SCIENCE CHINA
Mathematics

• ARTICLES •

November 2016 Vol. 59 No. 11: 2185–2198
doi: 10.1007/s11425-016-0067-7

Real hypersurfaces in the complex quadric with commuting Ricci tensor

SUH Young Jin* & HWANG Doo Hyun

Department of Mathematics, Kyungpook National University, Daegu 41566, Republic of Korea

Email: yjsuh@knu.ac.kr, engus0322@knu.ac.kr

Received January 30, 2016; accepted May 16, 2016; published online August 2, 2016

Abstract We introduce the notion of commuting Ricci tensor for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. It is shown that the commuting Ricci tensor gives that the unit normal vector field N becomes $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. Then according to each case, we give a complete classification of Hopf real hypersurfaces in $Q^m = SO_{m+2}/SO_mSO_2$ with commuting Ricci tensor.

Keywords commuting Ricci tensor, $\mathfrak{A}$-isotropic, $\mathfrak{A}$-principal, Kähler structure, complex conjugation, complex quadric

MSC(2010) 53C40, 53C55

Citation: Suh Y J, Hwang D H. Real hypersurfaces in the complex quadric with commuting Ricci tensor. Sci China Math, 2016, 59: 2185–2198, doi: 10.1007/s11425-016-0067-7

1 Introduction

Considering Hermitian symmetric spaces of rank 2, we can usually give examples of Riemannian symmetric spaces $SU_{m+2}/S(U_2U_m)$ and $SU_{2,m}/S(U_2U_m)$, which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians, respectively (see [13–16, 18]). These are regarded as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$ and they have rank 2.

Among the other different types of Hermitian symmetric spaces with rank 2 in the class of compact type, one can give the example of complex quadric $Q^m = SO_{m+2}/SO_mSO_2$, which is a complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$ (see [12, 17, 19]). We can also view the complex quadric as a kind of real Grassmann manifold of compact type with rank 2 (see [6]). Consequently, the complex quadric admits two important geometric structures, a complex conjugation structure A and a Kähler structure J, which anti-commute with each other, i.e., $AJ = -JA$. Then the triple (Q^m, J, g) $(m \geqslant 2)$ is an Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature equals 4 (see [5, 11]).

In the complex projective space $\mathbb{C}P^{m+1}$ and the quaternionic projective space $\mathbb{H}P^{m+1}$ some classifications related to commuting Ricci tensor or commuting structure Jacobi operator were investigated by Kimura [3, 4], Pérez [7] and Pérez and Suh [8, 9], respectively. Under the invariance of the shape operator along some distributions a new classification in the complex 2-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_mU_2)$ was investigated.

*Corresponding author

By the Kähler structure J of the complex quadric Q^m, we can decompose its action on any tangent vector field X on M in Q^m as follows:

$$JX = \phi X + \eta(X)N,$$

where $\phi X = (JX)^{\mathrm{T}}$ denotes the tangential component of JX, η denotes a 1-form defined by $\eta(X) = g(JX, N) = g(X, \xi)$ for the Reeb vector field $\xi = -JN$ and N a unit normal vector field on M in Q^m.

When the Ricci tensor Ric of M in Q^m commutes with the structure tensor ϕ, i.e., $\mathrm{Ric}\cdot\phi = \phi\cdot\mathrm{Ric}$, we say that M is *Ricci commuting* or *commuting Ricci tensor*.

Pérez and Suh [10] proved a non-existence property for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel and commuting Ricci tensor. Moreover, Suh [13] strengthened this result to hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor and gave a characterization of real hypersurfaces in $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_mU_2)$ as follows:

Theorem 1.1. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor, $m \geqslant 3$. Then M is locally congruent to a tube of radius r over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Moreover, Suh [18] studied another classification for Hopf hypersurfaces in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2U_m)$ with commuting Ricci tensor as follows:

Theorem 1.2. *Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2U_m)$ with commuting Ricci tensor, $m \geqslant 3$. Then M is locally congruent to an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$ or a horosphere whose center at infinity with $JX \in \mathfrak{J}X$ is singular.*

It is known that the Reeb flow on a real hypersurface in $G_2(\mathbb{C}^{m+2})$ is isometric if and only if M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1}) \subset G_2(\mathbb{C}^{m+2})$. Corresponding to this result, Suh [16] asserted that the Reeb flow on a real hypersurface in $SU_{2,m}/S(U_2U_m)$ is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1}) \subset SU_{2,m}/S(U_2U_m)$. Here, the Reeb flow on a real hypersurface in $SU_{m+2}/S(U_mU_2)$ or $SU_{2,m}/S(U_2U_m)$ is said to be *isometric* if the shape operator commutes with the structure tensor. Berndt and Suh [1], and Suh [19] have introduced this problem for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ and obtained the following result:

Theorem 1.3. *Let M be a real hypersurface of the complex quadric Q^m, $m \geqslant 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is an open part of a tube around a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$.*

In addition to the complex structure J there is another distinguished geometric structure on Q^m, namely a parallel rank two vector bundle $\mathfrak{A}$ which contains an S^1-bundle of real structures, i.e., complex conjugations A on the tangent spaces of Q^m. The set is denoted by $\mathfrak{A}_{[z]} = \{A_{\lambda\bar{z}} \mid \lambda \in S^1 \subset \mathbb{C}\}$, $[z] \in Q^m$, and means the set of all complex conjugations defined on Q^m. Then $\mathfrak{A}_{[z]}$ becomes a parallel rank 2-subbundle of $\mathrm{End}TQ^m$. This geometric structure determines a maximal $\mathfrak{A}$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of a real hypersurface M in Q^m.

When we consider the hypersurface M in the complex quadric Q^m, under the assumption of some geometric properties the unit normal vector field N of M in Q^m belongs to one of two classes, depending on whether N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal (see [17, 19]). In the first case where N is $\mathfrak{A}$-isotropic, it is known that M is locally congruent to a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k}. In the second case, when the unit normal N is $\mathfrak{A}$-principal, we proved that a contact hypersurface M in Q^m is locally congruent to a tube over a totally geodesic and totally real submanifold S^m in Q^m (see [19]).

Now at each point $z \in M$ let us consider a maximal $\mathfrak{A}$-invariant subspace $\mathcal{Q}_z$ of T_zM, $z \in M$, defined by

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Then when the unit normal vector field N is $\mathfrak{A}$-isotropic it can be easily checked that the orthogonal complement $\mathcal{Q}_z^{\perp} = \mathcal{C}_z \ominus \mathcal{Q}_z$, $z \in M$, of the distribution $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$, becomes $\mathcal{Q}_z^{\perp} = \mathrm{Span}\{A\xi, AN\}$. Here it can be easily checked that the vector fields $A\xi$ and AN belong to the tangent space T_zM, $z \in M$ if the unit normal vector field N becomes $\mathfrak{A}$-isotropic. Then motivated by the above result, Suh [19] gave a theorem for real hypersurfaces in the complex quadric Q^m with parallel Ricci tensor and $\mathfrak{A}$-isotropic unit normal vector field.

In the study of complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ or complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$, we considered hypersurfaces with parallel Ricci tensor and gave non-existence properties respectively (see [14, 23]). Suh [19] also considered the notion of parallel Ricci tensor $\nabla \mathrm{Ric} = 0$ for hypersurfaces M in Q^m. As a generalization of such facts, we consider the notion of harmonic curvature, i.e., $(\nabla_X \mathrm{Ric})Y = (\nabla_Y \mathrm{Ric})X$ for any tangent vector fields X and Y on M in Q^m and proved the following (see [20]).

Theorem 1.4. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geqslant 4$, with harmonic curvature and $\mathfrak{A}$-isotropic unit normal N. If the shape operator commutes with the structure tensor on the distribution $\mathcal{Q}^{\perp}$, then M is locally congruent to an open part of a tube around k-dimensional complex projective space $\mathbb{C}P^k$ in Q^m, $m = 2k$, or M has at most* 6 *distinct constant principal curvatures given by $\alpha, \gamma = 0(\alpha)$, $\lambda_1, \mu_1, \lambda_2$ and mu_2 with corresponding principal curvature spaces*

$$T_\alpha = [\xi], \quad T_\gamma = [A\xi, AN], \quad \phi(T_{\lambda_1}) = T_{\mu_1}, \quad \phi T_{\lambda_2} = T_{\mu_2},$$
$$\mathrm{dim} T_{\lambda_1} + \mathrm{dim} T_{\lambda_2} = m-2, \quad \mathrm{dim} T_{\mu_1} + \mathrm{dim} T_{\mu_2} = m-2.$$

Here four roots λ_i and μ_i, $i = 1, 2$ satisfy the equation

$$2x^2 - 2\beta x + 2 + \alpha\beta = 0,$$

where the function β denotes $\beta = \frac{\alpha^2+2\pm\sqrt{(\alpha^2+2)^2+4\alpha h}}{\alpha}$ and the function h is the mean curvature of M in Q^m. In particular, $\alpha = \sqrt{\frac{2m-1}{2}}$, $\gamma(=\alpha) = \sqrt{\frac{2m-1}{2}}$, $\lambda = 0$, $\mu = -\frac{2\sqrt{2}}{\sqrt{2m-1}}$, with multiplicities 1, 2, *$m-2$ and $m-2$, respectively.*

But from the assumption of harmonic curvature, it was impossible to derive the fact that either the unit normal N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. So Suh [20] gave a complete classification with the further assumption of $\mathfrak{A}$-isotropic as in Theorem 1.4. For the case where the unit normal vector field N is $\mathfrak{A}$-principal, we have proved that real hypersurfaces in Q^m with harmonic curvature do not exist.

However, when we consider a Ricci commuting real hypersurface M in Q^m, i.e., $\mathrm{Ric}\cdot\phi = \phi\cdot\mathrm{Ric}$ for hypersurfaces M in Q^m, we can assert that the unit normal vector field N becomes either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. Then motivated by such a result and using Theorem 1.3, in this paper we give a complete classification for real hypersurfaces in the complex quadric Q^m with commuting Ricci tensor, i.e., $\mathrm{Ric}\cdot\phi = \phi\cdot\mathrm{Ric}$ as follows:

Theorem 1.5. *Let M be a Hopf real hypersurface in the complex quadric Q^m, $m \geqslant 4$, with commuting Ricci tensor. If the shape operator commutes with the structure tensor on the distribution $\mathcal{Q}^{\perp}$, then M is locally congruent to an open part of a tube around totally geodesic $\mathbb{C}P^k$ in Q^{2k}, $m = 2k$, or M has* 3 *distinct constant principal curvatures given by*

$$\alpha = \sqrt{2(m-3)}, \quad \gamma = 0, \quad \lambda = 0, \quad \textit{and} \quad \mu = -\frac{2}{\sqrt{2(m-3)}} \quad \textit{or}$$
$$\alpha = \sqrt{\frac{2}{3}(m-3)}, \quad \gamma = 0, \quad \lambda = 0, \quad \textit{and} \quad \mu = -\frac{\sqrt{6}}{\sqrt{m-3}}$$

with corresponding principal curvature spaces, respectively

$$T_\alpha = [\xi], \quad T_\gamma = [A\xi, AN], \quad \phi(T_\lambda) = T_\mu, \quad \textit{and} \quad \mathrm{dim} T_\lambda = \mathrm{dim} T_\mu = m-2.$$

Remark 1.6. In Theorem 1.5, the second and the third cases can be explained geometrically as follows: the real hypersurface M is locally congruent to $M_1 \times \mathbb{C}$, where M_1 is a tube of radius $r = \frac{1}{\sqrt{2}} \tan^{-1} \sqrt{m-3}$ or respectively, of radius $r = \frac{1}{\sqrt{2}} \tan^{-1} \sqrt{\frac{m-3}{3}}$, around $(m-1)$-dimensional sphere S^{m-1} in Q^{m-1}, i.e., M_1 is a contact hypersurface defined by $S\phi + \phi S = k\phi$, $k = -\frac{2}{\sqrt{2(m-3)}}$, and $k = -\frac{\sqrt{6}}{\sqrt{m-3}}$, respectively (see [2,19]). By the Segre embedding, the embedding $M_1 \times \mathbb{C} \subset Q^{m-1} \times \mathbb{C} \subset Q^m$ is defined by $(z_0, z_1, \ldots, z_m, w) \to (z_0 w, z_1 w, \ldots, z_m w, 0)$. Here $(z_0 w)^2 + (z_1 w)^2 + \cdots + (z_m w)^2 = (z_0^2 + \cdots + z_m^2) w^2 = 0$, where $\{z_0, \ldots, z_m\}$ denotes a coordinate system in Q^{m-1} satisfying $z_0^2 + \cdots + z_m^2 = 0$.

Our paper is organized as follows. In Section 2, we present basic material about the complex quadric Q^m, including its Riemannian curvature tensor and a description of the singular vectors of Q^m like $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic unit normal vector field. In Section 3, we investigate the geometry of the subbundle $\mathcal{Q}$ for hypersurfaces in Q^m and some equations including Codazzi and fundamental formulas related to the vector fields ξ, N, $A\xi$ and AN for the complex conjugation A of M in Q^m.

In Section 4, the first step is to derive the formula of Ricci commuting from the equation of Gauss for real hypersurfaces M in Q^m and to get a key lemma that the unit normal vector field N is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal, and show that a real hypersurface in Q^m, $m = 2k$, which is a tube over a totally geodesic $\mathbb{C}P^k$ in Q^{2k} naturally has a commuting Ricci tensor. In Section 5, by the expressions of the shape operator S for real hypersurfaces M in Q^m, we present the proof of Theorem 1.5 with $\mathfrak{A}$-isotropic unit normal vector field.

In Section 6, we give a complete proof of Theorem 1.5 with $\mathfrak{A}$-principal unit normal vector field. The first part of this proof is devoted to giving some fundamental formulas from Ricci commuting and $\mathfrak{A}$-principal unit normal vector field. Then in the latter part of the proof, we will use the decomposition of two eigenspaces of the complex conjugation A in Q^m such that $T_z M = V(A) \oplus JV(A)$, where such eigenspaces are defined by $V(A) = \{X \in T_z Q^m \mid AX = X\}$ and $JV(A) = \{X \in T_z Q^m \mid AX = -X\}$, respectively.

2 The complex quadric

One can refer to [5,6,11,17,19] for more preliminaries. The complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ and defined by the equation $z_0^2 + \cdots + z_{m+1}^2 = 0$, where $z_0, \ldots, z_{m+1}$ are homogeneous coordinates on $\mathbb{C}P^{m+1}$. We equip Q^m with the Riemannian metric g which is induced from the Fubini-Study metric $\bar{g}$ on $\mathbb{C}P^{m+1}$ with constant holomorphic sectional curvature 4. Define the Fubini-Study metric $\bar{g}$ as $\bar{g}(X, Y) = \Phi(JX, Y)$ for any vector fields X and Y on $\mathbb{C}P^{m+1}$ and a globally closed $(1,1)$-form Φ given by $\Phi = -4\mathrm{i}\partial\bar{\partial}\log f_j$ on an open set $U_j = \{[z^0, z^1, \ldots, z^{m+1}] \in \mathbb{C}P^{m+1} \mid z^j \neq 0\}$, where the function f_j is denoted by $f_j = \sum_{k=0}^{m+1} t_j^k \bar{t}_j^k$, and $t_j^k = \frac{z^k}{z^j}$ for $j, k = 0, \ldots, m+1$. Consequently, the Kähler structure on $\mathbb{C}P^{m+1}$ induces canonically a Kähler structure (J, g) on the complex quadric Q^m.

Alternatively, the complex projective space $\mathbb{C}P^{m+1}$ is defined by using the Hopf fibration

$$\pi : S^{2m+3} \to \mathbb{C}P^{m+1}, \quad z \to [z],$$

which is said to be a Riemannian submersion. Then we naturally can consider the following diagram for the complex quadric Q^m:

$$\begin{array}{ccc} \tilde{Q} = \pi^{-1}(Q) & \xrightarrow{\ \tilde{i}\ } & S^{2m+3} \subset \mathbb{C}^{m+2} \\ \pi \big\downarrow & & \pi \big\downarrow \\ Q = Q^m & \xrightarrow{\ i\ } & \mathbb{C}P^{m+1}. \end{array}$$

The submanifold $\tilde{Q}$ of codimension 2 in S^{2m+3} is called the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{2m+4}$, which is given by

$$\tilde{Q} = \left\{ x + \mathrm{i}y \in \mathbb{C}^{m+2} \,\middle|\, g(x,x) = g(y,y) = \frac{1}{2} \text{ and } g(x,y) = 0 \right\},$$

where $g(x, y) = \sum_{i=1}^{m+2} x_i y_i$ for any $x = (x_1, \ldots, x_{m+2})$ and $y = (y_1, \ldots, y_{m+2}) \in \mathbb{R}^{m+2}$. Then the tangent space is decomposed as $T_z S^{2m+3} = H_z \oplus F_z$ and $T_z \tilde{Q} = H_z(Q) \oplus F_z(Q)$ at $z = x + \mathrm{i}y \in \tilde{Q}$, respectively, where the horizontal subspaces H_z and $H_z(Q)$ are given by $H_z = (\mathbb{C}z)^\perp$ and $H_z(Q) = (\mathbb{C}z \oplus \mathbb{C}\bar{z})^\perp$, and F_z and $F_z(Q)$ are fibers which are isomorphic to each other. Here $H_z(Q)$ becomes a subspace of H_z of real codimension 2 and orthogonal to the two unit normals $-\bar{z}$ and $-J\bar{z}$. Explicitly, at the point $z = x + \mathrm{i}y \in \tilde{Q}$ it can be described as

$$H_z = \{u + \mathrm{i}v \in \mathbb{C}^{m+2} \mid g(x,u) + g(y,v) = 0, g(x,v) = g(y,u)\}$$

and

$$H_z(Q) = \{u + \mathrm{i}v \in H_z \mid g(u,x) = g(u,y) = g(v,x) = g(v,y) = 0\},$$

where $\mathbb{C}^{m+2} = \mathbb{R}^{m+2} \oplus \mathrm{i}\mathbb{R}^{m+2}$, and $g(u,x) = \sum_{i=1}^{m+2} u_i x_i$ for any $u = (u_1, \ldots, u_{m+2})$, $x = (x_1, \ldots, x_{m+2}) \in \mathbb{R}^{m+2}$.

These spaces can be naturally projected by the differential map π_* as $\pi_* H_z = T_{\pi(z)}\mathbb{C}P^{m+1}$ and $\pi_* H_z(Q) = T_{\pi(z)}Q$, respectively. Hence, at the point $\pi(z) = [z]$ the tangent subspace $T_{[z]}Q^m$ becomes a complex subspace of $T_{[z]}\mathbb{C}P^{m+1}$ with complex codimension 1 and has two unit normal vector fields $-\bar{z}$ and $-J\bar{z}$ (see [11]).

Now let us denote by $A_{\bar{z}}$ the shape operator of Q^m in $\mathbb{C}P^{m+1}$ with respect to the unit normal $-\bar{z}$. Then, by virtue of the Weingarten equation, it is defined by $A_{\bar{z}}w = \bar{\nabla}_w \bar{z} = \bar{w}$ for a complex Euclidean connection $\bar{\nabla}$ induced from $\mathbb{C}^{m+2}$ and all $w \in T_{[z]}Q^m$, i.e., the shape operator $A_{\bar{z}}$ is just a complex conjugation restricted to $T_{[z]}Q^m$. Moreover, it satisfies the following: For any $w \in T_{[z]}Q^m$ and any $\lambda \in S^1 \subset \mathbb{C}$,

$$\begin{aligned}A^2_{\lambda\bar{z}}w &= A_{\lambda\bar{z}}A_{\lambda\bar{z}}w = A_{\lambda\bar{z}}\lambda\bar{w} \\ &= \lambda A_{\bar{z}}\lambda\bar{w} = \lambda\bar{\nabla}_{\lambda\bar{w}}\bar{z} = \lambda\bar{\lambda}\bar{\bar{w}} \\ &= |\lambda|^2 w = w.\end{aligned}$$

Accordingly, $A^2_{\lambda\bar{z}} = I$ for any $\lambda \in S^1$. So the shape operator $A_{\bar{z}}$ becomes an anti-commuting involution such that $A^2_{\bar{z}} = I$ and $AJ = -JA$ on the complex vector space $T_{[z]}Q^m$ and

$$T_{[z]}Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where $V(A_{\bar{z}}) = \mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the $(+1)$-eigenspace and $JV(A_{\bar{z}}) = \mathrm{i}\mathbb{R}^{m+2} \cap T_{[z]}Q^m$ is the (-1)-eigenspace of $A_{\bar{z}}$, i.e., $A_{\bar{z}}X = X$ and $A_{\bar{z}}JX = -JX$, respectively, for any $X \in V(A_{\bar{z}})$.

The Gauss equation for $Q^m \subset \mathbb{C}P^{m+1}$ implies that the Riemannian curvature tensor $\bar{R}$ of Q^m can be described using the complex structure J and the complex conjugations $A \in \mathfrak{A}$:

$$\begin{aligned}\bar{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY.\end{aligned}$$

Note that J and each complex conjugation A anti-commute, i.e., $AJ = -JA$ for each $A \in \mathfrak{A}$.

Recall that a nonzero tangent vector $W \in T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m:

1. If there exists a conjugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.

2. If there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/\|W\| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic.

3 Some general equations

Let M be a real hypersurface in Q^m and denote the induced almost contact metric structure by (ϕ, ξ, η, g). Note that $\xi = -JN$ with N being a (local) unit normal vector field of M and η the corresponding 1-form defined by $\eta(X) = g(\xi, X)$ for any tangent vector field X on M. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

At each point $z \in M$ we define a maximal $\mathfrak{A}$-invariant subspace of T_zM, $z \in M$ as follows:

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}.$$

Then we want to introduce an important lemma which will be used in the proof of our Theorem 1.5 in Introduction.

Lemma 3.1 (See [17, 19])**.** *For each $z \in M$ we have*

(i) *If N_z is $\mathfrak{A}$-principal, then $\mathcal{Q}_z = \mathcal{C}_z$.*

(ii) *If N_z is not $\mathfrak{A}$-principal, there exist a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $N_z = \cos(t)X + \sin(t)JY$ for some $t \in (0, \pi/4]$. Then we have $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$.*

Assume that M is a Hopf hypersurface. Then the shape operator S of M in Q^m satisfies

$$S\xi = \alpha\xi,$$

where $\alpha = g(S\xi, \xi)$ denotes the Reeb function on M. Considering the transform JX by the Kähler structure J on Q^m for any vector field X on M in Q^m, we get

$$JX = \phi X + \eta(X)N$$

for a unit normal N to M. Setting $Z = \xi$ in the following Codazzi equation:

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) = {} & \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ & + g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) \\ & + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z), \end{aligned} \tag{3.1}$$

we can eventually get the following:

$$\begin{aligned} 0 = {} & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ & + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ & - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ & + 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ & - 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned} \tag{3.2}$$

At each point $z \in M$, one can choose $A \in \mathfrak{A}_z$ such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leqslant t \leqslant \frac{\pi}{4}$ (see [11, Proposition 3]). Note that t is a function on M. Since $\xi = -JN$, we have

$$\begin{aligned} N &= \cos(t)Z_1 + \sin(t)JZ_2, \\ AN &= \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi &= \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi &= \sin(t)Z_2 + \cos(t)JZ_1. \end{aligned} \tag{3.3}$$

It implies that $g(\xi, AN) = 0$ and hence (3.2) becomes

$$\begin{aligned} 0 = {} & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ & + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ & - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ & - 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned} \tag{3.4}$$

4 Ricci commuting and a key lemma

By the equation of Gauss, the curvature tensor $R(X,Y)Z$ for a real hypersurface M in Q^m induced from the curvature tensor $\bar{R}$ of Q^m in Section 2 can be described in terms of the complex structure J and the complex conjugation $A \in \mathfrak{A}$ as follows:

$$\begin{aligned}R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\\ &\quad + g(AY,Z)AX - g(AX,Z)AY + g(JAY,Z)JAX - g(JAX,Z)JAY\\ &\quad + g(SY,Z)SX - g(SX,Z)SY\end{aligned}$$

for any $X,Y,Z \in T_zM$, $z \in M$.

Now let us put

$$AX = BX + \rho(X)N,$$

for any vector field $X \in T_zQ^m$, $z \in M$, $\rho(X) = g(AX,N)$, where BX and $\rho(X)N$ respectively denote the tangential and normal components of the vector field AX. Then $A\xi = B\xi + \rho(\xi)N$ and $\rho(\xi) = g(A\xi,N) = 0$. Then it follows that

$$\begin{aligned}AN &= AJ\xi = -JA\xi = -J(B\xi + \rho(\xi)N)\\ &= -(\phi B\xi + \eta(B\xi)N),\end{aligned}$$

where we have used $N = J\xi$ from the Reeb vector field $\xi = -JN$ and $J^2 = -I$. The equation gives $g(AN,N) = -\eta(B\xi)$. From this, together with the definition of the Ricci tensor, we have

$$\begin{aligned}\mathrm{Ric}(X) &= (2m-1)X - 3\eta(X)\xi - g(AN,N)AX + g(AX,N)AN\\ &\quad + \eta(AX)A\xi + (\mathrm{Tr}S)SX - S^2X.\end{aligned}$$

Then, summing up the above formulas, we have

$$\begin{aligned}\mathrm{Ric}(X) &= (2m-1)X - 3\eta(X)\xi + \eta(B\xi)\{BX + \rho(X)N\}\\ &\quad + \rho(X)\{-\phi B\xi - \eta(B\xi)N\} + \eta(BX)B\xi + (\mathrm{Tr}S)SX - S^2X.\end{aligned}$$

From this, together with the assumption of Ricci commuting, i.e., $\phi\cdot\mathrm{Ric}(X) = \mathrm{Ric}\cdot\phi X$, it follows that

$$\begin{aligned}&(2m-1)\phi X + \eta(B\xi)\phi BX - \rho(X)\phi^2B\xi + \eta(BX)\phi B\xi + (\mathrm{Tr}S)\phi SX - \phi S^2X\\ &\quad = (2m-1)\phi X + \eta(B\xi)B\phi X - \rho(\phi X)\phi B\xi + \eta(B\phi X)B\xi + (\mathrm{Tr}S)S\phi X - S^2\phi X. \qquad (4.1)\end{aligned}$$

Here we want to use the following formulas:

$$\begin{aligned}&\eta(BX) = g(A\xi,X),\\ &\eta(B\phi X) = g(A\xi,\phi X) = g(A\xi, JX - \eta(X)N) = g(AJ\xi,X)\\ &\qquad\quad = g(AN,X) = \rho(X),\\ &\rho(\phi X) = g(A\phi X,N) = g(AJ\phi X,\xi) = g(J\phi X,A\xi)\\ &\qquad\quad = g(\phi^2X + \eta(\phi X)N, A\xi) = -g(X,A\xi) + \eta(X)g(\xi,A\xi),\\ &\rho(X) = \eta(B\phi X).\end{aligned}$$

Substituting these formulas into (4.1), we have

$$\begin{aligned}&\eta(B\xi)\phi BX - \eta(B\phi X)\eta(B\xi)\xi + (\mathrm{Tr}S)\phi SX - \phi S^2X\\ &\quad = \eta(B\xi)B\phi X - \eta(X)\eta(B\xi)\phi B\xi + (\mathrm{Tr}S)S\phi X - S^2\phi X. \qquad (4.2)\end{aligned}$$

Then, by taking the inner product of (4.2) with ξ and using that M is Hopf, it follows that

$$\eta(B\xi)\phi B\xi = 0. \qquad (4.3)$$

Then the formula (4.2) becomes

$$\eta(B\xi)(\phi B - B\phi)X + (\mathrm{Tr}S)(\phi S - S\phi)X - (\phi S^2 - S^2\phi)X = 0. \qquad (4.4)$$

Remark 4.1. Let M be a real hypersurface over a totally geodesic $\mathbb{C}P^k \subset Q^{2k}$, $m = 2k$. Then in [17, 19], the structure tensor commutes with the shape operator, i.e., $S\phi = \phi S$. Moreover, the unit normal vector field N becomes $\mathfrak{A}$-isotropic. This gives $\eta(B\xi) = g(A\xi, \xi) = 0$. So it naturally satisfies the formula (4.2), i.e., the condition of Ricci commuting is satisfied.

On the other hand, from (4.3) we assert an important lemma as follows:

Lemma 4.2. *Let M be a real hypersurface in Q^m, $m \geqslant 3$, with commuting Ricci tensor. Then the unit normal vector field N becomes singular, i.e., N is $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal.*

Proof. From (4.3) we get

$$\eta(B\xi) = 0 \quad \text{or} \quad \phi B\xi = 0.$$

The first case gives that $\eta(B\xi) = g(A\xi, \xi) = \cos 2t = 0$, i.e., $t = \frac{\pi}{4}$. This implies that the unit normal N becomes

$$N = \frac{X + JY}{\sqrt{2}},$$

which means that N is $\mathfrak{A}$-isotropic.

The second case gives that

$$\rho(X) = g(AX, N) = \eta(B\phi X) = -g(X, \phi B\xi) = 0,$$

which means that $AX \in T_zM$ for any $A \in \mathfrak{A}$, $X \in T_zM$, $z \in M$. This implies $\mathcal{Q}_z = \mathcal{C}_z$, $z \in M$, and N is $\mathfrak{A}$-principal, i.e., $AN = N$. □

In order to prove Theorem 1.5 in Section 1, by virtue of Lemma 4.2, we can consider two classes of hypersurfaces in Q^m with the unit normal N being $\mathfrak{A}$-principal or $\mathfrak{A}$-isotropic. When M has an $\mathfrak{A}$-isotropic unit normal N, in Section 5 we will give the proof in detail and in Section 6 we will give the remainder proof for the case that M has an $\mathfrak{A}$-principal unit normal vector field.

5 Proof of Theorem 1.5 for $\mathfrak{A}$-isotropic unit normal vector field

In this section, we want to prove our Theorem 1.5 for real hypersurfaces M in Q^m with commuting Ricci tensor when the unit normal vector field becomes $\mathfrak{A}$-isotropic.

Since we assume that the unit normal N is $\mathfrak{A}$-isotropic, by the definition in Section 3 we know that $t = \frac{\pi}{4}$. Then by the expression of $\mathfrak{A}$-isotropic unit normal vector field, (3.3) gives $N = \frac{1}{\sqrt{2}}Z_1 + \frac{1}{\sqrt{2}}JZ_2$. Since the unit normal N is $\mathfrak{A}$-isotropic, we know that $g(\xi, A\xi) = 0$. Moreover, by (3.4) and using the anti-commuting property $AJ = -JA$ between the complex conjugation A and the Kähler structure J, we can prove the following (see also [17, Lemma 4.2]) lemma.

Lemma 5.1. *Let M be a Hopf real hypersurface in Q^m with (local) $\mathfrak{A}$-isotropic unit normal vector field N. For each point $z \in M$ we choose $A \in \mathfrak{A}_z$ such that $N_z = \cos(t)Z_1 + \sin(t)JZ_2$ holds for some orthonormal vectors $Z_1, Z_2 \in V(A)$ and $0 \leqslant t \leqslant \frac{\pi}{4}$. Then*

$$\begin{aligned} 0 = {} & 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ & + 2g(X, AN)g(Y, A\xi) - 2g(Y, AN)g(X, A\xi) \\ & + 2g(\xi, A\xi)\{g(Y, AN)\eta(X) - g(X, AN)\eta(Y)\} \end{aligned}$$

holds for all vector fields X, Y on M.

Then by virtue of $\mathfrak{A}$-isotropic unit normal, from Lemma 5.1 we obtain

$$2S\phi SX = \alpha(S\phi + \phi S)X + 2\phi X - 2g(X, AN)A\xi + 2g(X, A\xi)AN. \tag{5.1}$$

Now let us consider the distribution $\mathcal{Q}^{\perp}$, which is an orthogonal complement of the maximal $\mathfrak{A}$-invariant subspace $\mathcal{Q}$ in the complex subbundle $\mathcal{C}$ of T_zM, $z \in M$ in Q^m. Then by Lemma 3.1 in Section 3, the orthogonal complement $\mathcal{Q}^{\perp} = \mathcal{C} \ominus \mathcal{Q}$ becomes $\mathcal{C} \ominus \mathcal{Q} = \text{Span}[AN, A\xi]$. From the assumption of $S\phi = \phi S$ on the distribution $\mathcal{Q}^{\perp}$ it can be easily checked that the distribution $\mathcal{Q}^{\perp}$ is invariant by the shape operator S. Then (5.1) gives the following for $SAN = \lambda AN$,

$$\begin{aligned}(2\lambda-\alpha)S\phi AN &= (\alpha\lambda+2)\phi AN - 2A\xi\\ &= (\alpha\lambda+2)\phi AN - 2\phi AN\\ &= \alpha\lambda\phi AN.\end{aligned}$$

Then $A\xi = \phi AN$ gives the following:

$$SA\xi = \frac{\alpha\lambda}{2\lambda-\alpha}A\xi. \tag{5.2}$$

Then from the assumption $S\phi = \phi S$ on $\mathcal{Q}^{\perp} = \mathcal{C}\ominus\mathcal{Q}$, it follows that $\lambda = \frac{\alpha\lambda}{2\lambda-\alpha}$ gives

$$\lambda = 0 \quad \text{or} \quad \lambda = \alpha. \tag{5.3}$$

On the other hand, on the distribution $\mathcal{Q}$ we know that $AX \in T_zM$, $z \in M$, because $AN \in Q$. So (5.1), together with the fact that $g(X, A\xi) = 0$ and $g(X, AN) = 0$ for any $X \in \mathcal{Q}$, implies that

$$2S\phi SX = \alpha(S\phi + \phi S)X + 2\phi X. \tag{5.4}$$

Then we can take an orthonormal basis $X_1, \ldots, X_{2(m-2)} \in \mathcal{Q}$ such that $SX_i = \lambda_i X_i$ for $i = 1, \ldots, m-2$. Then by (5.1) we know that

$$S\phi X_i = \frac{\alpha\lambda_i + 2}{2\lambda_i - \alpha}\phi X_i.$$

Accordingly, by (5.3) the shape operator S can be expressed as

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 0(\alpha) & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 0 & 0(\alpha) & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \lambda_1 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & \lambda_{m-2} & 0 & \cdots & 0\\ 0 & 0 & 0 & 0 & \cdots & 0 & \mu_1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \mu_{m-2} \end{bmatrix},$$

where $\mu_i = \frac{\alpha\lambda_i+2}{2\lambda_i-\alpha}$ for $i = 1, \ldots, m-2$. From (4.4), together with $\eta(B\xi) = g(A\xi, \xi) = 0$, we have that

$$h(\phi S - S\phi)X = (\phi S^2 - S^2\phi)X, \tag{5.5}$$

where $h = \mathrm{Tr}S$ denotes the trace of the shape operator of M in Q^m.

Now let us consider the Ricci commuting property with $\mathfrak{A}$-isotropic normal vector field for $SX = \lambda X$, $X \in \mathcal{C}$. Then by (5.5), it follows that from $SX = \lambda X$, $X \in \mathcal{C}$,

$$(\lambda - \mu)\{h - (\lambda + \mu)\}\phi X = 0, \tag{5.6}$$

where we have used $S\phi X = \mu\phi X$ with $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$, $X \in \mathcal{Q}$ and $\mu = \frac{\alpha\lambda}{2\lambda-\alpha}$, $X \in Q^{\perp}$ respectively. Then (5.6) gives

$$\lambda = \mu \quad \text{or} \quad h = \lambda + \mu. \tag{5.7}$$

On the other hand, for such an $SX = \lambda X$, $X \in \mathcal{C}$, (5.5) gives

$$h\lambda\phi X - hS\phi X = \lambda^2\phi X - S^2\phi X. \tag{5.8}$$

Here we decompose $X \in \mathcal{C} = \mathcal{Q} \oplus \mathcal{Q}^\perp$ as

$$X = Y + Z,$$

where $Y \in \mathcal{Q}$ and $Z \in \mathcal{Q}^\perp$. Then $SX = \lambda X = \lambda Y + \lambda Z$ gives the following:

$$SY = \lambda Y \quad \text{and} \quad SZ = \lambda Z,$$

because the distributions $\mathcal{Q}$ and $\mathcal{Q}^\perp$ are invariant by the shape operator. Then by using the matrix representation of the shape operator, (5.8) gives the following decompositions:

$$h\lambda\phi Y - h\left(\frac{\alpha\lambda+2}{2\lambda-\alpha}\right)\phi Y = \lambda^2\phi Y - \left(\frac{\alpha\lambda+2}{2\lambda-\alpha}\right)^2\phi Y, \quad Y \in \mathcal{Q}, \tag{5.9}$$

$$h\lambda\phi Z - h\left(\frac{\alpha\lambda}{2\lambda-\alpha}\right)\phi Z = \lambda^2\phi Z - \left(\frac{\alpha\lambda}{2\lambda-\alpha}\right)^2\phi Z, \quad Z \in \mathcal{Q}^\perp. \tag{5.10}$$

By taking inner products of (5.9) and (5.10) with the vector fields ϕY and ϕZ, respectively, we have

$$\lambda^2 - h\lambda + \frac{\alpha\lambda+2}{2\lambda-\alpha}\left\{h - \frac{\alpha\lambda+2}{2\lambda-\alpha}\right\} = 0, \tag{5.11}$$

$$\lambda^2 - h\lambda + \frac{\alpha\lambda}{2\lambda-\alpha}\left\{h - \frac{\alpha\lambda}{2\lambda-\alpha}\right\} = 0. \tag{5.12}$$

Then subtracting (5.12) from (5.11) gives

$$h = \frac{2\alpha\lambda+2}{2\lambda-\alpha}. \tag{5.13}$$

Now from (5.7) let us consider the following two cases:

Case 1. $\lambda = \mu$.

From the matrix representation of the shape operator, $\lambda = \frac{\alpha\lambda+2}{2\lambda-\alpha}$ gives that

$$\lambda^2 - \alpha\lambda - 1 = 0.$$

Since the discriminant $D = \alpha^2 + 4 > 0$, we have two distinct solutions $\lambda = \cot r$ and $\mu = -\tan r$ with the multiplicities $(m-2)$ and $(m-2)$, respectively, i.e., the shape operator S can be expressed as

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0(\alpha) & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0(\alpha) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cot r & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cot r & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\tan r & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -\tan r \end{bmatrix}.$$

This means that the shape operator S commutes with the structure tensor ϕ, i.e., $S\cdot\phi = \phi\cdot S$. Then by Theorem 1.3, $m = 2k$, and M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{C}P^k$ in Q^{2k}.

Case 2. $\lambda \neq \mu$.

Now we only consider $\lambda \neq \mu$ on the distribution $\mathcal{Q}$. Since on the distribution $\mathcal{Q}^\perp$ we have assumed that $S\phi = \phi S$, it follows that $\lambda = \frac{\alpha\lambda}{2\lambda-\alpha}$. This gives $\lambda = 0$ or $\lambda = \alpha$ on the distribution $\mathcal{Q}^\perp$. Moreover, by the Ricci commuting property, we have the following from (5.3), together with (5.6), (5.7) and (5.13),

$$h = \lambda + \mu = \lambda + \frac{\alpha\lambda+2}{2\lambda-\alpha} = \frac{2\alpha\lambda+2}{2\lambda-\alpha}. \tag{5.14}$$

This gives $\lambda = 0$ or $\lambda = \alpha$. Then we have two subcases.

Subcase 2.1. $\lambda = 0$.

Then we can arrange the matrix of the shape operator such that

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0(\alpha) & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0(\alpha) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{2}{\alpha} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -\frac{2}{\alpha} \end{bmatrix}.$$

In this case the formula $h = \lambda + \mu$ and the notion of trace h of the shape operator S give

$$h = 0 - \frac{2}{\alpha} = \alpha + (m-2)\left(-\frac{2}{\alpha}\right).$$

Then it gives that $\alpha^2 = 2(m-3)$, i.e., $\alpha = \sqrt{2(m-3)}$.

Now let us consider the case $h = 3\alpha + (m-2)(-\frac{2}{\alpha})$. Then, from this, together with $h = \lambda + \mu = -\frac{2}{\alpha}$, we know $\alpha^2 = \frac{2}{3}(m-3)$. Then $\alpha = \sqrt{\frac{2}{3}(m-3)}$.

Subcase 2.2. $\lambda = \alpha$.

In this subcase, the expression of the shape operator becomes

$$S = \begin{bmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \alpha(0) & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \alpha(0) & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \alpha & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\alpha^2+2}{\alpha} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \frac{\alpha^2+2}{\alpha} \end{bmatrix}.$$

In this case also the formula $h = \lambda + \mu$ and the notion of trace h of the shape operator S yield

$$h = \alpha + \frac{\alpha^2+2}{\alpha} = (m+1)\alpha + (m-2)\frac{\alpha^2+2}{\alpha}.$$

Then it implies that $(2m-3)\alpha^2 = -2m+4$, which gives us a contradiction for $m \geqslant 3$.

Next, we consider the case that $h = (m-1)\alpha + (m-2)\frac{\alpha^2+2}{\alpha}$ in the above expression. Then

$$h = \lambda + \mu = \alpha + \frac{\alpha^2+2}{\alpha}$$

gives

$$(2m-5)\alpha^2 + 2(m-3) = 0,$$

which also gives a contradiction for $m \geqslant 4$.

Summing up the above discussions, we assert the following:

Theorem 5.2. *Let M be a real hypersurface in complex quadric Q^m, $m \geqslant 4$, with commuting Ricci tensor and $\mathfrak{A}$-isotropic normal. If the shape operator commutes with the structure tensor on the distribution $\mathcal{Q}^{\perp}$, then M is locally congruent to a tube of radius r over a totally geodesic $\mathbb{C}P^k$, $m = 2k$, in Q^{2k} or M has 3 distinct constant principal curvatures given by*

$$\alpha = \sqrt{2(m-3)}, \quad \gamma = 0, \quad \lambda = 0, \quad \text{and} \quad \mu = -\frac{2}{\sqrt{2(m-3)}} \quad \text{or}$$
$$\alpha = \sqrt{\frac{2}{3}(m-3)}, \quad \gamma = 0, \quad \lambda = 0, \quad \text{and} \quad \mu = -\frac{\sqrt{6}}{\sqrt{m-3}}$$

with corresponding principal curvature spaces

$$T_\alpha = [\xi], \quad T_\gamma = [A\xi, AN], \quad \phi(T_\lambda) = T_\mu, \quad \dim T_\lambda = \dim T_\mu = m-2.$$

6 Proof of Theorem 1.5 for $\mathfrak{A}$-principal unit normal vector field

In this section, we want to prove our Theorem 1.5 for real hypersurfaces with commuting Ricci tensor and $\mathfrak{A}$-principal unit normal vector field.

From the basic formulas for the real structure A and the Kähler structure J we have the following:

$$JAX = J\{BX + \rho(X)N\} = \phi BX + \eta(BX)N - \rho(X)\xi,$$
$$AJX = A\{\phi X + \eta(X)N\} = B\phi X + \rho(\phi X)N - \eta(X)\phi B\xi - \eta(X)\eta(B\xi)N.$$

From this, the anti-commuting structure $AJ = -JA$ gives the following:

$$\phi BX + \eta(BX)N - \rho(X)\xi = -B\phi X - \rho(\phi X)N + \eta(X)\phi B\xi + \eta(X)\eta(B\xi)N. \tag{6.1}$$

Then comparing the tangential and normal component of (6.1) gives the following:

$$\eta(BX) = -\rho(\phi X) + \eta(X)\eta(B\xi) \tag{6.2}$$

and

$$\phi BX = -B\phi X + \rho(X)\xi + \eta(X)\phi B\xi. \tag{6.3}$$

Since N is $\mathfrak{A}$-principal, i.e., $AN = N$, we know that $B\xi = -\xi$, and $\phi BX = -B\phi X$. Then (6.3) yields

$$-2\phi BX + (\text{Tr}S)(\phi S - S\phi)X - (\phi S^2 - S^2\phi)X = 0, \tag{6.4}$$

where we have used $\rho(X) = g(AX, N) = 0$. When N is $\mathfrak{A}$-principal, on the distribution $\mathcal{C} = \mathcal{Q}$ we have

$$2S\phi S - \alpha(\phi S + S\phi) = 2\phi. \tag{6.5}$$

So if we put $SX = \lambda X$ in (6.5), we have

$$S\phi X = \mu\phi X = \frac{\alpha\lambda + 2}{2\lambda - \alpha}\phi X. \tag{6.6}$$

Then from (6.3) and (6.5), it follows that

$$-2\phi BX + (\lambda - \mu)\{h - (\lambda + \mu)\}\phi X = 0. \tag{6.7}$$

It is well known that the tangent space T_zQ^m of the complex quadric Q^m is decomposed as

$$T_zQ^m = V(A) \oplus JV(A),$$

where $V(A) = \{X \in T_zQ^m \mid AX = X\}$ and $JV(A) = \{X \in T_zQ^m \mid AX = -X\}$. So $SX = \lambda X$ for $X \in \mathcal{C}$. The vector field X can be decomposed as follows:

$$X = Y + Z, \quad Y \in V(A), \quad Z \in JV(A),$$

where $AY = BY = Y$ and $AZ = BZ = -Z$. So it follows that $BX = AX = AY + AZ = Y - Z$. Then $\phi BX = \phi Y - \phi Z$. From this, together with (6.7), it follows that

$$-2(\phi Y - \phi Z) + (\lambda - \mu)\{h - (\lambda + \mu)\}(\phi Y + \phi Z) = 0.$$

Then by taking inner products with ϕY and ϕZ, respectively, we get

$$(\lambda - \mu)\{h - (\lambda + \mu)\} - 2 = 0 \tag{6.8}$$

and

$$(\lambda - \mu)\{h - (\lambda + \mu)\} + 2 = 0. \tag{6.9}$$

This gives a contradiction. Accordingly, we conclude that real hypersurfaces in Q^m with commuting Ricci tensor and $\mathfrak{A}$-principal normal vector field do not exist.

Summing up the above discussions, we assert the following:

Theorem 6.1. *There does not exist any real hypersurface in complex quadric Q^m, $m \geqslant 4$, with commuting Ricci tensor and $\mathfrak{A}$-prinicipal normal vector field.*

From Theorems 5.2 and 6.1, together with Lemma 4.2, we give a complete proof of our Theorem 1.5 in Section 1.

Remark 6.2. In this paper, we have proved that a tube over a totally geodesic $\mathbb{C}P^k$ in Q^m, $m = 2k$, mentioned in our main theorem is Ricci commuting, i.e., $\mathrm{Ric}\cdot\phi = \phi\cdot\mathrm{Ric}$. But related to the notion of Ricci parallel, Suh [19] asserted that a tube over $\mathbb{C}P^k$ never has parallel Ricci tensor, i.e., the Ricci tensor does not satisfy $\nabla\mathrm{Ric} = 0$.

Remark 6.3. In [21], a non-existence property of parallel normal Jacobi operator $\nabla\bar{R}_N = 0$ for Hopf real hypersurfaces in Q^m was given. Motivated by this result, it is interesting to consider the classification problem of commuting normal Jacobi operator, i.e., $\bar{R}_N\cdot\phi = \phi\cdot\bar{R}_N$, where $\bar{R}$ denotes the curvature tensor of the complex quadric Q^m.

Remark 6.4. In [22], Suh has also given a classification of parallel structure Jacobi operator R_ξ, i.e., $\nabla R_\xi = 0$. Related to this fact, another problem is to consider a complete classification of commuting structure Jacobi operator, i.e., $R_\xi\cdot\phi = \phi\cdot R_\xi$, where the structure Jacobi operator R_ξ is defined by $R_\xi X = R(X,\xi)\xi$ for the Reeb vector field ξ and any vector field X on a real hypersurface M in Q^m. Here R denotes the curvature tensor of M in Q^m.

Remark 6.5. As another commuting problem between Jacobi operators $\bar{R}_N$, R_ξ and the Ricci tensor Ric, it will be interesting to study complete classifications of Hopf hypersurfaces in Q^m with commuting properties like $R_\xi\cdot\bar{R}_N = \bar{R}_N\cdot R_\xi$, $\mathrm{Ric}\cdot R_\xi = R_\xi\cdot\mathrm{Ric}$ or $\mathrm{Ric}\cdot\bar{R}_N = \bar{R}_N\cdot\mathrm{Ric}$.

Acknowledgements This work was supported by National Research Foundation of Korea (Grant No. NRF-2015-R1A2A1A-01002459). The authors express their deep gratitude to the referees for their valuable comments and suggestions to develop the first version of this manuscript.

References

1 Berndt J, Suh Y J. Real hypersurfaces with isometric Reeb flow in complex quadrics. Internat J Math, 2013, 24: 1350050

2 Berndt J, Suh Y J. Contact hypersurfaces in Kaehler manifold. Proc Amer Math Soc, 2015, 143: 2637–2649

3 Kimura M. Real hypersurfaces and complex submanifolds in complex projective space. Trans Amer Math Soc, 1986, 296: 137–149

4 Kimura M. Some real hypersurfaces of a complex projective space. Saitama Math J, 1987, 5: 1–5

5 Klein S. Totally geodesic submanifolds in the complex quadric. Differential Geom Appl, 2008, 26: 79–96

6 Kobayashi S, Nomizu K. Foundations of Differential Geometry: Vol II. New York: Wiley-Interscience, 1996

7 Pérez J D. Commutativity of Cho and structure Jacobi operators of a real hypersurface in a complex projective space. Ann Mat Pura Appl, 2015, 194: 1781–1794

8 Pérez J D, Suh Y J. Real hypersurfaces of quaternionic projective space satisfying $\nabla_{U_i} R = 0$. Differential Geom Appl, 1997, 7: 211–217

9 Pérez J D, Suh Y J. Certain conditions on the Ricci tensor of real hypersurfaces in quaternionic projective space. Acta Math Hungar, 2001, 91: 343–356

10 Pérez J D, Suh Y J. The Ricci tensor of real hypersurfaces in complex two-plane Grassmannians. J Korean Math Soc, 2007, 44: 211–235

11 Reckziegel H. On the geometry of the complex quadric. In: Geometry and Topology of Submanifolds, vol. 8. Singapore: World Scientific, 1995, 302–315

12 Smyth B. Differential geometry of complex hypersurfaces. Ann of Math, 1967, 85: 246–266

13 Suh Y J. Real hypersurfaces in complex two-plane Grassmannians with commuting Ricci tensor. J Geom Phys, 2010, 60: 1792–1805

14 Suh Y J. Real hypersurfaces in complex two-plane Grassmannians with parallel Ricci tensor. Proc Roy Soc Edinburgh Sect A, 2012, 142: 1309–1324

15 Suh Y J. Real hypersurfaces in complex two-plane Grassmannians with harmonic curvature. J Math Pures Appl, 2013, 100: 16–33

16 Suh Y J. Hypersurfaces with isometric Reeb flow in complex hyperbolic two-plane Grassmannians. Adv Appl Math, 2013, 50: 645–659

17 Suh Y J. Real hypersurfaces in the complex quadric with Reeb parallel shape operator. Internat J Math, 2014, 25: 1450059

18 Suh Y J. Real hypersurfaces in the complex hyperbolic two-plane Grassmannians with commuting Ricci tensor. Internat J Math, 2015, 26: 1550008

19 Suh Y J. Real hypersurfaces in the complex quadric with parallel Ricci tensor. Adv Math, 2015, 281: 886–905

20 Suh Y J. Real hypersurfaces in complex quadric with harmonic curvature. J Math Pures Appl, 2016, 106: 393–410

21 Suh Y J. Real hypersurfaces in the complex quadric with parallel normal Jacobi tensor. Math Nachr, 2016, doi: 10.1002/mana.201500428

22 Suh Y J. Real hypersurfaces in the complex quadric with parallel structure Jacobi tensor. ArXiv:1605.05316, 2016

23 Suh Y J, Woo C. Real hypersurfaces in complex hyperbolic two-plane Grassmannians with parallel Ricci tensor. Math Nachr, 2014, 55: 1524–1529