqrt{(\alpha^2+2)^2 + 4\alpha h}}{\alpha}.$$

So according to the function β the principal curvatures λ and μ are solutions of the quadratic equation

$$2x^2 - 2\beta x + 2 + \alpha\beta = 0.$$

Here we have shown that the functions α and $h\alpha - \alpha^2$ are constants. So in this case M has at most 6 distinct constant principal curvatures

Among them, let us check the situation when the principal curvature λ vanishes. Then the corresponding principal curvature becomes $\mu = -\frac{2}{\alpha}$. Now let us consider the shape operator

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0(\alpha) & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0(\alpha) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{2}{\alpha} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -\frac{2}{\alpha} \end{bmatrix}.$$

Summing up the above facts, the shape operator S has at most 4 distinct constant principal curvatures α, $0(\alpha)$, 0 and μ with multiplicities 1, 2, $m-2$ and $m-2$ respectively. By the quadratic equation above we know that $2 + \alpha\beta = 0$. So naturally from the function β it follows that

$$\alpha^2 + 4 = \sqrt{(\alpha^2+2)^2 + 4\alpha h}.$$

So the function α satisfies $\alpha^2 - h\alpha + 3 = 0$. In this case, first we consider the shape operator S has 3 distinct constant principal curvatures such that α, 0, 0 and $-\frac{2}{\alpha}$ with multiplicities 1, 2, $m-2$ and $m-2$ respectively. Then $h = \alpha - \frac{2(m-2)}{\alpha}$, so it follows that $\alpha^2 + 3 = h\alpha = \alpha^2 - 2(m-2)$, which gives a contradiction.

Next we consider the case that M has 3 distinct constant principal curvatures α, 0 and $-\frac{2}{\alpha}$ with multiplicities 3, $m-2$ and $m-2$ respectively. Then the trace h becomes $h = 3\alpha - \frac{2(m-2)}{\alpha}$. From this, together with $h\alpha = \alpha^2 + 3$, it follows that $2\alpha^2 = 2m-1$, that is, $\alpha = \sqrt{\frac{2m-1}{2}}$. This gives an example of real hypersurfaces in Q^m whose Ricci tensor is parallel, that is, $\nabla Ric = 0$, which is a special kind of harmonic curvature (see [21]).

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