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Partially normal composition operators relevant to weighted directed trees

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This paper is dedicated to the memory of Professor Takayuki Furuta

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A B S T R A C T

We characterize properties including p-hyponormality and p-paranormality for composition operators arising from measurable transformations on weighted directed trees, in terms of a test at each node v involving the masses at nodes in a neighborhood of nodes near v. Also constructed are certain graphs $\mathcal{E}$ "universal" for p-hyponormality in that the neighborhood of any node in any graph yielding a p-hyponormal composition operator is a certain limit of neighborhoods in $\mathcal{E}$. These results are applied to some examples with particularly regular graph structures.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable, infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. The gap between normal and hyponormal operators has been considered by many operator theorists, and this study continues. One property used in this study is p-hyponormality; an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$, $p \in (0, \infty)$. If $p = 1$, T is *hyponormal* and if $p = \frac{1}{2}$, T is *semi-hyponormal* [15]. As well, T is said to be *∞-hyponormal* if it is p-hyponormal for all $p > 0$ [10]. According to the Löwner–Heinz inequality [15,5], every q-hyponormal operator is p-hyponormal for $p \leq q$. And T is *p-paranormal* if $\| |T|^p U |T|^p x\| \geq \| |T|^p x\|^2$ for all unit vectors $x \in \mathcal{H}$. In particular, 1-paranormality is referred to as paranormality. Every q-paranormal operator is p-paranormal for $q \leq p$. It

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is obvious that every p-hyponormal operator is p-paranormal, $p \in (0, \infty)$. Hence any p-hyponormal operator is q-paranormal for all $q \in (0, \infty)$.

Previously Burnap–Jung–Lambert discussed composition operators C_T on L^2 via conditional expectations in [2] and [1], in which they proved that the classes of p-hyponormal operators are distinct for each positive real number p and the p-paranormal operators classes are as well. They used conditional expectations to detect p-hyponormality of C_T, which will be main tool of this note. Also, Jung–Lim–Park constructed examples induced by some block matrix operators in [8], in which they proved that the classes of those operators are distinct with respect to each positive real number p. In [4] a model of a block matrix operator induced by two sequences was introduced and p-hyponormality was characterized, and distinctions obtained, via conditional expectation. This paper is a continuation of the study [4] of p-hyponormalities of composition operators on ℓ^2.

Here is some notation and terminology related to conditional expectation. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let $T : X \to X$ be a nonsingular measurable transformation, i.e., $T^{-1}\mathcal{F} \subset \mathcal{F}$ and $\mu \circ T^{-1} \ll \mu$. We assume that the Radon–Nikodym derivative $h = d\mu \circ T^{-1}/d\mu$ is in L^∞. The composition operator C_T acting on $L^2 := L^2(X, \mathcal{F}, \mu)$ is defined by $C_T f = f \circ T$. The condition $h \in L^\infty$ assures that C_T is bounded. We consider here only bounded composition operators on L^2. We denote the conditional expectation of f with respect to $T^{-1}\mathcal{F}$ by $Ef = E(f|T^{-1}\mathcal{F})$. We recall some known results from [9,2,6], which will be used frequently through this paper. The interested reader can find a more extensive list of properties for conditional expectations in [6] and [12]. Every $T^{-1}\mathcal{F}$ measurable function has the form $F \circ T$, where F is $\mathcal{F}$-measurable function. Note that $F \circ T = G \circ T$ if and only if $hF = hG$; in fact, $F \circ T \geq G \circ T$ if and only if $F\chi_S \geq G\chi_S$ where $S = $ support h and χ_S is the characteristic function of S [3]. It is known that $C_T^* f = h(Ef) \circ T^{-1}$ (the previous two properties show that this expression is well-defined) and $h \circ T > 0$ a.e. Also, it is well-known that $C_T^* C_T f = hf$ for $f \in L^2$ and $C_T C_T^* f = (h \circ T)Ef$ for $f \in L^2$. In particular, we will have need of the following special case: if $\mathcal{A}$ is the purely atomic σ-subalgebra of $\mathcal{F}$ generated by the partition of X into sets of positive measure $\{A_k\}_{k=0}^\infty$, then

$$E(f|\mathcal{A}) = \sum_{k=0}^{\infty} \frac{1}{\mu(A_k)} \left(\int_{A_k} f(x) d\mu(x) \right) \chi_{A_k}. \tag{1.1}$$

The following known results will be crucial in this paper.

1° C_T is normal if and only if $T^{-1}\mathcal{F} = \mathcal{F}$ and $h = h \circ T$ [6].
2° C_T is quasinormal if and only if $h = h \circ T$ [14].
3° C_T is ∞-hyponormal if and only if $h \geq h \circ T$ [2].
4° C_T is p-hyponormal if and only if $h > 0$ and $E(1/h^p) \leq 1/(h^p \circ T)$, for $p \in (0, \infty)$ [2].
5° C_T is p-paranormal if and only if $E(h^p) \geq h^p \circ T$ [1].

The idea in [8] and [4] provides a good motivation to study composition operators on the usual Hardy space $\ell^2(V)$ defined by a node set in a weighted directed tree $\mathcal{G} = (V, E, \mu)$ (whose notation is in the next section). For a directed tree $\mathcal{G}$ with masses, a measurable transformation T can be defined on a node set V. A composition operator C_T can be defined by such a transformation T and can be analyzed using the results 1°–5°.

This paper consists of four sections. In Section 2, we introduce some fundamental definitions and properties from graph theory for our purposes. We define a weighted directed tree $\mathcal{G} = (V, E, \mu)$ which provides a measurable transformation T on $\mathcal{G}$, with associated composition operator C_T on $\ell^2(V)$. With those constructions we characterize normal, quasinormal, ∞-hyponormal, p-hyponormal, and p-paranormal composition operators C_T induced by such measurable transformations T on (V, μ). In Section 3, we consider all $C_T(\mathcal{G})$

arising from (rooted or unrooted) weighted directed trees $\mathcal{G}$ for which the degree of the nodes is bounded by D and $\|C_T\| \leq M$. For each p, we construct a single "universal" weighted directed tree $\mathcal{E}$ (with good control on its degree bound and on $\|C_T(\mathcal{E})\|$) so that any such $C_T(\mathcal{G})$ is p-hyponormal if and only if certain subgraphs of $\mathcal{G}$ are "scaled limits" of subgraphs $\mathcal{E}$ (in a sense made precise). We consider as well some related constructions. In Section 4, we apply our results to some graphs with highly regular structure (such as uniform constant degree).

2. Basic constructions and characterizations

A pair $\mathcal{G} := (V, E)$ is a *directed graph* if V is a nonempty set and E is a subset of $V \times V \backslash \{(v, v) : v \in V\}$. An element of V is called a *node* (or *vertex*, *junction*) of $\mathcal{G}$ and a member of E is called an *edge* of $\mathcal{G}$ (see [7] or [11] for more information about graph theory). If W is a nonempty subset of V, then the pair

$$\mathcal{G}_W := (W, (W \times W) \cap E)$$

is a directed subgraph of $\mathcal{G}$. A directed graph $\mathcal{G}$ is called *connected* if for any two distinct nodes u and v of $\mathcal{G}$ there exists a finite sequence $v_1, \ldots, v_n$ of nodes of $\mathcal{G}$ such that $u = v_1$, $v = v_n$, and either (v_j, v_{j+1}) or (v_{j+1}, v_j) is in E for all $j = 1, \ldots, n-1$. Set

$$\mathrm{Chi}(u) = \{v \in V : (u, v) \in E\}, \quad u \in V.$$

A member v of $\mathrm{Chi}(u)$ is called a *child* of u. For $u \in V$, the cardinality of $\mathrm{Chi}(u)$ is called the *degree* of u. Next, we put

$$\mathrm{Gen}^k(u) = \cup_{v \in \mathrm{Gen}^{k-1}(u)} \mathrm{Chi}(v), \quad u \in V,$$

where $\mathrm{Gen}^0(u) = \{u\}$. The set $\mathrm{Gen}^k(u)$ is called *generation* k (or *the* k*-th generation*) of u.

A triple (V, E, μ) is a *weighted directed graph* if (V, E) is a directed graph and $(V, \mathcal{P}(V), \mu)$ is a σ-finite measure space on V, where $\mathcal{P}(V)$ is the power set of V. We write $\mathcal{G} = (V, E, \mu)$ for the above directed graph (V, E) with measure μ. For convenience we assume henceforth that $\mu(\{v\}) > 0$ for each $v \in V$. In this case σ-finiteness implies that V is at most countable set. For a node $u \in V$, we write m_u for the point mass $\mu(\{u\})$. If, for a given node $u \in V$, there exists a unique node $v \in V$ such that $(v, u) \in E$, then we say that u has *parent* v and write $\mathrm{par}(u)$ for v. A finite sequence $\{u_j\}_{j=1}^n$ of distinct nodes is said to be a *circuit* of $\mathcal{G}$ if $n \geq 2$, either (u_j, u_{j+1}) or (u_{j+1}, u_j) in E, for all $j = 1, \ldots, n-1$, and either (u_n, u_1) or (u_1, u_n) in E. A node $u \in V$ is a *leaf* of $\mathcal{G}$ if $\mathrm{Chi}(u) = \varnothing$. A node v of $\mathcal{G}$ is called a *root* of $\mathcal{G}$, or briefly $v \in \mathrm{Root}(\mathcal{G})$, if there is no node u of $\mathcal{G}$ such that (u, v) is an edge of $\mathcal{G}$. If $\mathrm{Root} := \mathrm{Root}(\mathcal{G})$ is a one-element set, then its unique element is denoted by root. We write $V^\circ = V \setminus \mathrm{Root}$. A triple $\mathcal{G} = (V, E, \mu)$ is a *weighted directed tree* if $\mathcal{G}$ is a weighted directed graph such that $\mathcal{G}$ is connected, $\mathcal{G}$ has no circuits, and each node $v \in V^\circ$ has a parent. In this case, $\mathrm{Root}(\mathcal{G})$ has at most one element. From now on we assume $\mathcal{G}$ is a weighted directed tree without further mention.

We now consider a measurable transformation T on V defined by

$$Tv = \begin{cases} \mathrm{par}(v) & \text{if } v \in V^\circ, \\ \mathrm{root} & \text{if } v = \mathrm{root}, \end{cases}$$

with the second line relevant only if V has a root. Clearly, if $\mathcal{G}$ has no root, $Tv = \mathrm{par}(v)$ for every $v \in V$. Recall that $\ell^2(V)$ is the set of functions $\{\alpha_v\}_{v \in V}$ such that $\sum_{v \in V} |\alpha_v|^2 < \infty$. If we define $C_T : \ell^2(V) \to \ell^2(V)$ by $C_T(f) = f \circ T$, then C_T is bounded if and only if

$$\sup_{v \in V} |\mu(\mathrm{Chi}(v))/m_v| < \infty. \tag{2.1}$$

Such a bounded operator C_T is said to be the *composition operator* arising from the weighted directed tree $\mathcal{G} = (V, E, \mu)$.

Before beginning our work, we give some computations which will be used frequently in this paper. Let $\mathcal{G} = (V, E, \mu)$ be a weighted directed tree. Then, for $v \in V$,

$$h(v) = \begin{cases} \frac{\mu(\{\text{root}\} \cup \text{Chi}(\text{root}))}{m_{\text{root}}} & \text{for } v = \text{root}, \\ \frac{\mu(\text{Chi}(v))}{m_v} & \text{for } v \in V^\circ, \end{cases} \tag{2.2}$$

and

$$E(f)(v) = \frac{1}{\mu(T^{-1}(T(v)))} \sum_{u \in T^{-1}(T(v))} f(u) m_u, \qquad v \in V. \tag{2.3}$$

We first observe that if $V = \{\text{root}\}$, obviously C_T is the identity operator. To avoid the trivial case, we assume that $E \neq \varnothing$ throughout this paper.

Proposition 2.1. *Let $\mathcal{G} = (V, E, \mu)$ be a weighted directed tree. Then C_T is normal if and only if the following three conditions hold*:

(i) *$\mathcal{G}$ does not have a root,*
(ii) *$|\text{Chi}(v)| = 1$ for every $v \in V$,*
(iii) *$m_v^2 = m_{\text{Chi}(v)} m_{\text{par}(v)}$ for every $v \in V$.*

Proof. We now suppose that C_T is normal. Since $T^{-1}\mathcal{P}(V) = \mathcal{P}(V)$, $\mathcal{G}$ must not have a root and obviously $|\text{Chi}(v)| \leq 1$ for every $v \in V$. Since $h = h \circ T$, it follows from (2.2) that

$$\frac{\mu(\text{Chi}(v))}{m_v} = \frac{\mu(\text{Chi}(\text{par}(v)))}{m_{\text{par}(v)}},$$

which implies (ii) and (iii).

Conversely, by (i) and (ii), $\mathcal{G}$ has a single branch. So $T^{-1}\mathcal{P}(V) = \mathcal{P}(V)$. And the condition $h = h \circ T$ follows from (iii) obviously. $\square$

Proposition 2.2. *Let $\mathcal{G} = (V, E, \mu)$ be a weighted directed tree. If $\mathcal{G}$ has a root, then C_T is quasinormal if and only if for $v \in V^\circ$, the following two conditions hold*:

(i) *$m_v \mu(\{\text{root}\} \cup \text{Chi}(\text{root})) = m_{\text{root}} \mu(\text{Chi}(v))$ if $\text{par}(v)$ is root,*
(ii) *$m_v \mu(\text{Chi}(\text{par}(v))) = \mu(\text{Chi}(v)) m_{\text{par}(v)}$ otherwise.*

If $\mathcal{G}$ has no root, then C_T is quasinormal if and only if condition (ii) *holds.*

Proof. We first compare $h \circ T$ and h on V (use (2.2)) to consider the quasinormality of C_T. If $\mathcal{G}$ has a leaf w, since $h(w) \neq h \circ T(w)$, we may assume that $\mathcal{G}$ is leafless. Next we consider the three cases; the node v is root, $\text{par}(v)$ is root, or $\text{par}(v)$ is not root. If v is root, then $h(v) = h \circ T(v)$, which is always true. If $\text{par}(v)$ is root, we check the statement (i) easily. Finally, if $v \in V^\circ$ and $\text{par}(v) \neq \text{root}$, by applying (2.2) again, we can obtain the statement (ii). The rootless case is trivial. $\square$

Proposition 2.3. *Let $\mathcal{G} = (V, E, \mu)$ be a weighted directed tree. If $\mathcal{G}$ has a root, then C_T is p-hyponormal if and only if $\mathcal{G}$ is leafless and for $v \in V$, the following two conditions hold:*

(i) *if v or $\mathrm{par}(v)$ is root, then*

$$\sum_{u\in\mathrm{Chi(root)}} \left(\frac{m_u}{\mu(\mathrm{Chi}(u))}\right)^p \cdot m_u \le \left(\frac{m_{\mathrm{root}}}{\mu(\{\mathrm{root}\}\cup \mathrm{Chi(root)})}\right)^p \cdot \mu(\mathrm{Chi(root)}), \tag{2.4}$$

(ii) *if $\mathrm{par}(v)$ is not a root and $v \in V^\circ$, then*

$$\sum_{u\in\mathrm{Chi}(\mathrm{par}(v))} \left(\frac{m_u}{\mu(\mathrm{Chi}(u))}\right)^p \cdot m_u \le \left(\frac{m_{\mathrm{par}(v)}}{\mu(\mathrm{Chi}(\mathrm{par}(v)))}\right)^p \cdot \mu(\mathrm{Chi}(\mathrm{par}(v))). \tag{2.5}$$

In the case $\mathcal{G}$ has no root, C_T is p-hyponormal if and only if $\mathcal{G}$ is leafless and (2.5) holds.

Proof. We observe that if C_T is p-hyponormal, since $h(v) > 0$, by (2.2) $\mathcal{G}$ is leafless. First, assume that $\mathcal{G}$ has a root. For a node $v \in V$, we consider three cases as in the proof of Proposition 2.2. If the node v or $\mathrm{par}(v)$ is root, by (2.2) and (2.3) we obtain that

$$\begin{aligned} E\left(\frac{1}{h^p}\right)(v) &= \frac{1}{\mu(\{\mathrm{root}\}\cup \mathrm{Chi(root)})} \sum_{u\in\{\mathrm{root}\}\cup\mathrm{Chi(root)}} \frac{m_u}{h^p(u)} \\ &= \frac{1}{\mu(\{\mathrm{root}\}\cup \mathrm{Chi(root)})} \left(\frac{m_{\mathrm{root}}^{p+1}}{[\mu(\{\mathrm{root}\}\cup \mathrm{Chi(root)})]^p} + \sum_{u\in\mathrm{Chi(root)}} \frac{m_u^{p+1}}{[\mu(\mathrm{Chi}(u))]^p}\right). \end{aligned}$$

If $\mathrm{par}(v)$ is not a root and $v \in V^\circ$, we obtain that

$$\begin{aligned} E\left(\frac{1}{h^p}\right)(v) &= \frac{1}{\mu(\mathrm{Chi}(\mathrm{par}(v)))} \sum_{u\in\mathrm{Chi}(\mathrm{par}(v))} \frac{m_u}{h^p(u)} \\ &= \frac{1}{\mu(\mathrm{Chi}(\mathrm{par}(v)))} \sum_{u\in\mathrm{Chi}(\mathrm{par}(v))} \left(\frac{m_u}{\mu(\mathrm{Chi}(u))}\right)^p m_u. \end{aligned}$$

From the equivalent condition 4° in the introduction, we can obtain the main part of this proposition. The remaining part is straightforward. □

The inequality of (2.4) or (2.5) will be referred as "*p-hyponormality for node v*" (with respect to C_T) in what follows. We say that the node v is *p-hyponormal* (for C_T) if either (2.4) or (2.5) holds as appropriate for v.

Proposition 2.4. *Let $\mathcal{G} = (V, E, \mu)$ be a weighted directed tree. If $\mathcal{G}$ has a root, then C_T is ∞-hyponormal if and only if for $v \in V^\circ$, the following two conditions hold:*

(i) *if $\mathrm{par}(v)$ is a root, then*

$$\frac{\mu(\mathrm{Chi}(v))}{m_v} \ge \frac{\mu(\{\mathrm{root}\}\cup \mathrm{Chi(root)})}{m_{\mathrm{root}}}; \tag{2.6}$$

(ii) *if* $\mathrm{par}(v)$ *is not a root, then*

$$\frac{\mu(\mathrm{Chi}(v))}{m_v} \geq \frac{\mu(\mathrm{Chi}(\mathrm{par}(v)))}{m_{\mathrm{par}(v)}}. \tag{2.7}$$

In the case $\mathcal{G}$ has no root, C_T is ∞-hyponormal if and only if (ii) *holds.*

Proof. Applying $h \geq h \circ T$ and (2.2), we obtain this proposition. □

Proposition 2.5. *Let $\mathcal{G} = (V, E, \mu)$ be a weighted directed tree. Then C_T is p-paranormal if and only if for $v \in V$, the following two conditions hold:*

(i) *if v or* $\mathrm{par}(v)$ *is a root, then*

$$\sum_{u \in \mathrm{Chi}(\mathrm{root})} \left(\frac{\mu(\mathrm{Chi}(u))}{m_u} \right)^p \cdot m_u \geq \left(\frac{\mu(\mathrm{Chi}(\mathrm{root}) \cup \{\mathrm{root}\})}{m_{\mathrm{root}}} \right)^p \cdot \mu(\mathrm{Chi}(\mathrm{root})), \tag{2.8}$$

(ii) *if* $\mathrm{par}(v)$ *is not a root and $v \in V^\circ$, then*

$$\sum_{u \in \mathrm{Chi}(\mathrm{par}(v))} \left(\frac{\mu(\mathrm{Chi}(u))}{m_u} \right)^p \cdot m_u \geq \left(\frac{\mu(\mathrm{Chi}(\mathrm{par}(v)))}{m_{\mathrm{par}(v)}} \right)^p \cdot \mu(\mathrm{Chi}(\mathrm{par}(v))). \tag{2.9}$$

In the case $\mathcal{G}$ has no root, C_T is p-paranormal if and only if (ii) *holds.*

Proof. Mimic the proof of Proposition 2.3. □

The inequality of (2.8) or (2.9) will be referred as "*p-paranormality for node v*" (with respect to C_T) below. We say that the node v is *p-paranormal* (for C_T) if either (2.8) or (2.9) holds as appropriate for v. In what follows we will abuse language slightly and say that a weighted directed graph $\mathcal{G}$ is p-hyponormal (p-paranormal, ...) if its associated C_T is p-hyponormal (p-paranormal, ...), or, equivalently, if each of its nodes is p-hyponormal (p-paranormal, ...).

3. Universal weak hyponormal directed trees

The goal of this section is to construct a "universal" p-hyponormal (or p-paranormal) directed tree with masses. We call a (sub)directed tree consisting of one node not a root, its children and their children (including the point mass data) *a system* $\mathcal{S}$. We group the nodes of a system into generations 0, 1, and 2 in the obvious way. We say that, for any node n in a directed graph not a root, the subgraph system in which n falls in generation 1 is a *neighborhood* of n and may write $\mathcal{S}(n)$. (Informally, the neighborhood of n is the subgraph of n, the parent of n, the siblings of n, the children of n, and the "nieces and nephews" of n, with edge and mass data included.) We say that a node n is *p-hyponormal* (or *p-paranormal*, ...) if the test for p-hyponormality at node n is successful; observe that this test involves only the neighborhood of n.

Let $\mathcal{G} = (V, E, \mu)$ be a weighted directed tree. If we denote

$$M := M_{\mathcal{G}} = \sup_{u \in V} \left\{ \frac{\sum_{v \in T^{-1}u} m_v}{m_u} \right\},$$

then the norm of C_T relevant to the directed tree $\mathcal{G}$ is $M^{1/2}$ (cf. [13]), which we have assumed is finite. We consider in this section only trees such that the supremum of the degrees of the nodes is finite, and let D (or $D_{\mathcal{G}}$) be the maximum of the degrees.

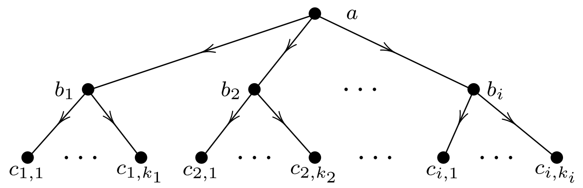

Fig. 3.1. Illustration of system $\mathcal{S}$.

We begin with the following fact:

"*Given D, the collection $\mathcal{S}_D$ of all systems $\mathcal{S}$ with each node having degree D or less, and such that each node in $\mathcal{S}$ has rational mass, is countable.*"

It obviously follows that for any $p > 0$ and any $M > 0$, the collection $\mathcal{S}^p_{D,M}$ of systems $\mathcal{S}$ in $\mathcal{S}_D$ such that each node in generation 1 of $\mathcal{S}$ is p-hyponormal, and for each node u in generation 0 or generation 1, $\frac{\sum_{v\in \mathrm{Chi}(u)} m_v}{m_u} \leq M$, is countable.

The next two lemmas concern the ability to take a system $\mathcal{S}$ in $\mathcal{S}^p_{D,M}$ and add a third generation so as to preserve properties of interest.

Lemma 3.1. *Suppose that $\mathcal{S}$ is a system in $\mathcal{S}^p_{D,M}$. Then we may add children to each member of generation* 2 *in $\mathcal{S}$ so that each (sub)system in the resulting tree $\mathcal{S}'$ is in $\mathcal{S}^p_{D,M}$. In particular, every node in generations* 1 *and* 2 *of $\mathcal{S}'$ is p-hyponormal.*

Proof. We will add a single child to each node in generation 2 of $\mathcal{S}$. To set the notation, let $\mathcal{S}$ be a system with the diagram in Fig. 3.1.

Consider node b_1 and its children $c_{1,1}, ..., c_{1,k_1}$. Observe that the rational number $(\sum_{j=1}^{k_1} m_{c_{1,j}})/m_{b_1}$ is less than or equal to M. Add $d_{1,1}$ a single child of $c_{1,1}$, $d_{1,2}$ a single child of $c_{1,2}$, and so on according to the rule

$$\frac{m_{d_{1,j}}}{m_{c_{1,j}}} = \frac{\sum_{l=1}^{k_1} m_{c_{1,l}}}{m_{b_1}}, \qquad 1 \leq j \leq k_1,$$

and note that we have preserved both M and D. Picking now some node c_{1,j_0} and its neighborhood, we have $\mathrm{par}(c_{1,j_0}) = b_1$ and $\mathrm{Chi}(\mathrm{par}\ (c_{1,j_0})) = \{c_{1,1}, ..., c_{1,k_1}\}$. Then

$$\begin{aligned}
\sum_{u\in \mathrm{Chi}(\mathrm{par}(c_{1,j_0}))} \left(\frac{m_u}{\mu(\mathrm{Chi}(u))}\right)^p \cdot m_u &= \sum_{j=1}^{k_1} \left(\frac{m_{c_{1,j}}}{m_{d_{1,j}}}\right)^p \cdot m_{c_{1,j}} \\
&= \sum_{j=1}^{k_1} \left(\frac{m_{b_1}}{\sum_{l=1}^{k_1} m_{c_{1,l}}}\right)^p \cdot m_{c_{1,j}} \\
&= \left(\frac{m_{b_1}}{\mu(\mathrm{Chi}(b_1))}\right)^p \cdot \sum_{j=1}^{k_1} m_{c_{1,j}} \qquad (3.1)\\
&= \left(\frac{m_{b_1}}{\mu(\mathrm{Chi}(b_1))}\right)^p \cdot \mu(\mathrm{Chi}(b_1)) \\
&= \left(\frac{m_{\mathrm{par}(c_{1,j_0})}}{\mu(\mathrm{Chi}(\mathrm{par}(c_{1,j_0})))}\right)^p \cdot \mu(\mathrm{Chi}(\mathrm{par}(c_{1,j_0})))
\end{aligned}$$

and this provides p-hyponormality for c_{1,j_0}. Obviously a similar result holds for each of the $c_{1,j}$; repeating the process for the nodes $b_2, ..., b_i$ and their children, we have the result. $\square$

By repeating the above construction, we get the following corollary.

Corollary 3.2. *Suppose that $\mathcal{S}$ is a system in $\mathcal{S}^p_{D,M}$. Then we may add successive generations to $\mathcal{S}$ indefinitely so as to create a tree in which every node has at least one child and every node in generations $1, 2, ...$ is p-hyponormal.*

We next modify the construction of Lemma 3.1 in a way that allows us to add a target node and children, up to scalar multiple.

Lemma 3.3. *Suppose that $\mathcal{S}$ is a system in $\mathcal{S}^p_{D,M}$ with $D \geq 2$, and let $\epsilon > 0$ be arbitrary. Consider some tree (not related to $\mathcal{S}$) as follows: let s be a node and $t_1, ..., t_k$ children of the node s for arbitrary $k \leq D$ such that $\left(\sum_{i=1}^k m_{t_i}\right)/m_s \leq M$ and $m_s, m_{t_1}, ..., m_{t_k}$ are all rational. Let c be any node in generation 2 of $\mathcal{S}$. We may add children to nodes in generation 2 of $\mathcal{S}$ and then to generation 3 such that the resulting tree has all rational masses, every neighborhood of a node in generations 2 and 3 is in $\mathcal{S}^p_{D,M+\epsilon}$, and there exists a child d of c with children $e_1, e_2, ..., e_k$ and a rational number r so that $m_d = rm_s$ and $m_{e_i} = rm_{t_i}$ $(1 \leq i \leq k)$.*

Proof. Without loss of generality, we assume that ϵ is rational. Let $\mathcal{S}$ be a system as in Fig. 3.1, and without loss of generality we take c to be $c_{1,1}$. Begin with the construction of Lemma 3.1, giving a node $c_{i,j}$ and a single child $d_{i,j}$ for every i and j so that

$$\frac{m_{d_{i,j}}}{m_{c_{i,j}}} = \frac{\sum_{l=1}^{k_i} m_{c_{i,l}}}{m_{b_i}} \quad (\leq M),$$

which yields equalities as in (3.1) for each of the $c_{i,j}$.

Now add another child to $c = c_{1,1}$, call it $\hat{d}_{1,1}$, and children $e_1, e_2, ..., e_k$ to $\hat{d}_{1,1}$, with $m_{\hat{d}_{1,1}} = \delta \cdot m_s$, $m_{e_i} = \delta \cdot m_{t_i}$ $(1 \leq i \leq k)$ with δ a small number to be determined. This clearly preserves the inequality already achieved in (3.1), and thus p-hyponormality is guaranteed for all the nodes $c_{i,j}$. (The equality is unchanged for any $c_{i,j}$ with $i \geq 2$; for any $c_{1,j}$, the right hand side is unchanged while the left hand side decreases because $\mu(\mathrm{Chi}(c_{1,1}))$ increases.) Now add a child to $d_{1,1}$, call it f, initially so that $m_f/m_{d_{1,1}} = m_{d_{1,1}}/m_{c_{1,1}}$. The test for p-hyponormality for $d_{1,1}$ or $\hat{d}_{1,1}$ has the left hand side

$$(LS) := \left(\frac{m_{d_{1,1}}}{m_f}\right)^p \cdot m_{d_{1,1}} + \left(\frac{m_{\hat{d}_{1,1}}}{\mu(\mathrm{Chi}(\hat{d}_{1,1}))}\right)^p \cdot m_{\hat{d}_{1,1}} \tag{3.2}$$

and the right hand side

$$(RS) := \left(\frac{m_{c_{1,1}}}{m_{d_{1,1}} + m_{\hat{d}_{1,1}}}\right)^p (m_{d_{1,1}} + m_{\hat{d}_{1,1}}). \tag{3.3}$$

If we take $\delta = 0$ temporarily, then $(LS) = (RS)$; if we increase m_f slightly, say by $\gamma > 0$, we get $(LS) < (RS)$; we may then increase δ slightly so as to preserve the inequality $(LS) < (RS)$, thus guaranteeing p-hyponormality for $d_{1,1}$ and $\hat{d}_{1,1}$ (and of course we may take δ and γ both rational).

By adding the child $\hat{d}_{1,1}$ to $c_{1,1}$, we have increased $\frac{\mu(\mathrm{Chi}(c_{1,1}))}{m_{c_{1,1}}}$, but may clearly choose δ so small

$$\frac{\mu(\mathrm{Chi}(c_{1,1}))}{m_{c_{1,1}}} < M + \epsilon.$$

Similarly by taking γ sufficiently small we may make

$$\frac{\mu(\mathrm{Chi}(d_{1,1}))}{m_{d_{1,1}}} < M + \epsilon.$$

Continue as in the construction of Lemma 3.1 by adding a single child $e_{l,j}$ to each of the $d_{l,j}$, except for $(l, j) = (1, 1)$, so as to ensure p-hyponormality for $d_{l,j}$. The resulting tree then has the desired properties with $r = \delta$. Hence the proof is complete. □

Note that if $D = 1$ in Lemma 3.3, we may obtain a similar result but with the resulting tree having the relevant systems in $\mathcal{S}^p_{D+1,M+\epsilon}$. Using Lemma 3.3 we are able to start with a finite tree p-hyponormal "so far" and extend it to a p-hyponormal tree including a given element of $\mathcal{S}^p_{D,M}$, up to scalar multiple.

Corollary 3.4. *Suppose that $\mathcal{S}$ is a system in $\mathcal{S}^p_{D,M}$ with $D \geq 2$, and let $\epsilon > 0$ be arbitrary. Let $\mathcal{S}'$ be any system in $\mathcal{S}^p_{D,M}$ with node s in generation 0, nodes $t_1, ..., t_q$ in generation 1, and nodes $u_{i,j}$, $1 \leq i \leq q$, $1 \leq j \leq k_i$, in generation 2, where $q \leq D$ and $k_i \leq D$, $1 \leq i \leq q$. Let c be arbitrary node in generation 2 of $\mathcal{S}$. We may add children, grandchildren, great-grandchildren, to nodes of generation 2 of $\mathcal{S}$ so the resulting tree has all nodes with rational masses, every node in generation 2, 3, and 4 has a neighborhood in $\mathcal{S}^p_{D,M+\epsilon}$, and there exists a child d of c, children $e_1, ..., e_q$ and grandchildren $f_{1,1}, ..., f_{1,k_1}, ...$ of d so that for some rational number r,*

$$rm_s = m_d, \tag{3.4a}$$

$$rm_{t_i} = m_{e_i}, \quad 1 \leq i \leq q, \tag{3.4b}$$

$$rm_{u_{i,j}} = m_{f_{i,j}}, \quad 1 \leq i \leq q, \ 1 \leq j \leq k_i. \tag{3.4c}$$

Proof. Use the construction of Lemma 3.3 and then add the $f_{i,j}$ according to (3.4c). Extend elsewhere to preserve p-hyponormality using Corollary 3.2. We have p-hyponormality at the e_i because $\mathcal{S}' \in \mathcal{S}^p_{D,M}$. Note all nodes have rational masses. □

We need some further definitions. Observe that the test at some node n for p-hyponormality (or p-paranormality) is unaffected if every mass in the neighborhood of n is multiplied by a positive constant. We say that two weighted directed graphs, $\mathcal{G}_1$ and $\mathcal{G}_2$ are *equivalent* if they are isomorphic as (unweighted) directed graphs and (with $i : \mathcal{G}_1 \to \mathcal{G}_2$ the isomorphism) there exists $r > 0$ so that $m_{i(v)} = rm_v$, for all $v \in \mathcal{G}_1$.[1] We will write $\mathcal{G}_1 \simeq \mathcal{G}_2$. (Informally, one graph is merely a "multiple" of the other.)

Given a weighted graph $\mathcal{G}$ and a sequence of weighted graphs $\{\mathcal{G}_n\}_{n=1}^{\infty}$, we say that $\lim_{n\to\infty} \mathcal{G}_n = \mathcal{G}$ if

i) each $\mathcal{G}_n$ is graph isomorphic to $\mathcal{G}$ (as unweighted graphs, with $i_n : \mathcal{G}_n \to \mathcal{G}$ the isomorphism) and
ii) $\lim_{n\to\infty} m_{i_n^{-1}(v)} = m_v$, for all $v \in \mathcal{G}$.

Observe that this is a weak condition if $\mathcal{G}$ is infinite, but if $\mathcal{G}$ is finite, it implies

$$\lim_{n\to\infty} \sup_{v\in\mathcal{G}} |m_{i_n^{-1}(v)} - m_v| = 0.$$

Given a weighted graph $\mathcal{G}$ and a sequence of weighted graphs $\{\mathcal{G}_n\}_{n=1}^{\infty}$, we say that $\mathcal{G}$ is an *equivalent-limit* of $\{\mathcal{G}_n\}_{n=1}^{\infty}$, and write $e\text{-}\lim \mathcal{G}_n = \mathcal{G}$, if there exist graphs $H_1, ..., H_n, ...$ such that

[1] We shall not always distinguish strictly between a graph and its node set. For example, we may speak of a node $v \in \mathcal{G}$ (rather than $v \in V$).

i) $\mathcal{G}_n \backsimeq H_n$, $n = 1, 2, \ldots$,
ii) $\lim_{n\to\infty} H_n = \mathcal{G}$.

Let $\mathcal{G}$ be a weighted directed tree and let T be the measurable transformation on V the vertex set of $\mathcal{G}$. Recall that the directed tree $\mathcal{G}$ is *p-hyponormal* (for C_T) if the composition operator C_T relevant to $\mathcal{G}$ is p-hyponormal. For $v \in V$, we recall that inequality of (2.4) or (2.5) [(2.8) or (2.9), resp.] is called "*p-hyponormality* [*p-paranormality*, resp.] *for node v*." We say the node v is *p-hyponormal* [*p-paranormal*, resp.].

The following proposition is obvious from the definition of "equivalent-limit" and from the conditions to test p-hyponormality (or p-paranormality, or some of the other conditions).

Proposition 3.5. *Let $\mathcal{G}$ and $\mathcal{G}_n$ be weighted directed trees. Then the following assertions hold.*

(i) *If $\mathcal{G} = e\text{-}\lim \mathcal{G}_n$ and each $\mathcal{G}_n$ is p-hyponormal (resp., p-paranormal, ...), then $\mathcal{G}$ is p-hyponormal (resp., p-paranormal, ...).*
(ii) *If n is a node of some tree $\mathcal{G}$, and $\mathcal{S}(n)$, the neighborhood system of n, is an* e-limit *of systems each p-hyponormal (resp., p-paranormal, ...) at their nodes in generation* 1, *then n is p-hyponormal (resp., p-paranormal, ...).*

In particular, the result of (ii) *holds if $\mathcal{S}(n) = e\text{-}\lim_{j\to\infty} \mathcal{S}_j$, where $\mathcal{S}_j \in \mathcal{S}^p_{D,M}$ for each j.*

We also have the following proposition.

Proposition 3.6. *Let $n \in \mathcal{G}$ be a node which is p-hyponormal. Suppose that $\mathcal{S}(n)$ is the neighborhood of n and the maximum degree of nodes in generations* 0 *and* 1 *of $\mathcal{S}(n)$ is D, denoted $D_{\mathcal{S}(n)}$, and suppose that*

$$M_{\mathcal{S}(n)} := \sup_{v\in\{\mathrm{par}(n)\}\cup\mathrm{Chi}(\mathrm{par}(n))} \frac{1}{m_v} \sum_{u\in\mathrm{Chi}(v)} m_u < \infty. \tag{3.5}$$

Then for any $\epsilon > 0$, $\mathcal{S}(n)$ is an e-limit of systems in $\mathcal{S}^p_{D_{\mathcal{S}(n)}, M_{\mathcal{S}(n)}+\epsilon}$ (in fact, a limit of them).

Proof. Consider the test for p-hyponormality at n. First, increase the mass of $\mathrm{par}(n)$ slightly to a rational number. This makes the inequality strict if it was not before; now perturb each mass at generation 1 slightly to a rational number while preserving the inequality. Finally, increase the mass of each node in generation 2 slightly to a rational number, which preserves the inequality, so the p-hyponormality for generation 1 is preserved. The resulting system has the right hand side of (3.5) changed to no worse than $M_{\mathcal{S}(n)} + \epsilon$ if we make our perturbations small. A sequence of such systems can obviously be constructed as required for $\mathcal{S}$ to be a limit of systems in $\mathcal{S}^p_{D,M_{\mathcal{S}(n)}+\epsilon}$. $\square$

From the second statement of Proposition 3.5 and Proposition 3.6, we get the following proposition.

Proposition 3.7. *Let n be a node of a weighted directed tree $\mathcal{G}$. Suppose $\epsilon > 0$. Then the following assertions hold.*

(i) *A node n is p-hyponormal if and only if $\mathcal{S}(n)$ is an* e-limit *of systems $\mathcal{S}_j$ in $\mathcal{S}^p_{D_{\mathcal{S}(n)}, M_{\mathcal{S}(n)}+\epsilon}$.*
(ii) *If the tree $\mathcal{G}$ containing n has bounds M and D, then n is p-hyponormal if and only if $\mathcal{S}(n)$ is an* e-limit *of systems $\mathcal{S}_j$ in $\mathcal{S}^p_{D,M+\epsilon}$.*

The point of this is that $\mathcal{S}^p_{D,M+\epsilon}$ serves as a "test set" to detect p-hyponormality for any node in a weighted directed tree with bounds D and M. We now assemble things to produce a single tree whose systems contain enough of $\mathcal{S}^p_{D,M+\epsilon}$ to test a weighted directed tree $\mathcal{G}$ with D and M for p-hyponormality.

Theorem 3.8. *Let $M > 0$, $D \in \mathbb{N}$, $D \geq 2$, $p > 0$ and $\epsilon > 0$ be arbitrary. There exists a p-hyponormal directed tree $\mathcal{E} = \mathcal{E}(M, D, \epsilon, p)$ with $D_{\mathcal{E}} = D$ and $M_{\mathcal{E}} < M + \epsilon$, and such that for all trees $\mathcal{G}$ with $D_{\mathcal{G}} \leq D$ and $M_{\mathcal{G}} \leq M$, the following assertions hold.*

(i) *If $\mathcal{G}$ has no root, $\mathcal{G}$ is p-hyponormal if and only if for every node n of $\mathcal{G}$, there exist $\mathcal{S}_j$, with $\mathcal{S}_j$ a system in $\mathcal{E}$ for all j, so that $\mathcal{S}(n) = e\text{-}\lim_{j\to\infty} \mathcal{S}_j$.*

(ii) *If $\mathcal{G}$ has a root, then the test (2.4) for p-hyponormality of the root r holds and $\mathcal{G}$ has the property in* (i).

Note that $\mathcal{E}$ has the additional property that $m_v \in \mathbb{Q}_+$ for all $v \in \mathcal{E}$. Also, if $D = 1$, then we may obtain the analogous result with $D_{\mathcal{E}} = 2$.

Proof. We have matters in place to construct the tree $\mathcal{E}$. First, enumerate the (countably many) elements of $\mathcal{S}^p_{D,M+\epsilon/2}$. Begin $\mathcal{E}_0 := \mathcal{E}$ with any root r, its children and their children satisfying the p-hyponormality test for the root r and $\mathrm{Chi}(r)$, and so that $D_{\mathcal{E}_0} \leq D$ and $M_{\mathcal{E}_0} \leq M$, and so that all masses are rational. Repeated use of Corollary 3.4 and Corollary 3.2 allow us to add, successively, nodes in our enumerated listing of $\mathcal{S}^p_{D,M+\epsilon/2}$ (up to equivalence) and fill out other parts of the tree preserving p-hyponormality at each step. (For example, to add the first node in our listing of $\mathcal{S}^p_{D,M+\epsilon/2}$ requires that we go to a tree $\mathcal{E}_1$ via Corollary 3.4 with no node having length more than 4 from r; we use Corollary 3.2 to "fill out" parts of the tree we aren't using to add the target node.) Each time we use Corollary 3.4 we are obliged to increase slightly the "M" of the tree we are building, but there are only countably many such uses so we can clearly ensure the final result is less than $M + \epsilon$. The resulting tree has (up to equivalence) every element of $\mathcal{S}^p_{D,M+\epsilon/2}$ using Proposition 3.7 (with ϵ there set to $\epsilon/2$) and we obtain the result.

Observe that if $D = 1$, we may obtain the analogous result if we are willing to accept $D_{\mathcal{E}} = 2$. Also note that $\mathcal{E}$ contains systems in $\mathcal{S}^p_{D,M+\epsilon}$ not in $\mathcal{S}^p_{D,M+\epsilon/2}$ and many systems not arrived at as one of our target systems from $\mathcal{S}^p_{D,M+\epsilon/2}$. However, since all nodes of $\mathcal{E}$ are p-hyponormal, e-limits of these systems (used to "check" p-hyponormality for some node g of $\mathcal{G}$) cannot cause difficulties. $\square$

We leave to the interested reader the modification of these arguments to produce a tree "universal" for p-paranormality.

The theorem is a little unpleasant in two ways: we must use e-limits, and not limits, and if $\mathcal{G}$ has a root r the p-hyponormality test for r and nodes in $\mathrm{Chi}(r)$ (which are all the same test) must be done separately (that is, are not to be found by comparison with parts of the graph $\mathcal{E}$). Both of these can be fixed to some extent.

First, consider some root r, its children and grandchildren as shown in Fig. 3.2.

The p-hyponormality test for r and any of $b_1, .., b_i$ is

$$\sum_{j=1}^{i} \left(\frac{m_{b_j}}{\mu(\mathrm{Chi}(b_j))}\right)^p \cdot m_{b_j} \leq \left(\frac{m_r}{m_r + \mu(\mathrm{Chi}(r))}\right)^p \cdot \mu(\mathrm{Chi}(r)). \tag{3.6}$$

Consider now the following system, where the (leading) r is not a root and repeated letters indicate the same mass in Fig. 3.3, and $m_r = m_{r'} = m_{r''}$. Consider the test for p-hyponormality for any of $r', b_1, ..., b_i$ in Fig. 3.3, which is

$$\sum_{u \in \{r', b_1, \ldots, b_i\}} \left(\frac{m_u}{\mu(\mathrm{Chi}(u))}\right)^p \cdot m_u \leq \left(\frac{m_r}{\mu(\mathrm{Chi}(r))}\right)^p \cdot \mu(\mathrm{Chi}(r)). \tag{3.7}$$

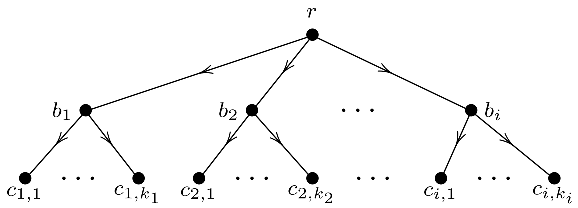

Fig. 3.2. Illustration of a system including root r.

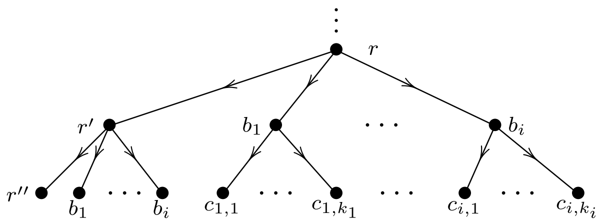

Fig. 3.3. Illustration of a tree repeated system.

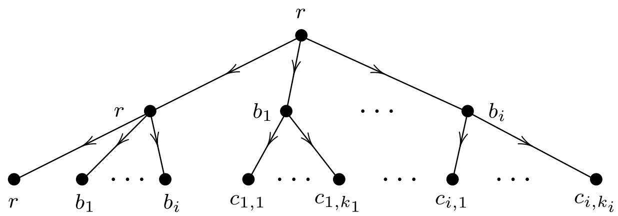

Fig. 3.4. An example of tree illustrating a root equivalent.

But this is

$$\begin{aligned}\sum_{j=1}^{i}\left(\frac{m_{b_j}}{\mu(\mathrm{Chi}(b_j))}\right)^p \cdot m_{b_j} + \left(\frac{m_{r'}}{\mu(\mathrm{Chi}(r'))}\right)^p \cdot m_{r'} \\ \leq \left(\frac{m_r}{m_{r'}+\sum_{j=1}^{i} m_{b_j}}\right)^p \cdot \sum_{j=1}^{i} m_{b_j} \\ + \left(\frac{m_r}{\mu(\mathrm{Chi}(r))}\right)^p \cdot m_{r'}.\end{aligned} \tag{3.8}$$

Using that the tree at r' looks exactly like the tree of r for the first generation and $m_r = m_{r'} = m_{r''}$, we may cancel the second terms on each side of the inequality in (3.8) and what results is exactly (3.6). Thus Fig. 3.2 is p-hyponormal (as far as $r, b_1, ..., b_i$ are concerned) if and only if Fig. 3.3 is p-hyponormal (as far as $r', b_1, ..., b_i$ are concerned).

Given an initial portion of a rooted tree as in Fig. 3.2, we call the (unrooted) system in Fig. 3.4 a "root equivalent". From the argument above nodes $b_1, ..., b_i$ and r of Fig. 3.2 are p-hyponormal if and only if the nodes at generation 1 of Fig. 3.4 are p-hyponormal. We may then modify Theorem 3.8 where now $D_{\mathcal{E}} = D + 1$, and so that clause (ii) is replaced by

(ii′) If $\mathcal{G}$ has a root, form its root equivalent $\mathcal{G}'$, and then check if $\mathcal{G}'$ is an e-limit of systems in $\mathcal{E}$, along with condition Theorem 3.8 (i).

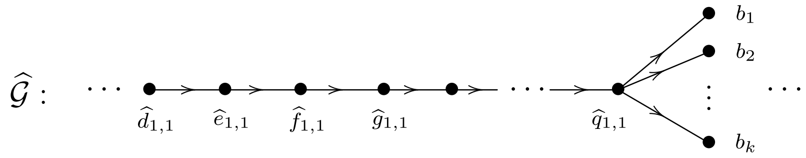

Fig. 3.5. Illustration of $\widehat{\mathcal{G}}$ including system $\mathcal{S}$ and node $\widehat{d}_{1,1}$.

It is possible, but not necessary, to modify the construction of $\mathcal{E}$ so that it contains systems of this "root equivalent" type. We leave this construction to the interested reader. In either case, $\mathcal{E}$ now contains enough systems to test both rooted and unrooted trees.

The matter of desiring limits as opposed to e-limits can also be improved somewhat but an interesting problem remains.

Suppose first that we have some system as in Fig. 3.1 in $\mathcal{S}^p_{D,M+\epsilon/2}$ we wish to add (exactly, and not merely up to scalar multiple) to the weighted directed tree we are building, and so that also

$$M \geq \frac{\sum_{j=1}^{i} m_{b_j}}{m_a} > 1. \tag{3.9}$$

Define $R := \frac{1}{m_a}\sum_{j=1}^{i} m_{b_j}$ and choose $r > 0$, rational, so that $R > r + 1$. In the construction of Lemma 3.3, when we adjoin $\hat{d}_{1,1}$, choose L a positive integer so large that $m_a/(1+r)^L$ is a small enough mass for $\widehat{d}_{1,1}$. Then we may adjoin nodes with masses as in Fig. 3.5, where $m_{\widehat{e}_{1,1}} = (1+r)m_{\hat{d}_{1,1}}$, $m_{\widehat{f}_{1,1}} = (1+r)m_{\widehat{e}_{1,1}}$, and so on, so that $m_{\widehat{q}_{1,1}} = (1+r)^L m_{\hat{d}_{1,1}} = m_a$. We observe $M_{\widehat{\mathcal{G}}}$ of the weighted directed tree $\widehat{\mathcal{G}}$ in Fig. 3.5 is less than or equal to $M+\epsilon$ as the bound, and the weighted directed tree $\widehat{\mathcal{G}}$ has p-hyponormality (in fact, for any p) by a computation in which $R > 1+r$ is used for p-hyponormality at $\widehat{q}_{1,1}$. This gives us the equivalent of Lemma 3.3 by getting the "start" of our target $\mathcal{S}'$ and then completes as in Corollary 3.4.

Observe that if $m_a, m_{b_1}, \ldots, m_{b_i}, m_{c_{1,1}}, \ldots$ are rational, then each mass in $\mathcal{G}'$ is rational as well.

Remark 3.9. We don't know how to find a construction if the target system has $\frac{\sum_{j=1}^{i} m_{b_j}}{m_a} < 1$. Perhaps one can somehow "back up" $\mathcal{S}'$ to a preceding system whose beginning has (3.9) with the needed inequality.

We now close this section with the following simple proposition.

Proposition 3.10. *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are weighted directed trees and that either $\mathcal{G}_1$ has no root or its root has p-hyponormality. Assume that for all $v \in \mathcal{G}_1$, there exists $w \in \mathcal{G}_2$ such that $\mathcal{S}(v) \simeq \mathcal{S}(w)$ and $\mathcal{G}_2$ is p-hyponormal. Then $\mathcal{G}_1$ is p-hyponormal.*

4. Some computational results

In this section we obtain some computational results related to graphs with very regular structure.

Proposition 4.1. *Let $\mathcal{G}$ be a weighted directed tree with root. Suppose $\mathcal{G}$ has regular degree (i.e., all nodes have same degree n) as in the tree in Fig. 4.1. Suppose that $m_{\mathrm{root}} = 1$ and in every generation masses are constant (so, we may say that*

$$\begin{aligned}
&m_I := m_{a_1} = m_{a_2} = \cdots = m_{a_n};\\
&m_{II} := m_{b_{1,1}} = \cdots = m_{b_{1,n}} = m_{b_{2,1}} = \cdots = m_{b_{n,n}};\\
&m_K := m_v \ \textit{for} \ v \in \mathrm{Gen}^K(\mathrm{root}),\ K \geq 3).
\end{aligned}$$

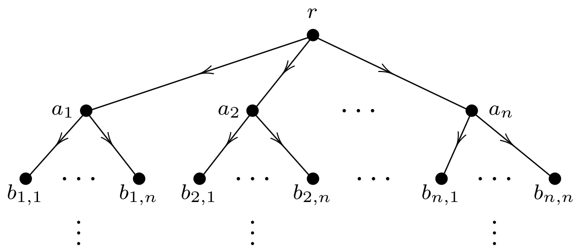

Fig. 4.1. Illustration of a rooted tree $\mathcal{G}$ of regular degree.

Then the following are equivalent:

(i) $\mathcal{G}$ *is p-hyponormal for some* $p \in (0, \infty)$;
(ii) $\mathcal{G}$ *is p-hyponormal for all* $p \in (0, \infty)$ (*i.e.,* $\mathcal{G}$ *is* ∞*-hyponormal*);
(iii) $\mathcal{G}$ *is p-paranormal for some* $p \in (0, \infty)$;
(iv) $\mathcal{G}$ *is p-paranormal for all* $p \in (0, \infty)$;
(v) *it holds that*

$$\frac{n \cdot m_{II}}{m_I} \geq 1 + n \cdot m_I \text{ and } \frac{m_K}{m_{K-1}} \geq \frac{m_{K-1}}{m_{K-2}} \quad (K \geq 3).$$

Proof. (i) $\Leftrightarrow$ (v) $\Leftrightarrow$ (ii) If par(v) or v is root, it follows from (2.4) that

$$\sum_{i=1}^{n} \left(\frac{m_I}{n \cdot m_{II}}\right)^p \cdot m_I \leq \left(\frac{1}{1 + n \cdot m_I}\right)^p \cdot n \cdot m_I,$$

which is equivalent to $\left(\frac{m_I}{n \cdot m_{II}}\right)^p \leq \left(\frac{1}{1+n \cdot m_I}\right)^p$, i.e., $\frac{n \cdot m_{II}}{m_I} \geq 1 + n \cdot m_I$. Next, if par($v$) is not a root and $v \in V^\circ$, then node v belongs to generation $K - 1$ of root ($K \geq 3$). So by the test of p-hyponormality (2.5), we have

$$\sum_{i=1}^{n} \left(\frac{m_{K-1}}{n \cdot m_K}\right)^p \cdot m_{K-1} \leq \left(\frac{m_{K-2}}{n \cdot m_{K-1}}\right)^p \cdot n \cdot m_{K-1},$$

that is, $\frac{m_K}{m_{K-1}} \geq \frac{m_{K-1}}{m_{K-2}}$ for all $K \geq 3$. To finish note these are independent of p.

(iii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (iv) If par(v) or v is root, then the inequality (2.8) for p-paranormality for $\mathcal{G}$ holds. Hence, in this case we have that

$$\sum_{i=1}^{n} \left(\frac{n \cdot m_{II}}{m_I}\right)^p \cdot m_I \geq \left(\frac{1 + n \cdot m_I}{1}\right)^p \cdot \frac{n \cdot m_I}{1},$$

i.e., $\frac{n \cdot m_{II}}{m_I} \geq 1 + n \cdot m_I$. Otherwise, if par($v$) is not a root and $v \in V^\circ$, the inequality (2.9) holds. Hence if a node v belongs to generation $K - 1$ of root ($K \geq 3$), then we obtain

$$\sum_{i=1}^{n} \left(\frac{n \cdot m_K}{m_{K-1}}\right)^p \cdot m_{K-1} \geq \left(\frac{n \cdot m_{K-1}}{m_{K-2}}\right)^p \cdot n \cdot m_{K-1},$$

that is, $\frac{m_K}{m_{K-1}} \geq \frac{m_{K-1}}{m_{K-2}}$ for all $K \geq 3$. $\square$

The following corollary comes from Proposition 4.1 immediately.

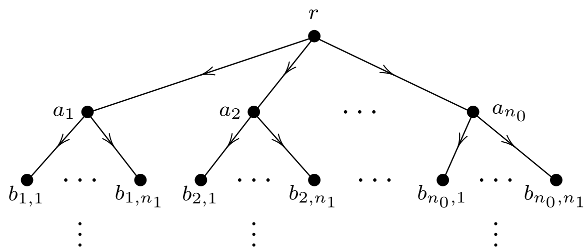

Fig. 4.2. Illustration of a rooted tree with constant degree in each generation.

Corollary 4.2. *If $\mathcal{G}$ is a weighted directed tree as in Proposition 4.1 with regular degree n and satisfying one of* (i)–(v)*, and $\mathcal{G}'$ is as in Proposition 4.1 with the same masses at each generation as $\mathcal{G}$ but with regular degree $m \geq n$, then $\mathcal{G}'$ satisfies one (hence all) of* (i)–(v)*.*

Proposition 4.1 generalizes in a reasonable way if degrees are merely constant at each generation.

Proposition 4.3. *Let $\mathcal{G}$ be a weighted directed tree with root. Let n_i denote the constant degree of children per node at generation i with constant mass on each node of a generation as in Proposition 4.1. Let $\mathcal{G}$ be the tree as in Fig. 4.2. Then the following are equivalent:*

(i) *$\mathcal{G}$ is p-hyponormal for some $p \in (0, \infty)$;*
(ii) *$\mathcal{G}$ is p-hyponormal for all $p \in (0, \infty)$;*
(iii) *$\mathcal{G}$ is p-paranormal for some $p \in (0, \infty)$;*
(iv) *$\mathcal{G}$ is p-paranormal for all $p \in (0, \infty)$;*
(v) *it holds that*

$$m_{II} \geq \frac{n_0}{n_1} \cdot m_I^2 + \frac{m_I}{n_1} \text{ and } \frac{m_K}{m_{K-1}} \geq \frac{n_{K-2}}{n_{K-1}} \cdot \frac{m_{K-1}}{m_{K-2}} \quad (K \geq 3).$$

Proof. In this proof we only consider the implications (i) $\Leftrightarrow$ (v) $\Leftrightarrow$ (ii) because other cases are similar to Proposition 4.1. If $\mathrm{par}(v)$ or v is root, it follows from (2.4) that

$$\sum_{i=1}^{n_0} \left(\frac{m_I}{n_1 \cdot m_{II}} \right)^p \cdot m_I \leq \left(\frac{1}{1 + n_0 \cdot m_I} \right)^p \cdot n_0 \cdot m_I,$$

which is equivalent to $m_{II} \geq \frac{n_0}{n_1} \cdot m_I^2 + \frac{1}{n_1} \cdot m_I$. Next, if $\mathrm{par}(v)$ is not a root and $v \in V^\circ$, by the test of p-hyponormality (2.5), similarly we have

$$\sum_{i=1}^{n_{K-2}} \left(\frac{m_{K-1}}{n_{K-1} \cdot m_K} \right)^p \cdot m_{K-1} \leq \left(\frac{m_{K-2}}{n_{K-2} \cdot m_{K-1}} \right)^p \cdot n_{K-2} \cdot m_{K-1},$$

which is equivalent to $\frac{m_K}{m_{K-1}} \geq \frac{n_{K-2}}{n_{K-1}} \cdot \frac{m_{K-1}}{m_{K-2}}$ for all $K \geq 3$. $\square$

We leave to the interested reader the formulation in the situation of Proposition 4.3 of the analog of Corollary 4.2 about increasing degrees. Note, however, that it may not be safe to decrease the (regular) degree of a p-hyponormal tree while preserving masses and expect to retain p-hyponormality. Indeed, the following is immediate from Proposition 4.1.

Corollary 4.4. *Let $\mathcal{G}$ be a weighted directed tree with root of regular degree n, with $m_{\mathrm{root}} = 1$ and $m_v \equiv m$ for all $v \in V^\circ$. Then $\mathcal{G}$ satisfies one (hence all) of* (i)–(v) *of Proposition 4.3 if and only if $\frac{n-1}{n} \geq m$.*

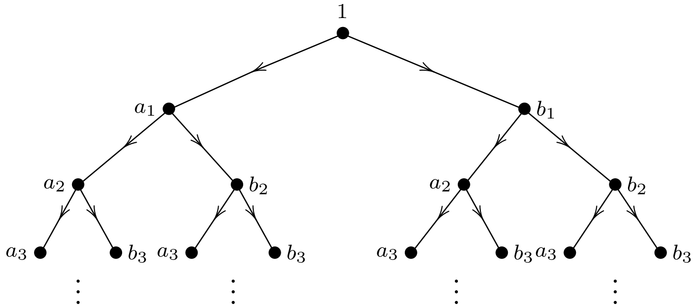

Fig. 4.3. Illustration of a rooted tree with regular degree 2.

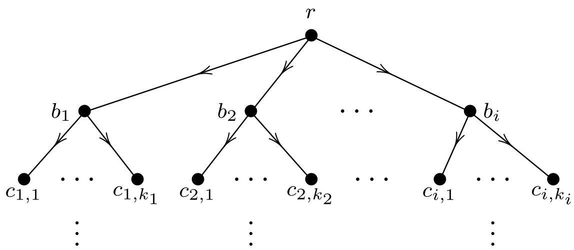

Fig. 4.4. Illustration of a tree $\mathcal{G}$ with root.

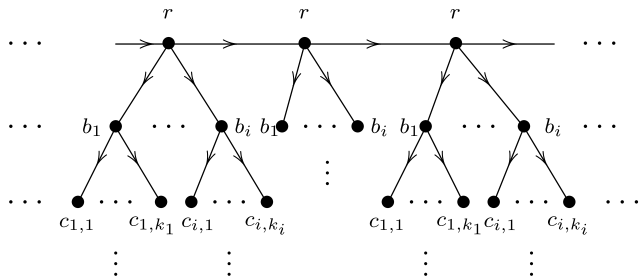

Fig. 4.5. Illustration of a tree $\mathcal{G}'$ without root.

It is easy from this to construct trees for which preserving masses and reducing degree do not preserve p-hyponormality. The consideration of trees with all weights in a particular generation the same is limiting, but we present next an example to suggest that such trees are minimal in a certain sense. Consider a tree as in Fig. 4.3, where the masses are as indicated. Suppose we desire p-hyponormality for some p with "slow growth" of the masses. Take as the measure of growth the sum of the masses at each generation (so the measure at generation 1 is $a_1 + b_1$, at generation 2 is $2(a_2 + b_2)$, and so on).

A computation using (2.5) shows that if $a_1 + b_1$ is fixed, to minimize $2(a_2 + b_2)$ it is necessary to take $a_1 = b_1$. If $a_2 + b_2$ is fixed, one should take $a_2 = b_2$ to minimize $4(a_3 + b_3)$. Continuing, it follows that the p-hyponormal tree for such slow growth has masses constant at each generation.

Finally, throughout this note the presence of a root has often made for special cases. We leave the reader to verify that the (rooted) graph $\mathcal{G}$ in Fig. 4.4 is p-hyponormal if and only if the (unrooted) weighted directed tree $\mathcal{G}'$ in Fig. 4.5 is p-hyponormal, so we may exchange the difficulties associated with a root for a (more complicated) rootless weighted directed tree if we wish.

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