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ORIGINAL PAPER

Yamabe Solitons on Three-Dimensional $N(k)$-Paracontact Metric Manifolds

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Abstract We prove that if a three-dimensional $N(k)$-paracontact metric manifold admits a Yamabe soliton (g, ξ), then the scalar curvature is constant and the manifold is a paraSasakian manifold. Moreover, we show that if a three-dimensional $N(k)$-paracontact metric manifold admits a Yamabe soliton (g, V), then either the manifold is a space of constant curvature, or the flow vector field V is Killing.

Keywords $N(k)$-paracontact manifold · Yamabe soliton · Scalar curvature · Constant curvature

Mathematics Subject Classification Primary 53C15; Secondary 53C25 · 53D10

1 Introduction

The study of the Yamabe flow appeared in the work of Hamilton [18] as a tool to construct Yamabe metrics on compact Riemannian manifolds. A time-dependent metric $g(\cdot, t)$ on a Riemannian or, pseudo-Riemannian manifold M is said to evolve by the Yamabe flow if the metric g satisfies

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$$\frac{\partial}{\partial t} g(t) = -r g(t), \quad g(0) = g_0 \tag{1.1}$$

on M where r is the scalar curvature. Ye [29] has found a point-wise elliptic gradient estimate for the Yamabe flow on a locally conformally flat compact Riemannian manifold. In the case of Ricci flow, Yamabe solitons or the singularities of the Yamabe flow appear naturally. A *Yamabe soliton* is defined on a Riemannian or, pseudo-Riemannian manifold (M, g) by a vector field V satisfying the equation [3]

$$\frac{1}{2} \pounds_V g = (r - \lambda) g, \tag{1.2}$$

where $\pounds_V g$ denotes the Lie derivative of the metric g along V, r stands for the scalar curvature, while λ is a constant. A Yamabe soliton is said to be *expanding, steady,* or *shrinking*, respectively, if $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$. Otherwise, it will be called *indefinite*.

When the vector field V is a gradient of a smooth function $f : M \to \mathbb{R}$, the equation (1.2) becomes

$$(r - \lambda) g = \operatorname{Hess} f, \tag{1.3}$$

where Hess f denotes the Hessian of f and in this case f is called the *potential function* of the Yamabe soliton and g is said to be a *gradient Yamabe soliton*.

There are several papers on the Yamabe flow by Barbosa and Ribeiro Jr [3], Brendle [4], Cao et al. [9], Chow [14], Yang and Zhang [27] and many others. According to Hsu [19], the metric of any compact gradient Yamabe soliton is a metric of constant scalar curvature (see also Daskalopoulos and Sesum [16]). Yamabe solitons on a three-dimensional Sasakian manifold were studied by Sharma [25]. A complete classification of Yamabe solitons of non-reductive homogeneous 4-spaces was given by Calvaruso and Zaeim [8]. Very recently, Wang [26] proved that a three-dimensional Kenmotsu manifold admitting a Yamabe soliton is of constant sectional curvature -1 and the Yamabe soliton is expanding with $\lambda = -6$.

Therefore, it is interesting to consider Yamabe solitons on three-dimensional $N(k)$-paracontact manifolds and in this direction we have the following theorems.

Theorem 1.1 *If a three-dimensional $N(k)$-paracontact metric manifold admits a Yamabe soliton (g, ξ), ξ being the Reeb vector field of the paracontact metric structure, then the scalar curvature is constant and the manifold is a paraSasakian manifold.*

Theorem 1.2 *If a three-dimensional $N(k)$-paracontact metric manifold admits a Yamabe soliton (g, V), V being an arbitrary vector field, then the manifold is a space of constant curvature, or the flow vector field V is Killing.*

The present paper is organized as follows: In Sect. 2 we give some well-known basic results on three-dimensional $N(k)$-paracontact metric manifolds. Last section is devoted to the detailed proof of Theorems 1.1 and 1.2.

2 Preliminaries

The study of nullity distribution on paracontact geometry is one among the most interesting topics in modern paracontact geometry. Kaneyuki and Kozai [20] initiated the study of paracontact geometry. Since then, many authors [1,2,10,15] contribute to the study of paracontact geometry. A systematic study of paracontact metric manifolds was carried out by Zamkovoy [30]. The importance of paracontact geometry interplays with the theory of para-Kähler manifolds and its role in pseudo-Riemannian geometry and mathematical physics. More recently, Cappelletti-Montano et al. [12] introduced a new type of paracontact geometry, so-called paracontact metric (k, μ)-spaces, where k and μ are some real constants. Martin-Molina [22,23] obtained some classification theorems on paracontact metric (k, μ)-spaces and constructed some examples (see also [6]).

A smooth manifold M^{2n+1} has an almost paracontact structure (ϕ, ξ, η) if it admits a $(1, 1)$-type tensor field ϕ, a vector field ξ (called the Reeb vector field) and a 1-form η satisfying the following conditions [21]

(i) $\phi^2 X = X - \eta(X)\xi$,
(ii) $\phi(\xi) = 0,\ \ \eta \circ \phi = 0,\ \ \eta(\xi) = 1$,
(iii) the tensor field ϕ induces an almost paracomplex structure on each fibre of $\mathcal{D} = \ker(\eta)$, that is, the eigendistributions $\mathcal{D}^+_\phi$ and $\mathcal{D}^-_\phi$ of ϕ corresponding to the eigenvalues 1 and -1, respectively, have same dimension n.

An almost paracontact manifold equipped with a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.1}$$

for all $X, Y \in \chi(M)$, is called *almost paracontact metric manifold* and (ϕ, ξ, η, g) is said to be an *almost paracontact metric structure*.

An almost paracontact structure is *normal* [30] if and only if the $(1, 2)$-type torsion tensor $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi = 0$, where $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$. An almost paracontact structure is called a *paracontact* structure if $g(X, \phi Y) = d\eta(X, Y)$ [30]. Any almost paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ admits (at least, locally) a ϕ-basis [30], that is, a pseudo-orthonormal basis of vector fields of the form $\{\xi, E_1, E_2, \ldots, E_n, \phi E_1, \phi E_2, \ldots, \phi E_n\}$, where $\xi, E_1, E_2, \ldots, E_n$ are space-like vector fields and then, by (2.1) the vector fields $\phi E_1, \phi E_2, \ldots, \phi E_n$ are time-like. For a three-dimensional almost paracontact metric manifold, any (local) pseudo-orthonormal basis of $\ker(\eta)$ determines a ϕ-basis, up to sign. If $\{e_2, e_3\}$ is a (local) pseudo-orthonormal basis of $\ker(\eta)$, with e_3, time-like, so by (2.1) vector field $\phi e_2 \in \ker(\eta)$ is time-like and orthogonal to e_2. Therefore, $\phi e_2 = \pm e_3$ and $\{\xi, e_2, \pm e_3\}$ is a ϕ-basis [7]. In a paracontact metric manifold, one can introduce a symmetric, trace-free $(1, 1)$-tensor $h = \frac{1}{2}\pounds_\xi \phi$ satisfying [13,30]

$$\phi h + h\phi = 0, \quad h\xi = 0, \tag{2.2}$$

$$\nabla_X \xi = -\phi X + \phi h X, \tag{2.3}$$

for all $X \in \chi(M)$. Note that, the condition $h = 0$ is equivalent to ξ being Killing vector field and then (ϕ, ξ, η, g) is said to be *K-paracontact structure*. An almost paracontact metric manifold is said to be *paraSasakian* manifold if and only if [30]

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X \tag{2.4}$$

holds, for any $X, Y \in \chi(M)$. A normal paracontact metric manifold is paraSasakian and satisfies

$$R(X, Y)\xi = -(\eta(Y)X - \eta(X)Y), \tag{2.5}$$

for any $X, Y \in \chi(M)$, but unlike contact metric geometry the relation (2.5) does not imply that the paracontact manifold is paraSasakian. It is well known that every paraSasakian manifold is K-paracontact. The converse is not always true, but it holds in the three-dimensional case [5].

According to Cappelletti-Montano and Di Terlizzi [13], we give the definition of paracontact metric (k, μ)-manifolds.

Definition 2.1 A paracontact metric manifold is said to be a *paracontact (k, μ)-manifold* if the curvature tensor R satisfies

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{2.6}$$

for all vector fields $X, Y \in \chi(M)$ and k, μ are real constants.

In particular, if $\mu = 0$, then the paracontact metric (k, μ)-manifold is called an $N(k)$-paracontact metric manifold. Thus, for a $N(k)$-paracontact metric manifold the curvature tensor satisfies

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y), \tag{2.7}$$

for all $X, Y \in \chi(M)$.

In a $N(k)$-paracontact metric manifold $(M^3, \phi, \xi, \eta, g)$, the following relations hold [17,24]:

$$QX = \left(\frac{r}{2} - k\right) X + \left(3k - \frac{r}{2}\right) \eta(X)\xi, \tag{2.8}$$

$$S(X, Y) = \left(\frac{r}{2} - k\right) g(X, Y) + \left(3k - \frac{r}{2}\right) \eta(X)\eta(Y), \tag{2.9}$$

$$\begin{aligned} R(X, Y)Z = &\left(\frac{r}{2} - 2k\right) \{g(Y, Z)X - g(X, Z)Y\} \\ &+ \left(3k - \frac{r}{2}\right) \{g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}, \end{aligned} \tag{2.10}$$

$$S(X, \xi) = 2k\eta(X), \tag{2.11}$$

where Q, S, R and r are the Ricci operator, Ricci tensor, curvature tensor and the scalar curvature, respectively. From (2.10) it follows that

$$R(\xi, X)Y = k\{g(X, Y)\xi - \eta(Y)X\}. \tag{2.12}$$

In addition, using (2.3) we have

$$(\nabla_X \eta)Y = g(X, \phi Y) - g(hX, \phi Y), \tag{2.13}$$

for all $X, Y \in \chi(M)$. We have the following result due to Cappelletti-Montano et al. ([12], p.686).

Lemma 2.2 *Any paracontact metric (k, μ)-manifold of dimension three is Einstein if and only if $k = \mu = 0$.*

Though any paracontact metric (k, μ)-manifold of dimension three is Einstein if and only if $k = \mu = 0$, it always admits some compatible Einstein metrics [11].

3 Proof of the Main Theorems

3.1 Proof of Theorem 1.1

Proof Let us consider a Yamabe soliton (g, ξ). Thus, from (1.2) we have

$$\frac{1}{2}\pounds_\xi g = (r - \lambda)g. \tag{3.1}$$

This implies

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 2(r - \lambda)g(X, Y). \tag{3.2}$$

Making use of (2.3) in the above equation yields

$$g(-\phi X + \phi hX, Y) + g(X, -\phi Y + \phi hY) = 2(r - \lambda)g(X, Y). \tag{3.3}$$

Applying (2.1) in (3.3) we obtain

$$g(\phi hX, Y) = (r - \lambda)g(X, Y). \tag{3.4}$$

Substituting $X = \xi$ in the above equation we have $r = \lambda$. Using this in (3.1) we have $\pounds_\xi g = 0$, thus ξ is a Killing vector field and consequently M is a K-paracontact manifold. Additionally, in dimension 3, a K-paracontact manifold is a paraSasakian manifold [5]. Moreover, since λ is constant, the scalar curvature r is constant. Thus, we have finished the proof of the theorem. □

Before we prove Theorem 1.2, we first present some key lemmas used later. From (1.2), we see that for a Yamabe soliton the vector field V is a *conformal vector field*, that is,

$$\pounds_V g = 2\rho g, \tag{3.5}$$

where ρ is called the *conformal coefficient* (here $\rho = r - \lambda$). Further, $\rho = 0$ is equivalent to V being Killing.

Lemma 3.1 [28] *On an n-dimensional Riemannian or, pseudo-Riemannian manifold (M^n, g) endowed with a conformal vector field V, we have*

$$(\pounds_V S)(X, Y) = -(n-2)g(\nabla_X D\rho, Y) + (\Delta\rho)g(X, Y),$$
$$\pounds_V r = -2\rho r + 2(n-1)\Delta\rho$$

for any vector fields X and Y, where D denotes the gradient operator and $\Delta = -divD$ denotes the Laplacian operator of g.

Lemma 3.2 *On any three-dimensional $N(k)$-paracontact metric manifold*

$$\xi(r) = 0. \tag{3.6}$$

Proof In a three-dimensional $N(k)$-paracontact manifold, we have

$$Q = \left(\frac{r}{2} - k\right) id + \left(3k - \frac{r}{2}\right) \eta \otimes \xi. \tag{3.7}$$

We have the following well-known formula on pseudo-Riemannian manifolds

$$trace\{Y \rightarrow (\nabla_Y Q)X\} = \frac{1}{2}\nabla_X r$$

for any vector field X. With the help of the above formula and Eqs. (2.3) and (2.11) we have from (3.7)

$$\xi(r)\eta(X) = 0. \tag{3.8}$$

Substituting $X = \xi$ in (3.8) implies that $\xi(r) = 0$ as we wanted to prove. □

3.2 Proof of Theorem 1.2

Proof It is well known that the Reeb vector field ξ is a unit vector field, that is, $g(\xi, \xi) = 1$. Taking the Lie derivative of this relation along the vector field V and using (1.2), we have

$$\eta(\pounds_V \xi) = -(\pounds_V \eta)(\xi) = \lambda - r. \tag{3.9}$$

Making use of $\rho = r - \lambda$ and $n = 3$ in Lemma 3.1 we obtain

$$(\pounds_V S)(X, Y) = -g(\nabla_X Dr, Y) + \Delta r g(X, Y), \tag{3.10}$$
$$\pounds_V r = -2r(r - \lambda) + 4\Delta r. \tag{3.11}$$

Taking Lie derivative of (2.9) along the vector field V we have

$$(\pounds_V S)(X, Y) = \tfrac{1}{2}(\pounds_V r)\{g(X, Y) - \eta(X)\eta(Y)\} + \left(\tfrac{r}{2} - k\right)(\pounds_V g)(X, Y)$$
$$+ \left(3k - \tfrac{r}{2}\right)\{(\pounds_V \eta)(X)\eta(Y) + \eta(X)(\pounds_V \eta)(Y)\}. \tag{3.12}$$

Equating the right hand sides of (3.10) and (3.12) and using Eqs. (1.2) and (3.11) we obtain

$$\begin{aligned} -g(\nabla_X Dr, Y) =& (2\Delta r - r(r-\lambda))\{g(X,Y) - \eta(X)\eta(Y)\} \\ &+ \left(\frac{r}{2} - k\right) 2(r-\lambda)g(X,Y) - \Delta r g(X,Y) \\ &+ \left(3k - \frac{r}{2}\right)\{(\pounds_V \eta)(X)\eta(Y) + \eta(X)(\pounds_V \eta)(Y)\}. \end{aligned} \tag{3.13}$$

For $X = Y = \xi$, the above equation gives

$$\xi(\xi(r)) = \Delta r - 4k(r-\lambda). \tag{3.14}$$

Applying Lemma 3.2 in the foregoing equation we have

$$\Delta r = 4k(r-\lambda). \tag{3.15}$$

In addition, putting $Y = \xi$ in (3.13) and using Eqs. (2.3) and (3.9) we obtain

$$\left(3k - \frac{r}{2}\right)(\pounds_V \eta)(X) = (\lambda - r)\left(\frac{r}{2} - 3k\right)\eta(X) - (\phi X)(r) + (\phi h X)(r). \tag{3.16}$$

Making use of (3.16) in (3.13) yields

$$\begin{aligned} \nabla_X Dr =& (\Delta r)X + (r(r-\lambda) - 2\Delta r)\{X - \eta(X)\xi\} - (r-\lambda)(r-2k)X \\ &- (\lambda - r)(r-6k)\eta(X)\xi + (\phi X)(r)\xi - (\phi h X)(r)\xi \\ &- (\phi Dr)\eta(X) - (\phi h Dr)\eta(X). \end{aligned} \tag{3.17}$$

From (3.17) and (3.15) we have

$$\begin{aligned} \nabla_X Dr =& -2k(r-\lambda)\{X - \eta(X)\xi\} - g(\phi Dr, X)\xi - g(\phi h Dr, X)\xi \\ &- (\phi Dr)\eta(X) - (\phi h Dr)\eta(X). \end{aligned} \tag{3.18}$$

We now consider a local orthonormal frame $\{e_i : i = 1, 2, 3\}$ on M^3. Applying the formula $S(X, Dr) = g(R(e_i, X)Dr, e_i)$ and Eqs. (2.1)–(2.3) in (3.18), we compute

$$S(X, Dr) = -(1+\alpha)g(\phi\nabla_{e_i} Dr, e_i)\eta(X) = 0,$$

where $\alpha = \pm\sqrt{1+k}$ and $i = 1, 2, 3$. Using this in (2.9) and noticing $\xi(r) = 0$, we have $(r-2k)Xr = 0$, that is, $X(r-2k)^2 = 0$. Thus, the scalar curvature of the manifold is constant. Applying this result in (3.15) gives $k(r-\lambda) = 0$. Thus, either $k = 0$, or $r = \lambda$.

`Case 1:` Suppose $k = 0$, then from Lemma 2.2 we conclude that the manifold becomes Einstein and hence, being three-dimensional, a space of constant curvature.

`Case 2:` If $r = \lambda$, then the Eq. (1.2) reduces to $\pounds_V g = 0$, that is, V is a Killing vector field, which finishes the proof of the theorem. □

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